Chapter 2

Ordinary differential Equations

Introduction:

A differential equation is an equation that involves one or more derivatives, or differentials. Differential equations are classified by:

a) **Type** (namely, ordinary or partial),
b) **Order** (that of the highest order derivative that occurs in the equation),and
c) **Degree** (the exponent of the highest power of the highest order derivative, after the equation has been cleared of fractions and radicals in the dependent variable and its derivatives).

For Example:

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$$

is an ordinary differential equation, of order three and degree two.

Only "ordinary" derivatives occur when the dependant variable y is a function of a single independent variable x. On the other hand, if the dependant variable y is a function of two or more independent variables, say

$$y = f(x,t),$$

where x and t are independent variables, then partial derivatives of y may occur. For example,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

is a partial differential equation, of order two and degree one, (It is the one-dimensional "wave equation").

Many physical problems, when formulated in mathematical terms, lead to differential equations. For example,

$$m\frac{d^2x}{dt^2} = 0, \qquad m\frac{d^2y}{dt^2} = -mg$$

describes the motion of a projectile (neglecting air resistance).

Indeed, one of the chief sources of differential equation is Newton's second law:

$$F = \frac{d}{dt}(mv),$$

where F is the resultant of the forces acting on a body of mass m and v is its velocity.

Solutions of Differential Equations (D.E):

A function

$$y = f(x)$$

is said to be a solution of a D.E if the latter is satisfied when y and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives. For example, if C_1 and C_2 are any constants, then

$$y = C_1 \cos x + C_2 \sin x \tag{1a}$$

is a solution of the D.E

$$\frac{d^2y}{dx^2} + y = o \tag{1b}$$

A physical problem that translates into a D.E usually involves additional conditions not expressed by the D.E itself. In mechanics, for example, the initial position and velocity of the moving body are usually prescribed, as well as the forces. The D.E, or equations, of motion will usually have solutions in which certain arbitrary constants occur, as in (1a) above. Specific values are then assigned to these arbitrary constants to meet the prescribed initial conditions.

A D.E of order n will generally have a solution involving n arbitrary constants. This solution is called the ***general*** solution. Once the general solution is known, it is only a matter of algebra to determine specific values of the constants if initial conditions are also prescribed.

In the reminder of this chapter, the following topics will be treated. Throughout only ***ordinary*** differential equations will be considered.

1. **First order.**
   a) Variable separable.
   b) Exact differentials.
   c) Homogeneous.
   d) Linear.

2. **Special types of second order.**

3. **Linear equations with constant coefficients.**
   a) Homogeneous.
   b) Inhomogeneous.

1.1 First Order Ordinary Differential Equations

Integrable Forms:

In this section we will introduce a few forms of differential equations that we may solve through integration.

(1.1.1) Variable Separable Differential Equations:

Any D.E that can be written in the form

$$\boxed{P(x)+Q(y).y'=0}$$

Is a separable equation, (because the dependant and independent variables are separated). We can obtain an implicit by integrating with respect to *x*.

$$\int P(x).dx+\int Q(y).\frac{dy}{dx}.dx=c$$

$$\boxed{\int P(x).dx+\int Q(y)dy=c}$$

Example: Consider the D.E $y'=xy^2$. We separate the dependant and independent variables and integrate to find the solution.

$$\frac{dy}{dx}=x.y^2$$
$$y^{-2}dy=x.dx$$
$$\int y^{-2}dy=\int x.dx+c$$
$$-y^{-1}=\frac{x^2}{2}+c$$
$$\left[y=\frac{-1}{x^2/2+c}\right]$$

Example: The equation $y' = y - y^2$ is separable.

$$\left(\frac{y'}{y - y^2} = 1\right)$$

We expand in partial fraction and integrate.

$$\left(\frac{1}{y} - \frac{1}{y-1}\right).y' = 1$$

$$\ln|y| - \ln|y-1| = x + c$$

We have an implicit function for *y(x)*. Now we solve for *y(x)*.

$$\ln\left|\frac{y}{y-1}\right| = x + c$$

$$\left|\frac{y}{y-1}\right| = e^{x+c}$$

$$\frac{y}{y-1} = \pm e^{x+c}$$

$$\frac{y}{y-1} = Ce^{x}$$

Example: Consider the D.E $(xy^2 + x)dx + (yx^2 + y)dy = 0$.We separate the dependant and independent variables and integrate to find the solution.

$$x(y^2 + 1).dx + y.(x^2 + 1)dy = 0$$

$$\frac{x.dx}{x^2 + 1} = -\frac{y.dy}{y^2 + 1}$$ Multiply by 2 and Integrate

$$\ln(x^2 + 1) + \ln(y^2 + 1) = c$$

$$\ln(x^2 + 1).(y^2 + 1) = c \Rightarrow (x^2 + 1).(y^2 + 1) = e^c = C$$

Example: Solve the following D.E ?

$$(4y-\cos y).\frac{dy}{dx}-3x^2=0$$
$$(4y-\cos y).\frac{dy}{dx}=3x^2$$
$$(4y-\cos y).dy=3x^2.dx$$
$$\int(4y-\cos y).dy=\int 3x^2.dx$$
$$\frac{4y^2}{2}-\sin y=\frac{3x^3}{3}+c$$
$$2y^2-\sin y=x^3+c$$

(1.1.2) Exact differential equations:

Any first order ordinary D.E's of the first degree can be written as the total D.E,

$$P(x,y).dx+Q(x,y).dy=0.$$

If this equation can be integrated directly, that is if there is a primitive, *u(x,y)*, such that,

$$du=P.dx+Q.dy\,,$$

then this equation is called *exact*. The (implicit) solution of the D.E is

$$u(x,y)=c\,,$$

where c is an arbitrary constant. Since the differential of a function, *u(x,y)*, is

$$du\equiv\frac{\partial u}{\partial x}.dx+\frac{\partial u}{\partial y}.dy\,,$$

P and Q are the partial derivatives of u:

$$P(x,y)=\frac{\partial u}{\partial x}\quad,\qquad Q(x,y)=\frac{\partial u}{\partial y}\quad.$$

In an alternative notation, the D.E

$$\frac{du}{dx} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial x} = P(x,y) + Q(x,y).\frac{dy}{dx}$$

The solution of the D.E is *u(x,y) = c*.

Example:

$$x + y.\frac{dy}{dx} = 0$$

is an exact D.E since

$$\frac{d}{dx}\left(\frac{1}{2}\left(x^2 + y^2\right)\right) = x + y.\frac{dy}{dx}$$

The solution of the D.E is

$$\frac{1}{2}(x^2 + y^2) = c$$

Example: Let *f(x)* and *g(x)* be known functions.

$$g(x).y' + g'(x).y = f(x)$$

is an exact D.E since

$$\frac{d}{dx}(g(x).y(x)) = g\,y' + g'\,y \ .$$

The solution of D.E is

$$g(x).y(x) = \int f(x).dx + c$$

$$y(x) = \frac{1}{g(x)}.\int f(x).dx + \frac{c}{g(x)}.$$

A necessary condition for exactness. The solution of the Exact equation $P+Q.y'=0$ is $u=c$ where u is the primitive of the equation $\frac{du}{dx}=P+Q.y'$. At present the only method we have for determining the primitive is guessing. This is fine for simple equations, but for more difficult cases we would like a method more concrete than inspiration. As a first step toward this goal we determine a criterion for determining if an equation is exact.

Consider the exact equation,

$$\boldsymbol{P + Q \,.\, y' = 0\,,}$$

with primitive u, where we assume that the function P and Q are continuously differentiable. Since the mixed partial derivatives of u are equal,

$$\frac{\partial^2 u}{\partial x.\partial y}=\frac{\partial^2 u}{\partial y.\partial x},$$

a necessary condition for exactness is

$$\boxed{\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.}$$

Example: Prove that the following D.E is exact?

$$y^2dx+2xydy=0$$

$$P=y^2 \qquad Q=2xy$$

$$\frac{\partial P}{\partial y}=2y \qquad \frac{\partial Q}{\partial x}=2y$$

$$\because \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$$

Example: Prove that the following D.E is exact and find the general solution?

$$(3x^2 y + 2xy).dx + (x^3 + x^2 + 2y).dy = 0$$

$$P = 3x^2 y + 2xy \qquad Q = x^3 + x^2 + 2y$$

$$\frac{\partial P}{\partial y} = 3x^2 + 2x \qquad \frac{\partial Q}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \therefore \quad D.E \quad is \quad exact$$

$$\frac{\partial u}{\partial x} = P = 3x^2 y + 2xy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial u}{\partial y} = Q = x^3 + x^2 + 2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

by integrating eq. (1) with respect to x we get

$$u = x^3 y + x^2 y + c$$

$$u = x^3 y + x^2 y + \phi(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

by deriving eq. (3) with respect to y we get

$$\frac{\partial u}{\partial y} = x^3 + x^2 + \phi'(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

and by equalizing eq. (4) with eq. (2) we get

$$x^3 + x^2 + \phi'(y) = x^3 + x^2 + 2y$$

$$\phi'(y) = 2y$$

$$\int \phi'(y) = \int 2y$$

$$\phi(y) = y^2 + c$$

and by substituting $\phi(y)$ *in eq. (3) we get the General solution as,*

$$u(x, y) = x^3 y + x^2 y + y^2 + c$$