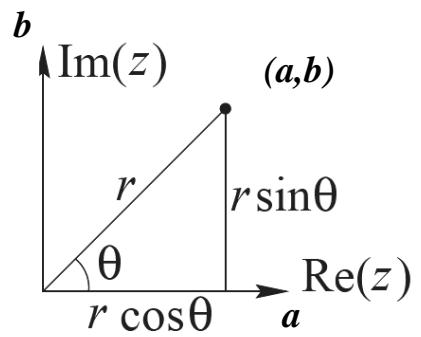


2.2 Polar form of a complex number:

$$\begin{aligned}z &= a + bi \\r &= \sqrt{a^2 + b^2} \\a &= r \cos \theta \\b &= r \sin \theta \\a + bi &= r (\cos \theta + i \sin \theta)\end{aligned}$$



r = the absolute value of a complex number.

θ = the capacity of a complex number.

$$a + bi = r [\cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k)]$$

where k is any integer number.

Example: convert $-3+3i$ into its polar form?

$$r = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$$

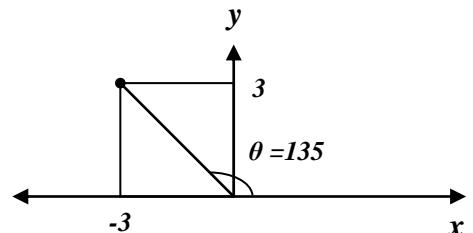
$$-3 + 3i = r (\cos \theta + i \sin \theta)$$

$$= 3\sqrt{2} (\cos 135 + i \sin 135)$$

Proof:

$$-3 + 3i = 3\sqrt{2} \left(\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$-3 + 3i = 3 (-1 + i) = -3 + 3i$$



Multiplication Using Polar Form:

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$z_1 \cdot z_2 = (r_1 (\cos \theta_1 + i \sin \theta_1)) \times (r_2 (\cos \theta_2 + i \sin \theta_2))$$

$$= [r_1 r_2 ((\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2))]$$

$$z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

**The absolute value of multiplying two complex numbers =
the 1st absolute value \times the 2nd absolute value = $r_1 \times r_2$.**

The capacity of the resulting complex number =

the sum of the first and second capacities = $\theta_1 + \theta_2$.

Example: $z_1 = 3(\cos 120 + i \sin 120)$, $z_2 = 2(\cos 60 + i \sin 60)$

$$z_1 \times z_2 = 3 \times 2 [\cos(120 + 60) + i \sin(120 + 60)] = 6 [-1 + 0] = -6$$

Proof:

$$z_1 = 3(\cos 120 + i \sin 120) = \left(\frac{-3}{2}, \frac{3\sqrt{3}}{2} \right)$$

$$z_2 = 2(\cos 60 + i \sin 60) = (1, \sqrt{3})$$

$$z_1 \times z_2 = \left(-\frac{3}{2} - \frac{9}{2} \right), \left(\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} \right) = \frac{-12}{2} = -6$$

Division Using Polar Form:

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Power of complex numbers:

Demoivre's Theorem:

$$z = r (\cos \theta + i \sin \theta)$$

$$z^2 = r^2 (\cos 2\theta + i \sin 2\theta)$$

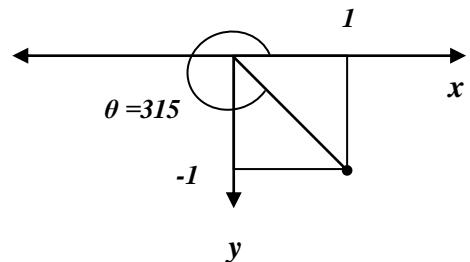
$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

Example: If $z = 1 - i$ find z^6 ?

$$z = 1 - i \Rightarrow \sqrt{2} (\cos 315 + i \sin 315)$$

$$z^6 = 8 (\cos 6 \times 315 + i \sin 6 \times 315)$$

$$z^6 = 8 (\cos 1890 + i \sin 1890)$$



The Nth Root of $q(\cos \phi + i \sin \phi)$:

Assume that $p(\cos \theta + i \sin \theta)$ is the Nth root.

$$[p(\cos \theta + i \sin \theta)]^n = q(\cos \phi + i \sin \phi)$$

$$p^n(\cos n\theta + i \sin n\theta) = q(\cos \phi + i \sin \phi)$$

$$p^n \cos n\theta = q \cos \phi$$

$p^n \sin n\theta = q \sin \phi$ by taking the second power and adding the two equation

$$q^2 (\sin^2 \phi + \cos^2 \phi) = p^{2n} (\cos^2 n\theta + \sin^2 n\theta)$$

$$q^2 = p^{2n}$$

$$q = p^n$$

$$p = \sqrt[n]{q}$$

$$n\theta = \phi + 2\pi k$$

$$\theta = \frac{\phi + 2\pi k}{n} \quad k = 0 \Leftrightarrow \text{one root}$$

$$k = 0,1 \Leftrightarrow \text{two roots}$$

$$k = 0,1,2 \Leftrightarrow \text{three roots}$$

$$k = 0,1,\dots,(n-1) \Leftrightarrow \text{for } n \text{ roots}$$

Example: find $\sqrt[3]{27(\cos 60 + i \sin 60)}$?

$$z = q(\cos \phi + i \sin \phi)$$

$$z = 27(\cos 60 + i \sin 60)$$

$$z_n = p(\cos \theta + i \sin \theta)$$

$$p = \sqrt[3]{27} = 3$$

$$\theta = \frac{\phi + 2\pi k}{3}$$

$$k = 0 \Rightarrow \theta = \frac{60 + 0}{3} = 20$$

$$k = 1 \Rightarrow \theta = \frac{60 + 360}{3} = 140$$

$$k = 2 \Rightarrow \theta = \frac{60 + 720}{3} = 260$$

Example: find the cubic root of 1 ?

$$z = 1 (\cos \phi + i \sin \phi)$$

$$z_n = p (\cos \theta + i \sin \theta)$$

$$p = \sqrt[3]{1} = 1$$

$$\theta = \frac{\phi + 2\pi k}{3}$$

$$k = 0 \Rightarrow \theta = \frac{0+0}{3} = 0$$

$$k = 1 \Rightarrow \theta = \frac{0+360}{3} = 120$$

$$k = 2 \Rightarrow \theta = \frac{0+720}{3} = 240$$

$$z_1 = 1 (\cos \theta + i \sin \theta) = 1$$

$$z_2 = 1 (\cos 120 + i \sin 120) = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$$

$$z_3 = 1 (\cos 240 + i \sin 240) = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$$

Check :

$$\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i \right) \times \left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i \right) = \left(\frac{1}{4} + \frac{3}{4} \right) + \left(\frac{\sqrt{3}}{4} i - \frac{\sqrt{3}}{4} i \right) = 1$$