

**Example:** Prove that the following D.E is exact and find the general solution?

$$(\cos x + y \sin x).dx = \cos x dy$$

$$(\cos x + y \sin x).dx - \cos x dy = 0$$

$$P = \cos x + y \sin x \quad Q = -\cos x$$

$$\frac{\partial P}{\partial y} = \sin x \quad \frac{\partial Q}{\partial x} = -(-\sin x) = \sin x$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \text{D.E is exact}$$

$$\frac{\partial u}{\partial x} = P = \cos x + y \sin x \quad \dots \quad (1)$$

$$\frac{\partial u}{\partial y} = Q = -\cos x \quad \dots \quad (2)$$

by integrating eq. (1)

$$u = \sin x - y \cos x + \phi(y) \quad \dots \quad (3)$$

by deriving eq. (3) partially to y

$$\frac{\partial u}{\partial y} = -\cos x + \phi'(y) \quad \dots \quad (4)$$

by equalizing eq. (4) to eq. (2) we get

$$-\cos x + \phi'(y) = -\cos x \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c$$

by substituting  $\phi(y)$  in eq. (3)

$u = \sin x - y \cos x + c$  general solution

**Example:** Prove that the following D.E is exact and find the general solution?

$$(x \cos xy + \sin xy).dx + (x^2 \cos xy + e^y).dy = 0$$

$$P = x \cos xy + \sin xy \quad Q = x^2 \cos xy + e^y$$

$$\frac{\partial P}{\partial y} = -x.y.x \sin xy + x \cos xy + x \cos xy \quad \frac{\partial Q}{\partial x} = -x^2.y \sin xy + 2x \cos xy$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \text{D.E is exact}$$

$$P = \frac{\partial u}{\partial x} = x \cos xy + \sin xy \quad \dots \quad (1)$$

$$Q = \frac{\partial u}{\partial y} = x^2 \cos xy + e^y \quad \dots \quad (2)$$

by integrating eq. (2) with respect to  $y$

$$u = x \cdot \sin xy + e^y + \phi(x) \dots \quad (3)$$

drive eq. (3) partially with respect to  $x$

$$\frac{\partial u}{\partial x} = x \cdot y \cdot \cos xy + \sin xy + \phi'(x) \dots \quad (4)$$

by equalizing eq. (4) with eq. (1) we get

$$x \cdot y \cdot \cos xy + \sin xy + \phi'(x) = x y \cos xy + \sin xy \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c$$

$\therefore u = x \cdot \sin xy + e^y + c$  which is the general solution.

### (1.1.3) Homogeneous differential equations:

Homogeneous coefficient, first order D.E's form another class of soluble eqs. We will find that a change in dependant variable will make such eqs. separable or we can determine an integrating factor that will make such eqs. exact. First we define homogeneous functions.

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

**Example:** Solve the homogeneous D.E?

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \dots \quad (1)$$

$$\text{Let } v = \frac{y}{x} \Rightarrow y = v \cdot x$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx} = f(v)$$

$$x \cdot \frac{dv}{dx} = f(v) - v$$

$$\frac{x}{dx} = \frac{f(v) - v}{dv}$$

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\int \frac{dx}{x} + c = \int \frac{dv}{F(v) - v}$$

**Example:** Solve the homogeneous D.E?

$$y' = \frac{x^2 + y^2}{x \cdot y}$$

$$\frac{dy}{dx} = \frac{x^2}{x.y} + \frac{y^2}{x.y}$$

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} \quad \dots \dots \dots \quad (1)$$

$$Let \quad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

*substitute in Eq. (1)*

$$v + x \cdot \frac{dv}{dx} = \frac{1}{v} + v$$

$$x \frac{dv}{dx} = \frac{1}{v} \quad \Rightarrow \quad \frac{x}{dx} = \frac{1}{v \cdot dv}$$

$$\int \frac{dx}{x} = \int v.dv$$

$$\ln x = \frac{v^2}{2} \Rightarrow 2(\ln x + c_1) = v^2$$

$$2(\ln x + c_1) = \left(\frac{y}{x}\right)^2 \Rightarrow 2\ln x + 2c_1 = \frac{y^2}{x^2}$$

$$\ln x^2 + C = \frac{y^2}{x^2} \Rightarrow y^2 = x^2(\ln x^2 + C)$$

**Example:** Solve the homogeneous D.E?

$$2xyy' - y^2 + x^2 = 0$$

$$\frac{2xy}{x^2} \cdot \frac{dy}{dx} - \frac{y^2}{x^2} + \frac{x^2}{x^2} = 0$$

$$\frac{2y}{x} \cdot \frac{dy}{dx} - \left(\frac{y}{x}\right)^2 + 1 = 0 \quad \dots \dots \dots \quad (1)$$

$$\text{Let } v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx} \Rightarrow 2v \left( v + x \cdot \frac{dv}{dx} \right) - v^2 + 1 = 0$$

$$2.v^2 + 2.x.v \cdot \frac{dv}{dx} - v^2 + 1 = 0 \Rightarrow 2.x.v \cdot \frac{dv}{dx} + v^2 + 1 = 0$$

$$\begin{aligned}
 2.x.v \cdot \frac{dv}{dx} = -(v^2 + 1) &\Rightarrow \frac{x}{dx} = \frac{-(v^2 + 1)}{2.v.dv} \\
 \frac{dx}{x} = \frac{2.v.dv}{-(v^2 + 1)} &\Rightarrow -\int \frac{dx}{x} = \int \frac{2.v.dv}{v^2 + 1} \\
 -\ln x + c = \ln(v^2 + 1) &\Rightarrow \ln x^{-1} + c = \ln(v^2 + 1) \\
 e^{(\ln x^{-1} + c)} = e^{\ln(v^2 + 1)} &\Rightarrow e^{\ln x^{-1}} \cdot e^c = e^{\ln(v^2 + 1)} \\
 x^{-1} \cdot c = v^2 + 1 &\Rightarrow \frac{c}{x} = \left( \frac{y}{x} \right)^2 + 1 \\
 \frac{c}{x} = \frac{y^2}{x^2} + 1 &
 \end{aligned}$$

### (1.1.3.1) Equations reducible to homogeneous form:

Certain equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

can be reduced to the homogeneous form by substitution of

$$\begin{aligned}
 x = X + h \quad y = Y + k, \quad \text{then} \quad \frac{dy}{dx} &= \frac{dY}{dX} \\
 \frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{A(X + h) + B(Y + k) + C} &= \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + C} \quad \text{where,}
 \end{aligned}$$

$$ah + bk + c = 0$$

$$Ah + Bk + C = 0$$

**Example:** Solve the following D.E?

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{x + 2y - 3}{2x + y - 3} \\
 \text{assume } x &= X + h \quad y = Y + k \\
 \frac{dY}{dX} &= \frac{X + h + 2(Y + k) - 3}{2(X + h) + Y + k - 3} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3}
 \end{aligned}$$

$$h + 2k - 3 = 0$$

$$\underline{2h + k - 3 = 0}$$

$$3k - 6 + 3 - 3 = 0 \Rightarrow k = 1 \Rightarrow h = 1$$

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y}, \quad Y = VX$$

$$V + X \frac{dV}{dX} = \frac{X + 2VX}{2X + VX} = \frac{1+2V}{2+V}$$

$$X \frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1-V^2}{2+V}$$

$$\frac{(2+V)dV}{1-V^2} = \frac{dX}{X} \Rightarrow \frac{2+V}{1-V^2} = \frac{A}{1-V} + \frac{B}{1+V} = \frac{A(1+V) + B(1-V)}{1-V^2}$$

$$2+V = A(1+V) + B(1-V)$$

$$2+V = V(A-B) + A+B$$

$$A-B=1$$

$$\underline{A+B=2}$$

$$2A=3 \Rightarrow A=\frac{3}{2}, \quad B=\frac{1}{2}$$

$$\frac{\frac{3}{2}dV}{1-V} + \frac{\frac{1}{2}dV}{1+V} = \frac{dX}{X} \Rightarrow \frac{3dV}{2(1-V)} + \frac{dV}{2(1+V)} = \frac{dX}{X}$$

$$\frac{3dV}{(1-V)} + \frac{dV}{(1+V)} = \frac{2dX}{X}$$

$$-3\ln(1-V) + \ln(1+V) = 2\ln X$$

$$\ln(1-V)^{-3} + \ln(1+V) - \ln X^2 = C$$

$$\frac{\ln(1+V)}{\ln(1-V)^3} - \ln X^2 = C \Rightarrow \ln \frac{1+V}{X^2(1-V)^3} = C$$

$$\frac{1+V}{X^2(1-V)^3} = c \Rightarrow \frac{1+\frac{Y}{X}}{X^2(1-\frac{Y}{X})^3} = c \Rightarrow \frac{\frac{X+Y}{X}}{X^2 \frac{(X-Y)^3}{X^3}} = c \Rightarrow \frac{X+Y}{(X-Y)^3} = c$$

$$x = X + h = X + 1 \Rightarrow X = x - 1 \Rightarrow Y = y - 1$$

$$\therefore \frac{x-1+y+1}{[x-1-(y-1)]^3} = c \Rightarrow \frac{x+y-2}{(x-y)^3} = c$$

### (1.1.4) The First Order, Linear Differential Equations:

#### (1.1.4.1) Homogeneous Equations:

The first order, linear, homogeneous D.E has the form

$$\boxed{\frac{dy}{dx} + p(x).y = 0}$$

We can solve any equation of this type because it is separable.

$$\begin{aligned}\frac{y'}{y} &= -p(x)dx \\ \ln|y| &= -\int p(x).dx + c \\ y &= \pm e^{-\int p(x).dx+c}\end{aligned}$$

$$\boxed{y = ce^{-\int p(x).dx}}$$

**Example:** Consider the equation

$$\begin{aligned}\frac{dy}{dx} + \frac{1}{x}y &= 0 \\ y &= ce^{-\int p(x).dx} \Rightarrow y(x) = ce^{-\int \frac{1}{x}dx}, \quad \text{for } x \neq 0 \\ y(x) &= ce^{-\ln|x|} \\ y(x) &= \frac{c}{|x|} \Rightarrow y(x) = \frac{c}{x}\end{aligned}$$

#### (1.1.4.2) Inhomogeneous Equations:

The first order, linear, inhomogeneous D.E has the form

$$\boxed{\frac{dy}{dx} + p(x).y = f(x) \quad \dots \dots \dots \quad (1)}$$

The equation is not separable. Note that it is similar to the exact equation we solved previously,

$$g(x).y'(x) + g'(x).y(x) = f(x)$$

To solve equ. (1), we multiply by an *integrating factor*. Multiplying a D.E by its integrating factor changes it to an exact equation. Multiplying equ. (1) by the function,  $I(x)$ , yields,

$$I(x) \cdot \frac{dy}{dx} + p(x) \cdot I(x) \cdot y = f(x) \cdot I(x).$$

in order that  $I(x)$  be an Integrating factor, it must satisfy

$$\frac{d}{dx} \cdot I(x) = p(x) \cdot I(x).$$

This is a first order, linear, homogeneous equation with the solution

$$I(x) = c e^{\int p(x) dx}$$

This is an integrating factor for any constant  $c$ . For simplicity choose  $c=1$ .

To solve equation (1) we multiply by the integrating factor and integrate.

$$\text{Let } P(x) = \int p(x) dx$$

$$e^{P(x)} \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y = e^{P(x)} \cdot f(x)$$

$I(x) \cdot y = \int I(x) \cdot f(x) dx + c \quad \dots \dots \dots \quad \text{Rule}$	
$\frac{d}{dx} I(x) \cdot y = I(x) \cdot f(x) \quad \dots \dots \dots \quad \text{Rule}$	

$$\frac{d}{dx} (e^{P(x)} \cdot y) = e^{P(x)} \cdot f(x)$$

$$y = e^{-P(x)} \int e^{P(x)} \cdot f(x) dx + c e^{-P(x)}$$

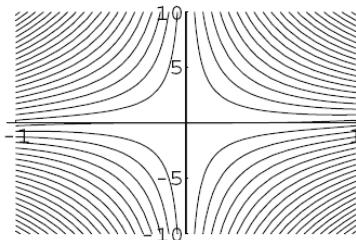
$$y \equiv y_p + c \cdot y_h$$

**Note** that the general solution is the sum of a particular solution,  $y_p$ , that satisfies  $y' + p(x) \cdot y = f(x)$ , and an arbitrary constant times a homogeneous solution,  $y_h$ , that satisfies  $y' + p(x) \cdot y = 0$ .

**Example:** consider the D.E

$$y' + \frac{1}{x} y = x^2 \quad , \quad x > 0.$$

First find the integrating factor.



**Solution to  $y' + y/x = x^2$**

$$I(x) = \exp\left(\int \frac{1}{x} dx\right) = e^{\ln x} = x$$

We multiply by the integrating factor and integrate.

$$I(x).y = \int I(x).f(x).dx + c$$

$$\frac{d}{dx} I(x).y = I(x).f(x) \Rightarrow$$

$$\frac{d}{dx}(x^y) = x^y$$

$$x y = \frac{1}{4} x^4 + c$$

The particular homogeneous solutions are

$$y_p = \frac{1}{4}x^3 \quad \text{and} \quad y_h = \frac{1}{x}.$$

Note that the general solution to the D.E is a one-parameter family of functions. The general solution is plotted in the figure above for various values of  $c$ .

**Example:** Solve the following linear D.E?

$$\frac{dy}{dx} + 2y = \cos x$$

$$p(x) = 2 \quad f(x) = \cos x$$

$$I(x) = e^{\int p(x).dx} = e^{\int 2.dx} = e^{2x}$$

$$I(x).y = \int I(x).f(x).dx + c$$

$$\int e^{2x} \cdot \cos x \cdot dx = u \cdot v - \int v \cdot du \quad \text{integration by part}$$

$$\text{Let } u = e^{2x} \Rightarrow du = 2e^{2x} \cdot dx$$

$$dv = \cos x \cdot dx \Rightarrow v = \int \cos x \cdot dx = \sin x$$

$$\therefore \int e^{2x} \cdot \cos x \cdot dx = e^{2x} \cdot \sin x - 2 \int e^{2x} \cdot \sin x \cdot dx$$

$$\text{Also } \int e^{2x} \cdot \sin x \cdot dx = u \cdot v - \int v \cdot du \quad \text{integration by part}$$

$$\text{Let } u = e^{2x} \Rightarrow du = 2e^{2x} \cdot dx$$

$$dv = \sin x \cdot dx \Rightarrow v = \int \sin x \cdot dx = -\cos x$$

$$\therefore \int e^{2x} \cdot \sin x \cdot dx = -e^{2x} \cdot \cos x - \int -\cos x \cdot 2e^{2x} \cdot dx$$

$$\int e^{2x} \cdot \sin x \cdot dx = -e^{2x} \cdot \cos x + 2 \int -\cos x \cdot e^{2x} \cdot dx$$

$$\int e^{2x} \cdot \cos x \cdot dx = e^{2x} \cdot \sin x - 2 \left[ -e^{2x} \cdot \cos x + 2 \int e^{2x} \cdot \cos x \cdot dx \right]$$

$$\int e^{2x} \cdot \cos x \cdot dx = e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x - 4 \int e^{2x} \cdot \cos x \cdot dx$$

$$\int e^{2x} \cdot \cos x \cdot dx + 4 \int e^{2x} \cdot \cos x \cdot dx = e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x$$

$$5 \int e^{2x} \cdot \cos x \cdot dx = e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x$$

$$\therefore \int e^{2x} \cdot \cos x \cdot dx = \frac{e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x}{5}$$

substitute in eq. (1)

$$e^{2x} \cdot y = \frac{e^{2x} \cdot \sin x + 2e^{2x} \cdot \cos x}{5} + c$$

### (1.1.4.3) Equation reducible to liner form (Bernoulli's eq.):

The Eq. of the form  $\frac{dy}{dx} + P \cdot y = Q \cdot y^n$  can be reduced to linear form by dividing by  $y^n$  and substituting  $z = y^{1-n}$ .

$$\frac{dy}{dx} + P \cdot y = Q \cdot y^n$$

$$y^{-n} \cdot y' + P \cdot y^{1-n} = Q,$$

$$\because z = y^{1-n} \quad \therefore \frac{dz}{dx} = (1-n) \cdot y^{-n} \cdot \frac{dy}{dx}$$

$$y^{-n} \cdot y' = \frac{1}{1-n} \cdot \frac{dz}{dx} \quad \Rightarrow \quad \frac{1}{1-n} \cdot \frac{dz}{dx} + P \cdot z = Q$$

**Example:** solve the following D.F?

$$\begin{aligned}\frac{dy}{dx} - x.y &= -y^3 \cdot e^{-x^2} \\ y^{-3} \cdot y' - x.y^{-2} &= -e^{-x^2} \\ z = y^{-2} \Rightarrow \frac{dz}{dx} &= -2y^{-3} \cdot y' \\ y^{-3} \cdot y' &= -\frac{1}{2} \cdot \frac{dz}{dx} \Rightarrow -\frac{1}{2} \cdot \frac{dz}{dx} - x.z &= -e^{-x^2} \\ \frac{dz}{dx} + 2x.z &= 2e^{-x^2} \quad \text{which is a linear eq.}\end{aligned}$$

Solve and re-substitute in the first eq.