

Eq (5-18) applied to all i at constant j (Fig 5-7b) results in a system of linear equations which has a tridiagonal matrix coefficient. Thus, Eq (5-18) can be written as

$$a_{ij} U_{i-1,j}^{k+1} + b_{ij} U_{i,j}^{k+1} + c_{ij} U_{i+1,j}^{k+1} = D_{ij} \quad (5-19)$$

where

$$a_{ij} = 1, \quad b_{ij} = -2(1 + \beta^2), \quad c_{ij} = 1$$

$$D_{ij} = -\beta^2 U_{i,j+1}^k - \beta^2 U_{i,j-1}^{k+1}$$

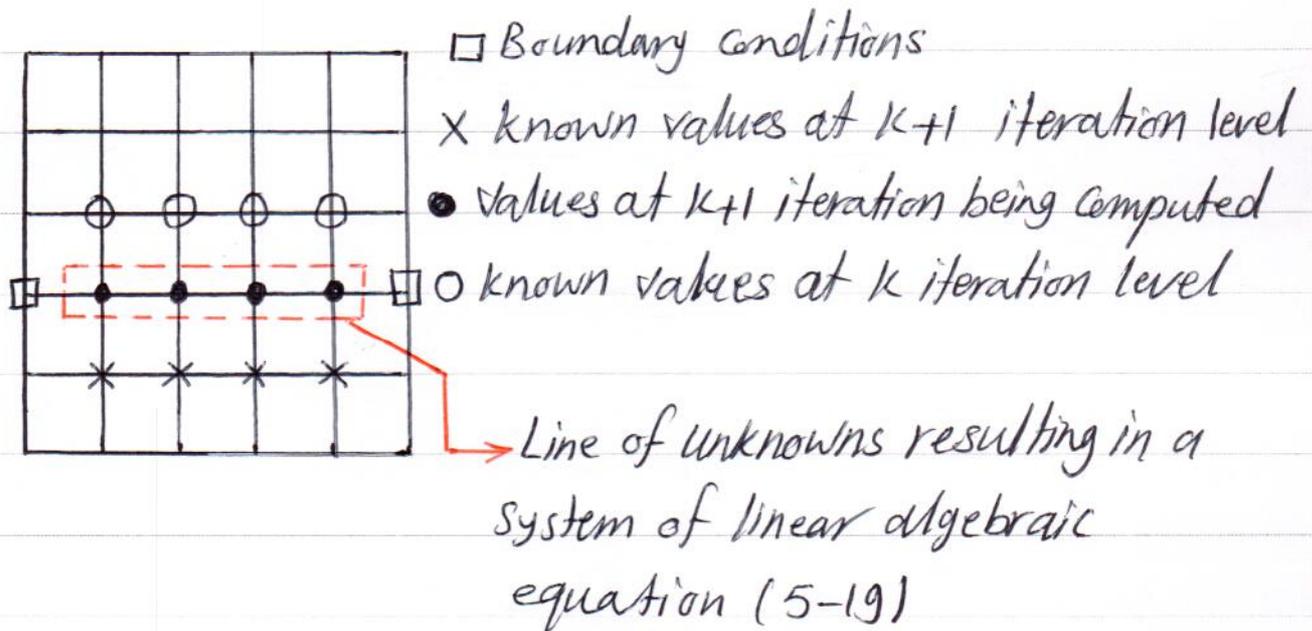


Fig (5-7) Grid points used in the line Gauss-Seidel iteration method.

This method converges faster than the point Gauss-Seidel method (by about a factor of 1/2), but it requires more computation time per iteration, since a system of simultaneous equations is being solved.

5-6 Point Successive over-Relaxation Method (SOR):

Consider the point Gauss-Seidel iteration method, given by

$$U_{ij}^{k+1} = \frac{1}{2(1+\beta^2)} \left[U_{i+1,j}^k + U_{i-1,j}^{k+1} + \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1}) \right]$$

Adding $U_{ij}^k - U_{ij}^k$ to the right-hand side, and collecting terms, one obtains:

$$U_{ij}^{k+1} = U_{ij}^k + \frac{1}{2(1+\beta^2)} \left[U_{i+1,j}^k + U_{i-1,j}^{k+1} + \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1}) - 2(1+\beta^2) U_{ij}^k \right] \quad (5-20)$$

As the solution proceeds, U_{ij}^k must approach U_{ij}^{k+1} . To accelerate the solution, the bracket term is multiplied by ω , the relaxation parameter, so that

$$U_{ij}^{k+1} = U_{ij}^k + \frac{\omega}{2(1+\beta^2)} \left[U_{i+1,j}^k + U_{i-1,j}^{k+1} + \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1}) - 2(1+\beta^2) U_{ij}^k \right] \quad (5-21)$$

For the solution to converge, it is necessary that $0 < \omega < 2$.

If $0 < \omega < 1$ under relaxation
 $\omega = 0$ Gauss-Seidel iteration method
 $1 < \omega < 2$ over relaxation

Eq (5-21) can be rearranged as

$$U_{ij}^{k+1} = (1-\omega) U_{ij}^k + \frac{\omega}{2(1+\beta^2)} \left[U_{i+1,j}^k + U_{i-1,j}^{k+1} + \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1}) \right] \quad (5-22)$$

5-7 Line Successive Over-Relaxation Method (LSOR):

The line Gauss-Seidel iteration method can be accelerated by introducing a relaxation parameter similar to the one introduced into the point Gauss-Seidel method to provide the point SOR method. The line Gauss-Seidel method for the model equation is given by Eq (5-18) as

$$U_{i-1,j}^{k+1} - 2(1+\beta^2) U_{ij}^{k+1} + U_{i+1,j}^{k+1} = -\beta^2 U_{i,j+1}^k - \beta^2 U_{i,j-1}^{k+1}$$

Introduction of the relaxation parameter and

rearranging terms results in

$$\omega U_{i-1,j}^{k+1} - 2(1+\beta^2) U_{i,j}^{k+1} + \omega U_{i+1,j}^{k+1} = - (1-\omega) [2(1+\beta^2)] U_{i,j}^k - \omega \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1}) \quad (5-23)$$

Eq (5-23) can be written in the following form:

$$a_{i,j} U_{i-1,j}^{k+1} + b_{i,j} U_{i,j}^{k+1} + c_{i,j} U_{i+1,j}^{k+1} = D_{i,j} \quad (5-24)$$

Where

$$a_{i,j} = \omega, \quad b_{i,j} = -2(1+\beta^2), \quad c_{i,j} = \omega$$

$$D_{i,j} = - (1-\omega) [2(1+\beta^2)] U_{i,j}^k - \omega \beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+1})$$

There is no simple way to determine the value of optimum ω . In practice, trial and error is used to compute ω_{opt} for a particular problem.

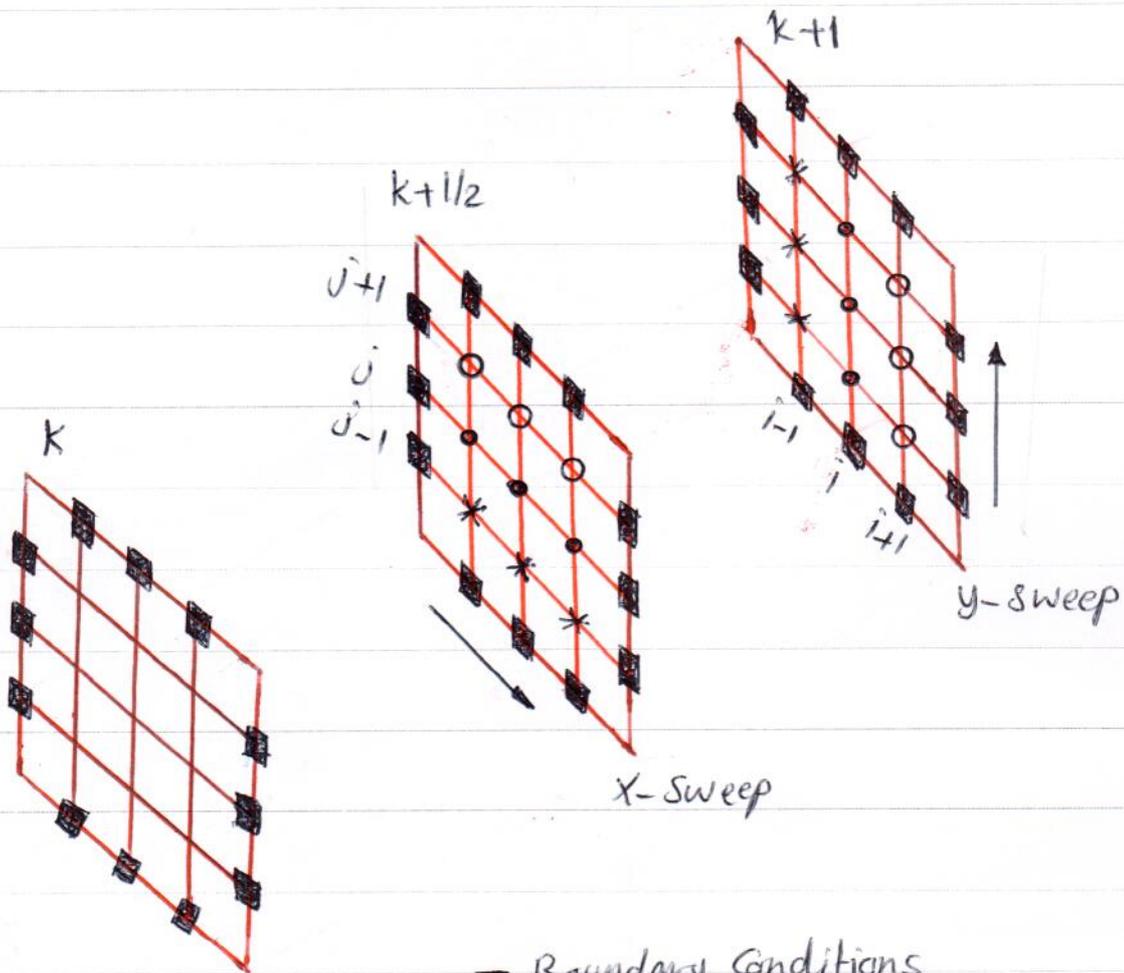
5-8 The Alternating Direction Implicit (ADI) Method:

Using ADI method, Eq (5-1) can be written as

$$U_{i-1,j}^{k+\frac{1}{2}} - 2(1+\beta^2) U_{i,j}^{k+\frac{1}{2}} + U_{i+1,j}^{k+\frac{1}{2}} = -\beta^2 (U_{i,j+1}^k + U_{i,j-1}^{k+\frac{1}{2}}), \quad (5-25)$$

and

$$\beta^2 U_{i,j}^{k+1} - 2(1+\beta^2) U_{i,j}^{k+1} + \beta^2 U_{i,j+1}^{k+1} = -U_{i+1,j}^{k+1/2} - U_{i-1,j}^{k+1} \quad (5-26)$$



- Boundary Conditions
- * Most recently computed values
- Values being computed
- Known values at previous iteration

Fig(5-8) Grid points used in Equations (5-25) and (5-26).

In these equations, (5-25) is solved implicitly for the unknown in the x-direction and (5-26) is solved implicitly in the y-direction, as shown in Fig (5-8):

The solution procedure can be accelerated by introducing a relaxation parameter ω into the ADI equations. The resulting formulations are:

$$\omega U_{i-1/2j}^{k+1/2} - 2(1+\beta^2) U_{ij}^{k+1/2} + \omega U_{i+1/2j}^{k+1/2} = -(1-\omega) [2(1+\beta^2)] U_{ij}^k - \omega \beta^2 (U_{ij+1}^k + U_{ij-1}^{k+1/2}) \quad (5-27)$$

and

$$\omega \beta^2 U_{ij-1}^{k+1} - 2(1+\beta^2) U_{ij}^{k+1} + \omega \beta^2 U_{ij+1}^{k+1} = -(1-\omega) [2(1+\beta^2)] U_{ij}^{k+1/2} - \omega (U_{i+1/2j}^{k+1/2} + U_{i-1/2j}^{k+1}) \quad (5-28)$$

However, Eqs (5-27) and (5-28) can be written in the following form

$$a_{xi} U_{i-1/2j}^{k+1/2} + b_{xi} U_{ij}^{k+1/2} + c_{xi} U_{i+1/2j}^{k+1/2} = d_{xi} \quad (5-29)$$

$$a_{yj} U_{ij-1}^{k+1} + b_{yj} U_{ij}^{k+1} + c_{yj} U_{ij+1}^{k+1} = d_{yj} \quad (5-30)$$

Where

$$ax_i = \omega$$

$$bx_i = -2(1 + \beta^2)$$

$$cx_i = \omega$$

$$dx_i = -(1 - \omega) [2(1 + \beta^2)] U_{ij}^k - \omega \beta^2 (U_{i+1,j}^k + U_{i-1,j}^k)$$

$$ay_j = \omega \beta^2$$

$$by_j = -2(1 + \beta^2)$$

$$cy_j = \omega \beta^2$$

$$dy_j = -(1 - \omega) [2(1 + \beta^2)] U_{ij}^{k+\frac{1}{2}} - \omega (U_{i+1,j}^{k+\frac{1}{2}} + U_{i-1,j}^{k+\frac{1}{2}})$$

The solution procedure starts with solution of tridiagonal system Eqs (5-29) and (5-30).

5-9 Convergence criterion:

Since the elliptic equations are solved iteratively, there will be an error between two iterations. The imposed convergence criterion is that if error \leq error_{max}, the solution has converged and the solution is stopped,

$$\text{Where } \sum_{\substack{j=2 \\ i=2}}^{\substack{j=n-1 \\ i=m-1}} |U_{ij}^{k+1} - U_{ij}^k| / U_{ij}^{k+1} \quad (5-31)$$

Example 5-1:

Consider steady-state conduction governed by Laplace's equation in the two-dimensional domain.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Boundary conditions are:

$$y=0, \quad T=T_1$$

$$y=H, \quad q_w = -k \frac{\partial T}{\partial y}$$

$$x=0, \quad T=T_2$$

$$x=L, \quad T=T_3$$

Where

$$T_1 = 200 \text{ } ^\circ\text{C}, \quad T_2 = 0, \quad T_3 = 0, \quad k = 13.4 \text{ W/m}\cdot\text{K}$$

$q_w = 10000 \text{ W/m}^2$, $L = 0.25$, $H = 0.5$. Determine the temperature distribution $T(x, y)$, using

(1) PSOR, (2) LSOR, (3) ADI

Take the grid size 31×61 and the maximum error is 0.001.

