

Chapter Five

Elliptic Equations

5-1 Introduction:

The governing equations in fluid mechanics and heat transfer can be reduced to elliptic form for particular applications. Such examples are the steady-state heat conduction equation, velocity potential equation for incompressible, inviscid flow, and the stream function equation. Typically elliptic equations in a two-dimensional Cartesian system are Laplace's equations,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (5-1)$$

and Poisson's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad (5-2)$$

5-2 Finite Difference Formulations:

Of the various existing finite difference formulations, the so-called "five-point formula" is the most

commonly used. In this representation of the PDE, central differencing which is second-order accurate is utilized. Therefore, model Eq (5-1) is approximated as

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{(\Delta y)^2} = 0 \quad (5-3)$$

The corresponding grid points are shown in Fig (5-1).

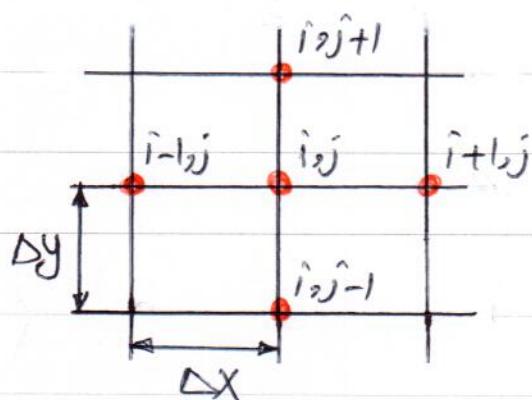


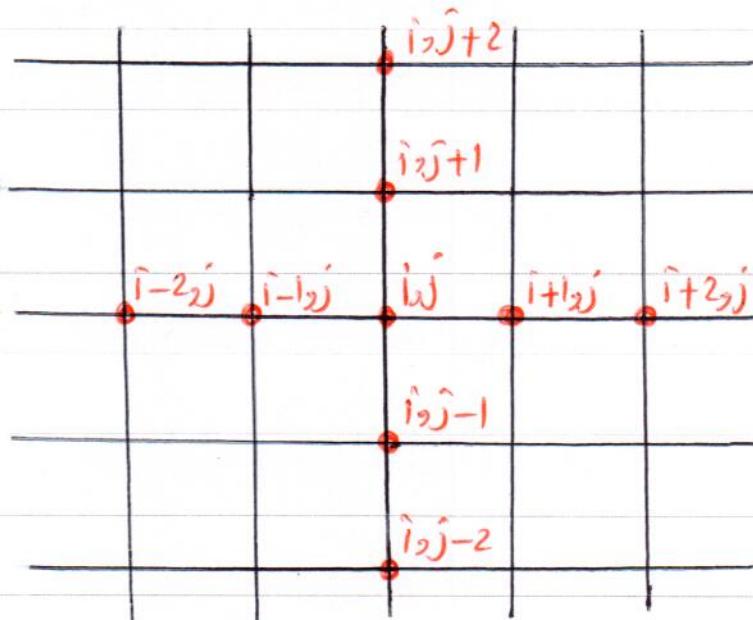
Fig (5-1) Grid points for a five-point formula.

A higher order formulation is the nine-point formula, which uses a fourth-order approximation for the derivatives. With this formulation, the FDE of model Eq (5-1) is:

$$\frac{-U_{i-2,j} + 16U_{i-1,j} - 30U_{i,j} + 16U_{i+1,j} - U_{i+2,j}}{12(\Delta x)^2} +$$

$$\frac{-U_{i,j-2} + 16U_{i,j-1} - 30U_{i,j} + 16U_{i,j+1} - U_{i,j+2}}{12(\Delta y)^2} = 0 \quad (5-41)$$

The grid points involved in above equation are shown in Fig(5-2).



Fig(5-2) Grid points for a nine-point formula.

One obvious difficulty with the application of this formula is the implementation of the boundary conditions. Thus, for problems where higher accuracy is required, it is easier to use the five-point formula with small grid sizes than the fourth-order accurate nine-point formula. Due to its simplicity, the five point formula represented by Eq(5-3) will be considered.

Rewrite Eq(5-3) as

$$U_{i+1,j} - 2U_{i,j} + U_{i-1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 (U_{i,j+1} - 2U_{i,j} + U_{i,j-1}) = 0 \quad (5-5)$$

Define the ratio of step sizes as β , so that

$\beta = \Delta x / \Delta y$. By rearranging the terms in above equation, gives

$$U_{i+1,j} + U_{i-1,j} + \beta^2 U_{i,j+1} + \beta^2 U_{i,j-1} - 2(1 + \beta^2) U_{i,j} = 0 \quad (5-6)$$

In order to explore various solution procedures, first consider a square domain

$$0 \leq X \leq 1, \quad 0 \leq Y \leq 1$$

subject to Dirichlet boundary conditions. In this example, we will use the five-point scheme, Eq(5-5), and let $\Delta x = \Delta y = 0.1$, resulting in a uniform 11×11 grid over the square domain. With $\Delta x = \Delta y$, the difference equation can be written as

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = 0 \quad (5-7)$$

In this example problem with Dirichlet boundary conditions, we have 81 grid points where U is unknown. For each one of those points, we can write the differen-

ce equation so that our problem is one of solving the system of 81 simultaneous linear algebraic equations for the 81 unknown U_{ij} . Mathematically, the problem can be expressed as

$$U_{3,2} + U_{1,2} + U_{2,3} + U_{2,1} - 4 U_{2,2} = 0$$

$$U_{4,2} + U_{2,2} + U_{3,3} + U_{3,1} - 4 U_{3,2} = 0$$

⋮

$$U_{11,10} + U_{9,10} + U_{10,11} + U_{10,9} - 4 U_{10,10} = 0$$

(5-8)

The above system of linear algebraic equations can be written as:

$$[A] U = C \quad (5-9)$$

Where

$[A]$ is the matrix of known coefficients

U is the column vector of unknowns

C is the column vector of known quantities.

Methods for solving systems of linear algebraic equations can be classified as either direct or iterative.

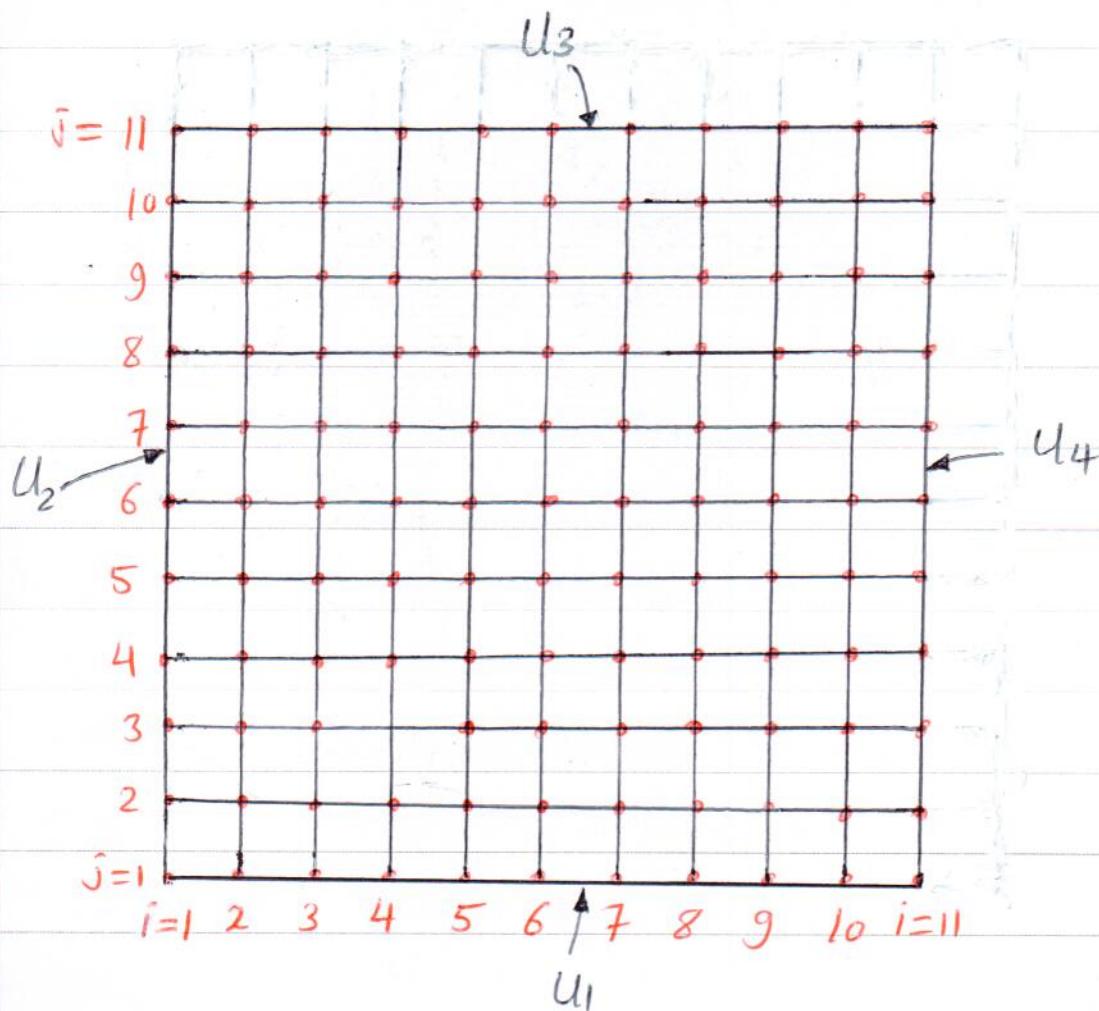


Fig (5-3) Grid system used for solution of Eq(5-7).

Some familiar direct methods are Cramer's rule and Gaussian elimination. The major disadvantage of these methods is the enormous amount of arithmetic operations required to produce a solution. Some advanced direct methods have been proposed which require moderate computation time, but almost all of them