

For stability consideration,  $U_i^n$  is replaced by its average at the neighboring points. Thus,

$$U_i^{n+1} = \frac{1}{2} (U_{i+1}^n + U_{i-1}^n) - \frac{\Delta t}{2\Delta x} (E_{i+1}^n - E_{i-1}^n)$$

or

$$U_i^{n+1} = \frac{1}{2} (U_{i+1}^n + U_{i-1}^n) - \frac{\Delta t}{4\Delta x} [(U_{i+1}^n)^2 - (U_{i-1}^n)^2] \quad (6-23)$$

The solution will be stable when

$$\left| \frac{\Delta t}{\Delta x} U_{\max} \right| \leq 1$$

since the method is first-order, it is expected that the errors will be dissipative.

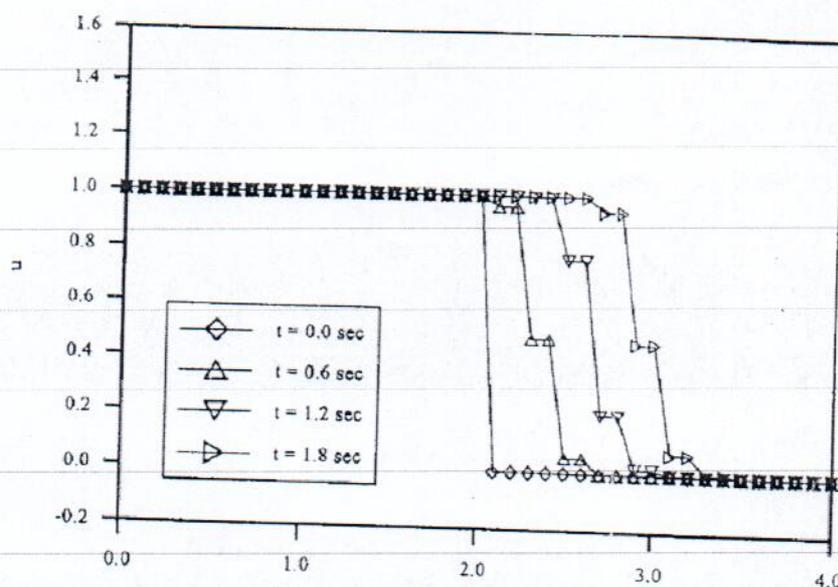


Fig (6-2) solution of the inviscid Burgers equation -

The solution for  $\Delta x = 0.1$  and  $\Delta t = 0.1$  at several time intervals is presented in Fig (6-2), which clearly reflects the dissipative nature of the solution. Note that the discontinuity is smeared over several grid points.

### 6.3.2. The Lax-Wendroff method:

Here the finite difference formulation of the method is derived from a Taylor series expansion, as in the linear case. Consider the expansion

$$U_i^{n+1} = U_i^n + \frac{\partial U}{\partial t} \Delta t + \frac{\partial^2 U}{\partial t^2} \frac{(\Delta x)^2}{2!} + \dots \quad (6-24)$$

The model equation is

$$\frac{\partial U}{\partial t} = - \frac{\partial E}{\partial x} \quad (6-25)$$

Therefore,

$$\frac{\partial^2 U}{\partial t^2} = - \frac{\partial}{\partial t} \left( \frac{\partial E}{\partial x} \right) = - \frac{\partial}{\partial x} \left( \frac{\partial E}{\partial t} \right)$$

But

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial U} \frac{\partial U}{\partial t} = \frac{\partial E}{\partial U} \left( - \frac{\partial E}{\partial x} \right) = -A \left( \frac{\partial E}{\partial x} \right)$$

But

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial u} \cdot \frac{\partial u}{\partial t} = \frac{\partial E}{\partial u} \left( -\frac{\partial E}{\partial x} \right) = -A \left( \frac{\partial E}{\partial x} \right)$$

where

$A = \frac{\partial E}{\partial u}$  is known as the Jacobian. Therefore,

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x} \left( -A \frac{\partial E}{\partial x} \right) = \frac{\partial}{\partial x} \left( A \frac{\partial E}{\partial x} \right) \quad (6-26)$$

For the model equation, where

$$E = \frac{1}{2} u^2$$

then

$$A = \frac{\partial E}{\partial u} = \frac{\partial}{\partial u} \left( \frac{1}{2} u^2 \right) = u$$

After the substitution of Eq(6-25) and Eq(6-26) into Eq(6-24), one obtains

$$u_i^{n+1} = u_i^n + \left( -\frac{\partial E}{\partial x} \right) \Delta t + \frac{\partial}{\partial x} \left( A \frac{\partial E}{\partial x} \right) \frac{(\Delta t)^2}{2!} + O(\Delta t)^3$$

or

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\left( \frac{\partial E}{\partial x} \right) + \frac{\partial}{\partial x} \left( A \frac{\partial E}{\partial x} \right) \frac{\Delta t}{2} + O(\Delta t)^2 \quad (6-27)$$

Now, the spatial derivatives are approximated by central differencing of order two, resulting in the FDE

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = - \frac{E_{i+1}^n - E_{i-1}^n}{2\Delta x} + \left[ \frac{(A \frac{\partial E}{\partial x})_{i+\frac{1}{2}}^n - (A \frac{\partial E}{\partial x})_{i-\frac{1}{2}}^n}{\Delta x} \right] \frac{\Delta t}{2} \quad (6-28)$$

At this point, the approximation

$$\frac{(A \frac{\partial E}{\partial x})_{i+\frac{1}{2}}^n - (A \frac{\partial E}{\partial x})_{i-\frac{1}{2}}^n}{\Delta x} = \frac{A_{i+\frac{1}{2}}^n \frac{E_{i+1}^n - E_i^n}{\Delta x} - A_{i-\frac{1}{2}}^n \frac{E_i^n - E_{i-1}^n}{\Delta x}}{\Delta x}$$

In above equation, the Jacobians are evaluated at the midpoints, which results in

$$\frac{\frac{1}{2\Delta x} (A_{i+\frac{1}{2}}^n + A_i^n) (E_{i+1}^n - E_i^n) - \frac{1}{2\Delta x} (A_i^n + A_{i-\frac{1}{2}}^n) (E_i^n - E_{i-1}^n)}{\Delta x} \quad (6-29)$$

Substituting  $A=U$  in Eq (6-29), and then in Eq (6-28), gives

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} (E_{i+1}^n - E_{i-1}^n)$$

$$+ \frac{\Delta t^2}{4\Delta x^2} [(u_{i+\frac{1}{2}}^n + u_i^n) (E_{i+\frac{1}{2}}^n - E_i^n) - (u_i^n + u_{i-\frac{1}{2}}^n) (E_i^n - E_{i-\frac{1}{2}}^n)] \quad (6-30)$$

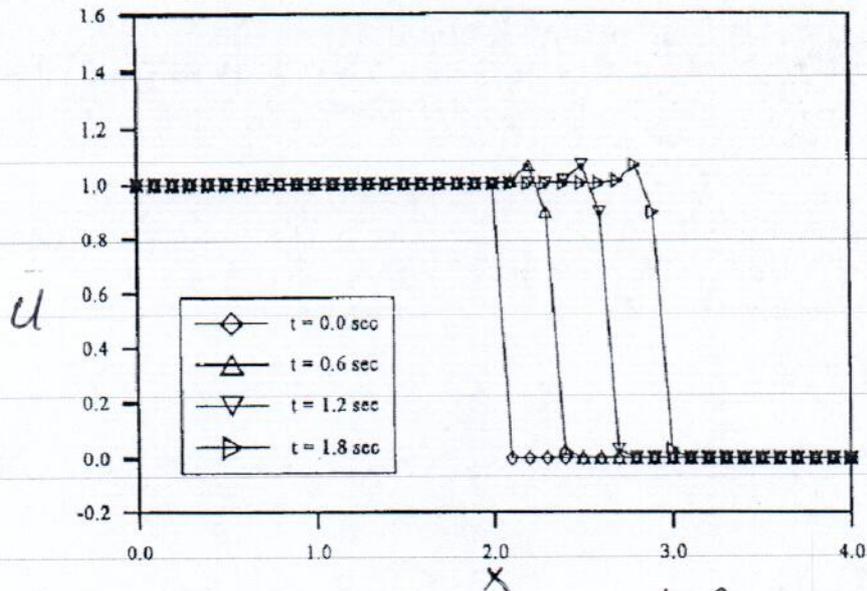


Fig (6-3) Solution of the inviscid Burgers equation by the Lax-Wendroff explicit method,  $\Delta x = 0.1$ ,  $\Delta t = 0.1$ .

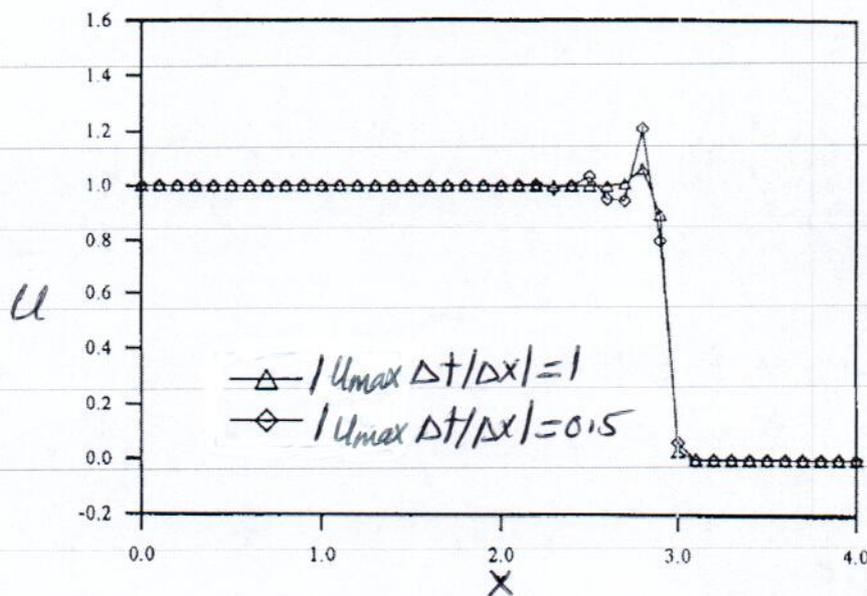


Fig (6-4) Effect of step sizes on the solution of Burgers equation by the Lax-Wendroff method.

The method is second-order, with a stability requirement of  $|u_{\max} \Delta t / \Delta x| \leq 1$ . Application of the method to the sample problem yields the solution shown in Fig (6-3), where the step sizes were  $\Delta x = 0.1$  and  $\Delta t = 0.1$ . These step sizes correspond to  $|u_{\max} \Delta t / \Delta x| = 1$ . The dispersion error is evident by the presence of oscillations in the neighborhood of the discontinuity. Two solutions obtained with various step sizes as shown in Fig (6-4). The best solution is obtained at  $|u_{\max} \Delta t / \Delta x| = 1$ .

### 6-3-3 The MacCormack method:

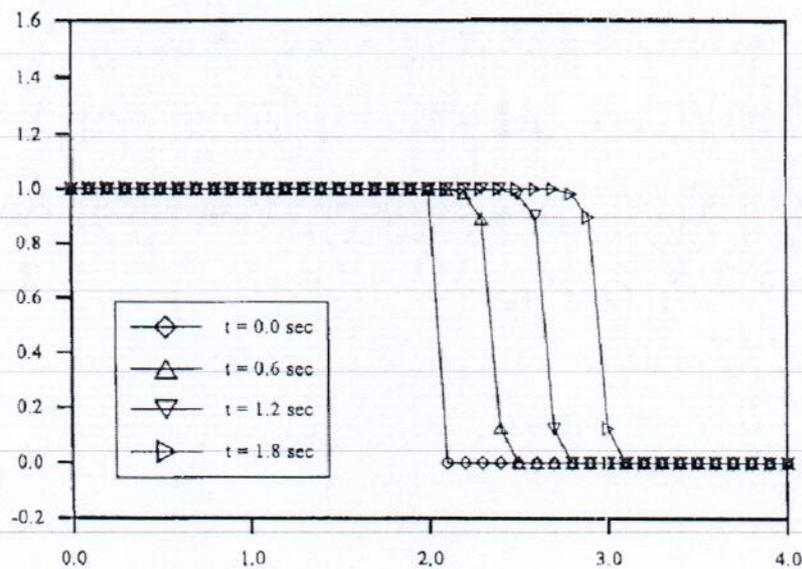
This multi-level method applied to the model equation yields the finite difference equations

$$u_i^* = u_i^n - \frac{\Delta t}{\Delta x} (E_{i+1}^n - E_i^n) \quad (6-31)$$

and

$$u_i^{n+1} = \frac{1}{2} \left[ u_i^n + u_i^* - \frac{\Delta t}{\Delta x} (E_i^* - E_{i-1}^*) \right] \quad (6-32)$$

The stability requirement of the method is  $|u_{\max} \Delta t / \Delta x| \leq 1$ .



Fig(6-5) Solution of the inviscid Burgers equation by the MacCormack explicit method,  $\Delta x = 0.1$  and  $\Delta t = 0.1$ .

The solution obtained at several time intervals with  $\Delta x = 0.1$  and  $\Delta t = 0.1$  is shown in Fig(6-5). This solution is well behaved. This is due to the splitting procedure and corresponding forward, backward differencing used to approximate the spatial derivative. As expected, the best solution is obtained when  $|U_{max} \Delta t / \Delta x| = 1$ .

## Example 6-1:

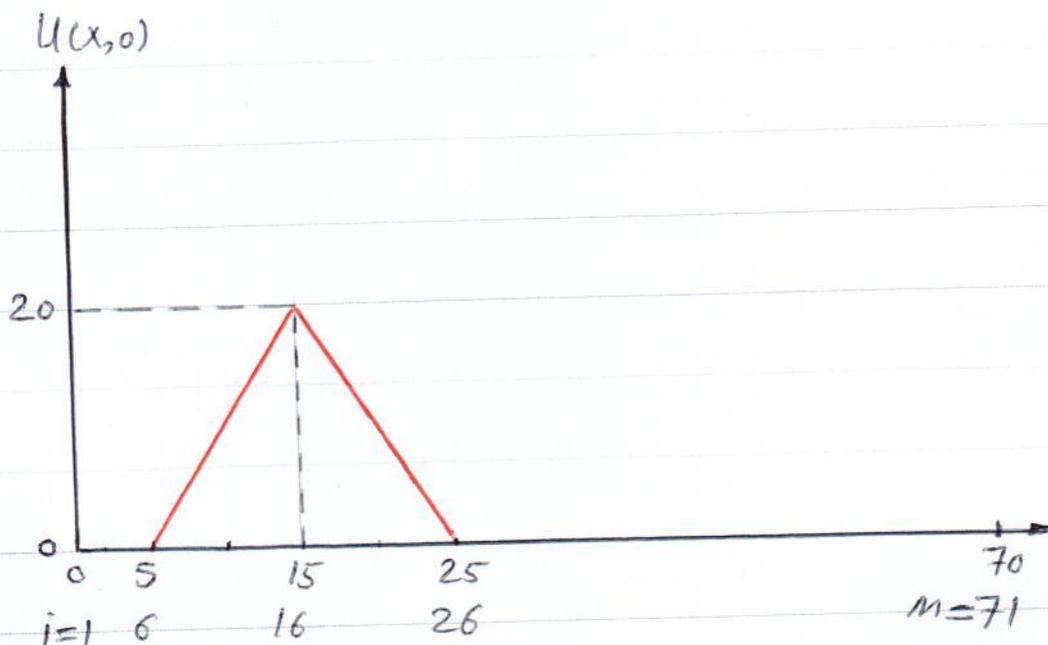
A wave is propagating in a closed-end tube. Compute the wave propagation up to  $t = 0.35$  sec by solving the first-order wave equation. Assume the speed of sound to be  $200 \text{ m/sec}$ . The wave has a triangular shape, as shown in figure, which is to be used as the initial condition at  $t = 0$ . Solve the problem by the following methods.

(a) First upwind differencing.

(b) Euler's FTCS Implicit.

The step sizes are specified as follows:

$$\Delta x = 1 \quad (M = 71), \quad \Delta t = 0.005$$



Example 6-2:

Use MacCormack method to solve the Burgers equation:

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x},$$

The initial condition is specified as:

$$u(x, 0) = 5 \quad 0 \leq x \leq 20$$

$$u(x, 0) = 0 \quad 20 \leq x \leq 40$$

print the solution at intervals of 0.15 sec up to  $t = 2.5$  sec. The following step-sizes are suggested:

$$\Delta x = 1.0, \quad \Delta t = 0.1$$