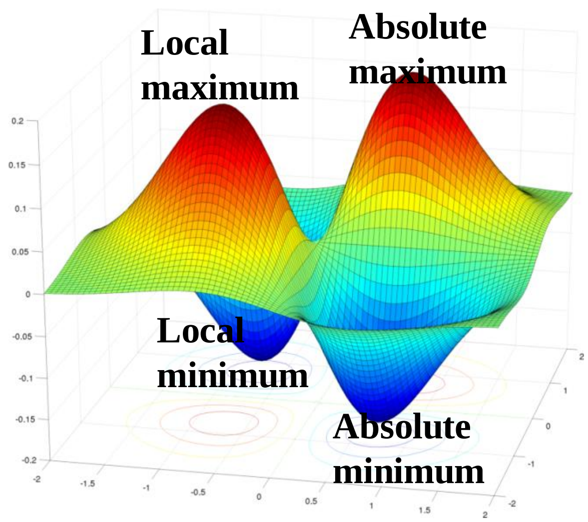
2.7. Maximum and minimum values

One of the main uses of ordinary derivatives is finding maximum and minimum values. In this section we are going to see how the partial derivatives are used to find the local maximum and minimum values of the function for two or more variables.

$$f_x = 0 \ and \ f_y = 0 \ \ at \ a \ point(a, b)$$

This point called critical point

Whether absolute point or local point



Its possible to test the function to know the critical point from this equation

$$D = f_{xx}|_{P(a.b)} \cdot f_{yy}\Big|_{p(a,b)} - \left(f_{xy}\Big|_{P(a,b)}\right)^2$$

(a) D > 0 and f_{xx} at (a.b) > 0 then f(a,b) is local minimum

(b) D > 0 and f_{xx} at (a.b) < 0 then f(a,b) is local maximum

(c) D < 0 then f(a,b) is called saddle point

Example 2.13

Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ **find the critical point**

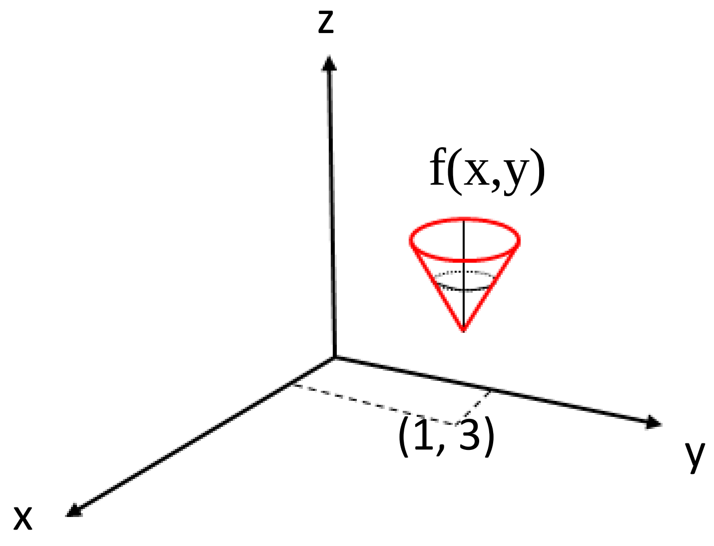
Solution

$$f_x = 2x - 2$$

$$f_y = 2y - 6$$

$$if\ f_x = 0\ then\ x = 1$$

$$if\ f_y = 0\ then\ y = 3$$



$$z|_{(1,3)} = 1^2 + 3^2 - 2 - 18 + 14 = 4$$

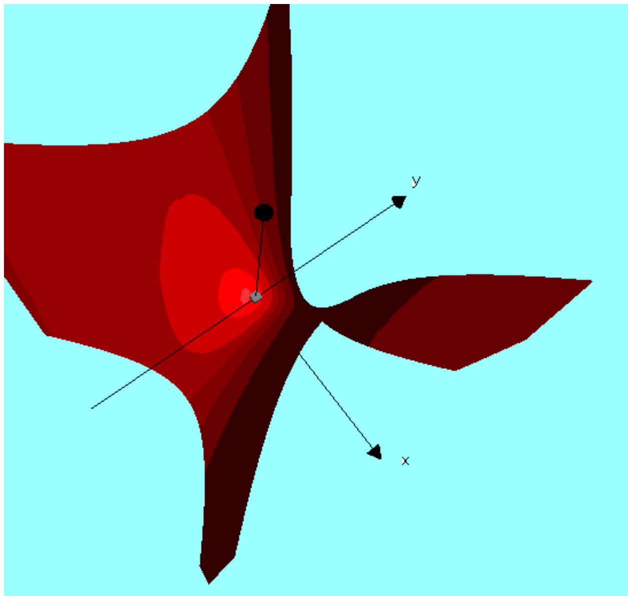
The critical point is (1,3,4)

Example 2.14

Find the critical point $f(x,y) = y^2 - x^2$

Solution

$$f_x = -2x\ and\ f_y = 2y$$



The critical point is (0,0)

For points on the x-axis (y=0) $f(x,y) = -x^2 < 0$

For points on the y-axis (x=0) $f(x,y) = y^2 > 0$

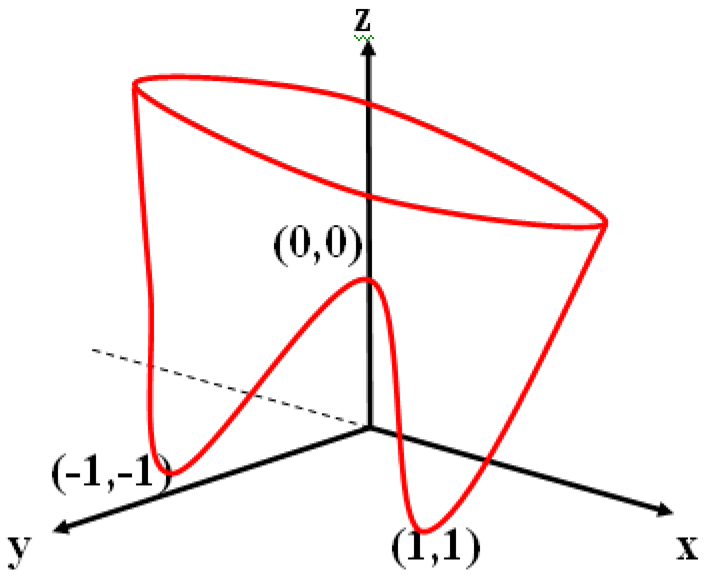
$f(0,0) = 0$ **is a maximum in the direction of x-axis and minimum in the Direction of y-axis.**

Neat the origin the graph has the shape of a saddle (0,0) here called saddle point

Example 2.15

Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$



Solution

To find the critical points

$$f_x = 4x^3 - 4y = 0, \qquad f_y = 4y^3 - 4x = 0$$

$$x^3 - y = 0 \quad and \quad y^3 - x = 0$$

$$y = x^3 \quad and \quad x = y^3$$

To find the X values

$$x^9 - x = 0 = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

x= 0,1,-1

y=0,1,-1 **(0,0), (1,1), (-1,-1)** **the critical points**

$$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = -4$$

$$D_{(x,y)} = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

$$D_{(0,0)} = 144x^2y^2 - 16 = -16 < 0 \; its\ a\ saddle\ point$$

$$D_{(1,1)} = 144 - 16 = 128 > 0 \quad f_{xx(1,1)} = 12 > 0 \; it\ is\ a\ local\ minimum$$

$$D_{(-1,-1)} = 144 - 16 = 128 > 0 \quad f_{xx(-1,-1)} = 12 > 0 \; it\ is\ a\ local\ minimum$$

2.8Absolute maximum and minimum values

To find the absolute maximum and minimum values of continuous function f (x,y) on a closed bounded set D.

1- Find the value of f at the critical point of f in D

2- Find the extreme values of f

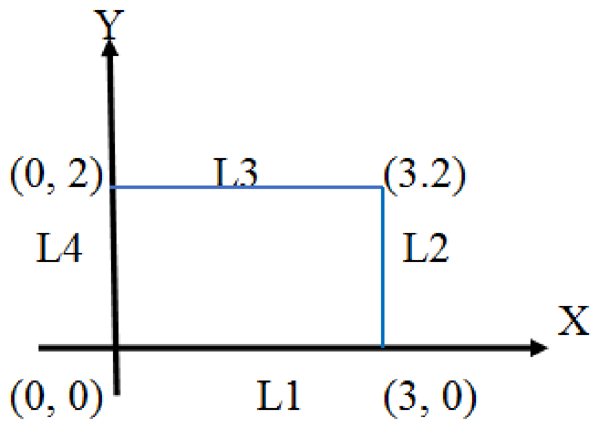
3- The largest of the values from steps 1 and 2 is the absolute maximum and the smallest of these values is the absolute minimum value.

Example 2.16

Find the absolute maximum and minimum values of the function
$f(x,y) = x^2 - 2xy + 2y$ **on the rectangular** $D = [(x,y)|0 \le x \le 3, \quad 0 \le y \le 2]$

Solution

To find the critical points



$f_x = 2x - 2y = 0 \; and \; f_y = -2x + 2 = 0$

$x = 1, \quad y = 1$ **The critical point is (1,1)**

To find the points on the boundary

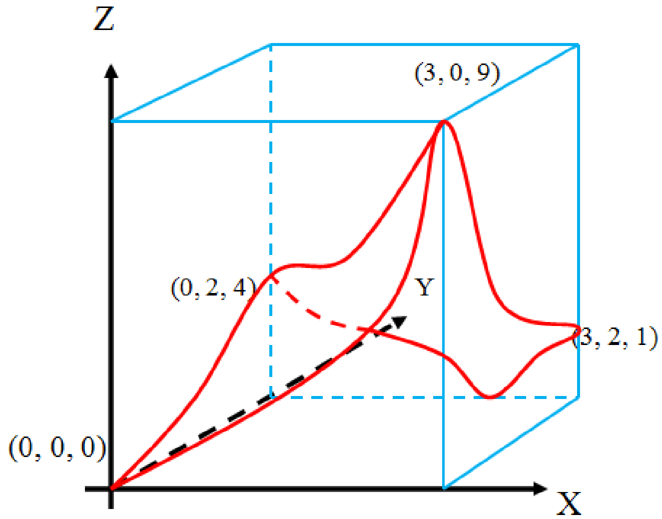
<u>L1</u>

$y = 0, \quad x = 0 \to 3$

$f(x,0) = x^2$

Maximum value is f(3,0)=9

Minimum value is f(0,0)=0



L2

$x = 3, \qquad y = 0 \to 2$

$f(3, y) = 9 - 4y$

Maximum value is f(3,0)=9
Minimum value is f(3,2)=1

L3

$y = 2, \qquad x = 0 \to 3$

$f(x, 2) = x^2 - 4x + 4 = (x - 2)^2$

Maximum value is f(0,2)=4
Minimum value is f(2,2)=0

L4

$x = 0, \qquad y = 0 \to 2$

$f(0, y) = 2y$

Maximum value is f(0,2)=4
Minimum value is f(0,0)=0

Absolute maximum value of f (x,y) on D is f(3,0)=9

Absolute minimum value of f(x,y) on D is f(0,0)=f(2,2)=0

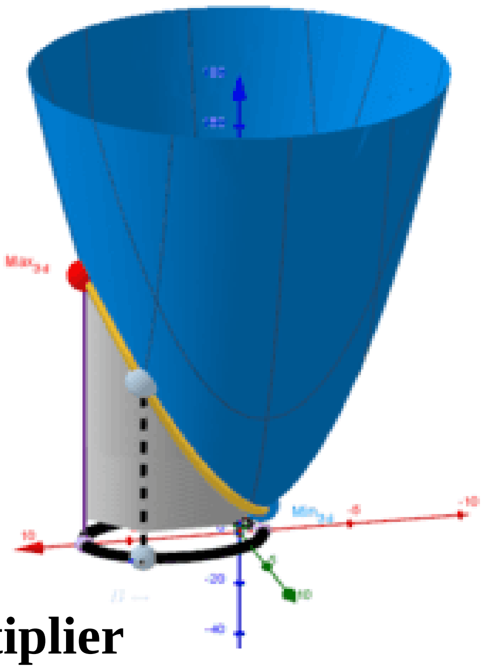
2.9 Lagrange Multipliers Method

This method is used to find the stationary points (maximum and minimum) of the function w=f(x,y,z) with constraint g(x,y,z)=k as shown in Figure below.

The figure shows a g(x,y) curve together with several curves of f(x,y). To maximize f(x,y) subject to g(x,y)=k to find largest value of C such that the level curve f(x,y)=c intersect g(x,y)=k. its appear from the figure that this happens when these curves just touch each other.

<u>This mean the normal lines at intersection point (xo,yo) are identical</u>

<u>The gradient vectors are parallel</u>



$$\nabla f(x_o, y_o) = \lambda \nabla g(x_o, y_o) \quad \lambda \text{ is a calar}$$

For 3D (three variables)

$$\nabla f(x_o, y_o, z_o) = \lambda \nabla g(x_o, y_o, z_o)$$

The number λ in the equation is called a Lagrange Multiplier

To find the maximum and minimum values of f(x,y,z) subject to the constraint g(x,y,z)=k

(a) Find all value of x,y,z and λ $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

(b) Evaluate f at all point (x,y,z) that result from step (a)

The largest of these values is <u>the maximum</u> and the smallest is <u>the minimum</u>

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

Example 2.17

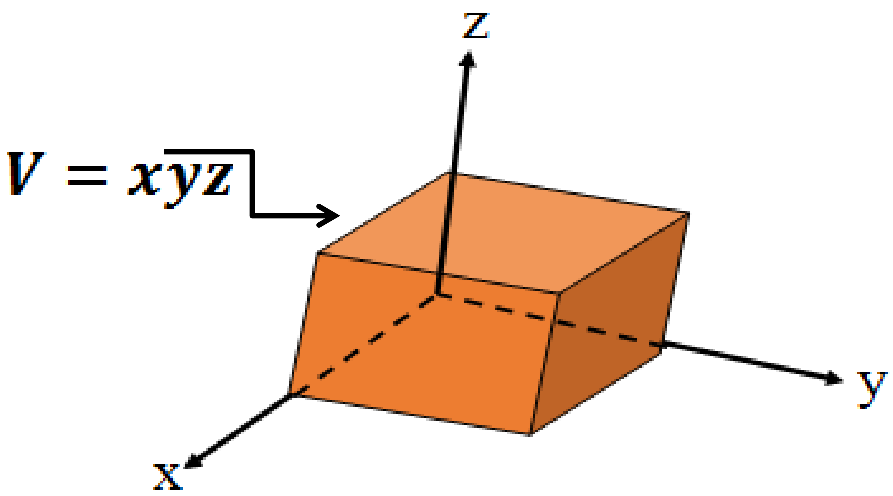
A rectangular box with out cover is to be made from $12m^2$ of cardboard, find the maximum value of such box.

Solution

$$V = xyz$$

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

$$\nabla f = \lambda \nabla g$$



$$v_x = \lambda g_x \quad v_y = \lambda g_y \quad v_z = \lambda g_z$$

$$yz = \lambda(2z + y) \qquad (1)$$

$$xz = \lambda(2z + x) \qquad (2)$$

$$xy = \lambda(2x + 2y) \qquad (3)$$

$$2xz + 2yz + xy = 12 \qquad (4)$$

Multiply eq. 1 by x, eq. 2 by y and eq. 3 by z

$$xyz = \lambda(2xz + xy) \qquad (5)$$

$$xyz = \lambda(2yz + xy) \qquad (6)$$

$$xyz = \lambda(2xz + 2yz) \qquad (7)$$

$$From\ Eqs. (5) and\ (6) \quad 2xz + yx = 2yz + xy\ then\ \ y = x$$

$$From\ Eqs. (6) and\ (7) \quad 2yz + yx = 2xz + 2yz\ then\ \ y = x = 2z$$

$$Sub. in\ eq. (4) \quad 4z^2 + 4z^2 + 4z^2 = 12$$

$$z^2 = 1\ \ then\ z = 1 \quad x = 2\ y = 2$$

$$V = 2 * 2 * 1 = 4m^2$$

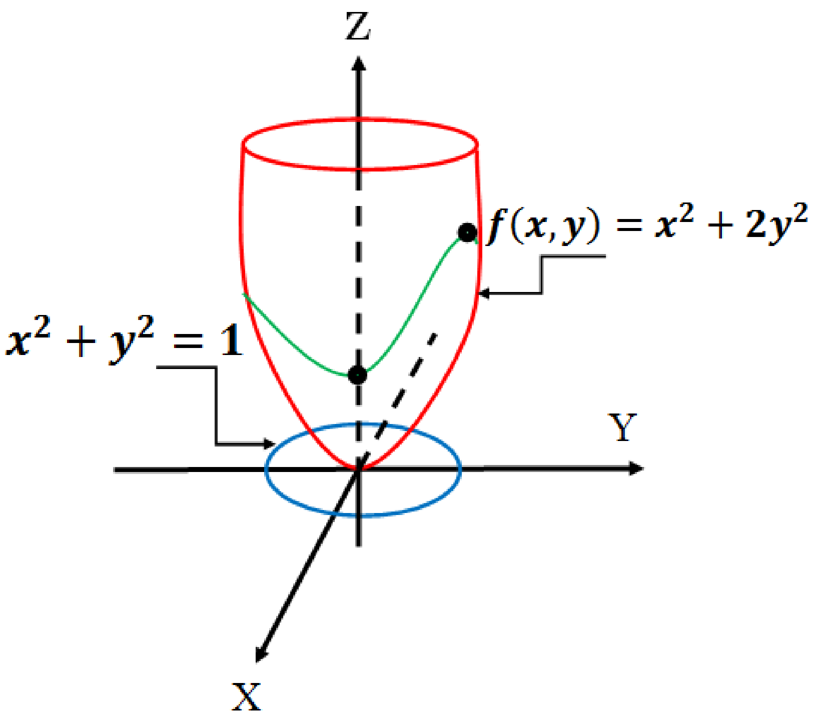
Example 2.18

Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

Solution

$g(x,y) = x^2 + y^2 = 1$

$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z$



$2x = \lambda 2x \qquad (1)$

$4y = \lambda 2y \qquad (2)$

$x^2 + y^2 = 1 \qquad (3)$

$From\ eq.(1)\ \ x = 0\ \ or\ \ \lambda = 1$

$if\ x = 0\ \ y = \pm 1\ \ from\ Eq.3$

$if\ \lambda = 1\ \ y = 0\ \ from\ Eq.2$

Therefore the possible extreme values at the points (0,1), (0,-1) (1,0) and (-1,0)

$f(0,1) = 2$

$f(0,-1) = 2$

$f(1,0) = 1$

$f(-1,0) = 1$

The maximum value of f is f(0,1) = f(0,-1) =2

The minimum value of f is f(1,0) =f(-1,0) =1