

chapter Six

Hyperbolic Equations

6.1 Introduction:

Hyperbolic equations and methods of solution are investigated in this chapter by considering simple model equations. This chapter presents various methods of solution by finite difference approximations of the hyperbolic PDEs. Both linear and nonlinear PDEs are investigated by applying numerical methods to the model equations.

6.2 Linear first-order hyperbolic equation:

The first model equation to consider is the linear first-order wave equation,

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \quad a > 0 \quad (6-1)$$

Where

a is the wave speed

The finite difference approximations, which are formulated in explicit or implicit forms, are used

to discretize Eq(6-1). Some of these methods and their applications are presented in this section.

6.2-1. Explicit Methods:

6.2-1-1. Euler's FTFS method:

In this method, forward time and forward space approximations are used. Thus, the FDE of Eq(6-1) can be expressed as:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -a \frac{U_{i+1}^n - U_i^n}{\Delta x} \quad (6-2)$$

stability analysis indicates that this method is unconditionally unstable.

6.2-1-2. Euler's FTCS method:

In this method, central differencing is used for the spatial derivative, resulting in

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -a \frac{U_{i+1}^n - U_{i-1}^n}{2 \Delta x} \quad (6-3)$$

This method is unconditionally unstable.

6.2.1.3- The First upwind differencing method:

Backward differencing of the spatial derivative produces a finite difference equation of the form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad (6-4)$$

Von Neumann stability analysis indicates that this method is stable when $|a \Delta t / \Delta x| \leq 1$

$$\frac{a \Delta t}{\Delta x} \leq 1$$

For the model Equation (6-1), a forward differencing for the spatial derivative must be used if $a \Delta t > 0$. Therefore, the FDE for a conditionally stable solution is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_{i+1}^n - u_i^n}{\Delta x} \quad (6-5)$$

6.2.1.4. Midpoint leapfrog method:

In this method, central differencing of the second order is used for both the time and space derivatives, resulting in FDE

$$\frac{U_i^{n+1} - U_i^n}{2 \Delta t} = -\alpha \frac{U_i^n + U_{i+1}^n - U_{i-1}^n}{2 \Delta x} \quad (6-6)$$

The method is stable when, $\frac{\alpha \Delta t}{\Delta x} \ll 1$.

As the formulation indicates, two sets of initial values are required to start the solution. The dependent variable at the advanced time level $n+1$ requires the values at time level $n-1$ and n . To provide two sets of initial data, a starter solution that requires only one set of initial data, say at $n-1$, is used. The use of a starter solution will affect the order of accuracy of the method.

6-2-2. Implicit Methods:

6-2-2-1. Euler's FTCS method:

In this method forward time and central space approximation are used. This approximation applied to model equation (6-1) yields:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{\alpha}{2 \Delta x} [U_{i+1}^{n+1} - U_{i-1}^{n+1}] \quad (6-7)$$

Re-arranging above equation gives,

$$\left(\frac{\alpha \Delta t}{2\Delta x}\right) U_{i-1}^{n+1} - U_i^{n+1} - \left(\frac{\alpha \Delta t}{2\Delta x}\right) U_{i+1}^{n+1} = -U_i^n \quad (6-8)$$

Once this equation is applied to all grid points at the unknown time level, a set of linear algebraic equations will result. Again, these equations can be represented in a matrix form, where the coefficient matrix is tridiagonal. Therefore, Eq (6-8) can be written as

$$a_i U_{i-1}^{n+1} + b_i U_i^{n+1} + c_i U_{i+1}^{n+1} = d_i \quad (6-9)$$

where

$$a_i = \frac{\alpha \Delta t}{2\Delta x}$$

$$b_i = -1$$

$$c_i = -\frac{\alpha \Delta t}{2\Delta x}$$

$$d_i = -U_i^n$$

6.2.2.2. Crank-Nicolson method:

This is a widely used implicit method for which the model equation takes the form

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -a \frac{1}{2} \left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2 \Delta x} + \frac{U_{i+1}^n - U_{i-1}^n}{2 \Delta x} \right] \quad (6-10)$$

This formulation also results in a tridiagonal system of equations as follow.

$$\frac{a \Delta t}{4 \Delta x} U_{i-1}^{n+1} - U_i^{n+1} - \frac{a \Delta t}{4 \Delta x} U_{i+1}^{n+1} = -U_i^n + \frac{a \Delta t}{4 \Delta x} (U_{i+1}^n - U_{i-1}^n)$$

Thus

$$a_i U_{i-1}^{n+1} + b_i U_i^{n+1} + c_i U_{i+1}^{n+1} = d_i \quad (6-11)$$

where

$$a_i = \frac{a \Delta t}{4 \Delta x}, \quad b_i = -1, \quad c_i = \frac{-a \Delta t}{4 \Delta x}$$

$$d_i = -U_i^n + \frac{a \Delta t}{4 \Delta x} (U_{i+1}^n - U_{i-1}^n)$$

6.2.3. Multi-step methods:

These methods may be referred to as predictor-corrector methods as well. In the first step, a temporary

Value for the dependent variable is predicted, and in the second step, a corrected value is computed to provide the final value of the dependent variable. Some of these methods are introduced in this section and their applications to linear model equations illustrated. Later, these methods are extended to nonlinear problems.

6-2-3-1 Lax-Wendroff Multi-step Method:

The Lax-Wendroff method is split into two time levels. However, this method can be expressed as

$$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} (U_i^n + U_{i+1}^n) - \frac{a \Delta t}{2 \Delta x} (U_{i+1}^n - U_i^n) \quad (6-12)$$

and

$$U_i^{n+1} = U_i^n - \frac{a \Delta t}{\Delta x} (U_{i+\frac{1}{2}}^{n+\frac{1}{2}} - U_{i-\frac{1}{2}}^{n+\frac{1}{2}}) \quad (6-13)$$

The stability condition is $\frac{a \Delta t}{\Delta x} \leq 1$

6-2-3-2 MacCormack Method:

In this multi-level method, the first equation uses forward differencing resulting in the FDE

$$\frac{U_i^* - U_i^n}{\Delta t} = -a \frac{U_{i+1}^n - U_i^n}{\Delta x} \quad (6-14)$$

where * represents a temporary value of the dependent variable at the advanced level. The second equation uses backward differencing. Thus,

$$\frac{U_i^{n+1} - U_i^{n+\frac{1}{2}}}{\frac{1}{2} \Delta t} = -a \frac{U_i^* - U_{i-1}^*}{\Delta x} \quad (6-15)$$

The value of $U_i^{n+\frac{1}{2}}$ is replaced by an average value, as follows.

$$U_i^{n+\frac{1}{2}} = \frac{1}{2} (U_i^n + U_i^*) \quad (6-16)$$

The two-level MacCormack method is organized as

~~predictor step:~~ $U_i^* = U_i^n - \frac{a \Delta t}{\Delta x} (U_{i+1}^n - U_i^n)$, $(6-17)$

and

~~corrector step:~~ $U_i^{n+1} = \frac{1}{2} [(U_i^n + U_i^*) - \frac{a \Delta t}{\Delta x} (U_i^* - U_{i-1}^*)]$ $(6-18)$

The stability requirement of $\frac{a \Delta t}{\Delta x} \leq 1$

6-3 Nonlinear first-order hyperbolic equations:

The majority of partial differential equations in fluid mechanics and heat transfer are nonlinear. A classical nonlinear first-order hyperbolic equation is the inviscid Burgers equation, which can be expressed as

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} \quad (6-19)$$

which, in a conservative form, may be expressed as

$$\frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) \quad (6-20)$$

or

$$\frac{\partial u}{\partial t} = - \frac{\partial E}{\partial x} \quad (6-21)$$

where

$$E = u^2/2$$

Equation (6-19) can be interpreted as the propagation of a wave with each point having a different velocity and eventually forming a discontinuity in the domain. A discontinuity described by the function

$$U(x, 0) = 1, \quad 0 \leq x \leq 2$$

$$U(x, 0) = 0, \quad 2 \leq x \leq 4$$

Shown in Fig(6-1) is to be used as initial data to investigate its propagation throughout the domain.

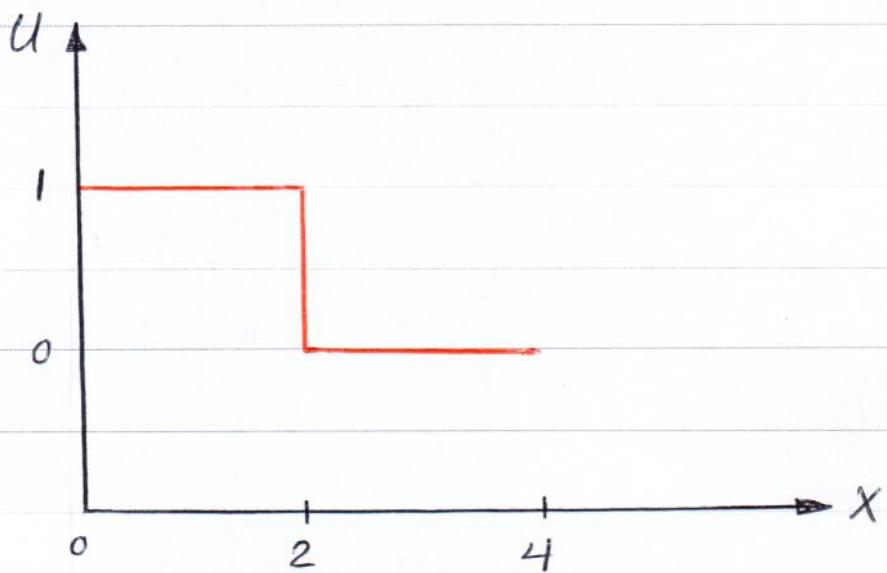


Fig (6-1) Discontinuity used as initial condition-

6.3-1 The Lax Method:

This explicit method uses forward time differencing and central space differencing. The corresponding FDE for model equation is

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = - \frac{E_{i+1}^n - E_{i-1}^n}{2 \Delta x} \quad (6-22)$$