Applications

Of

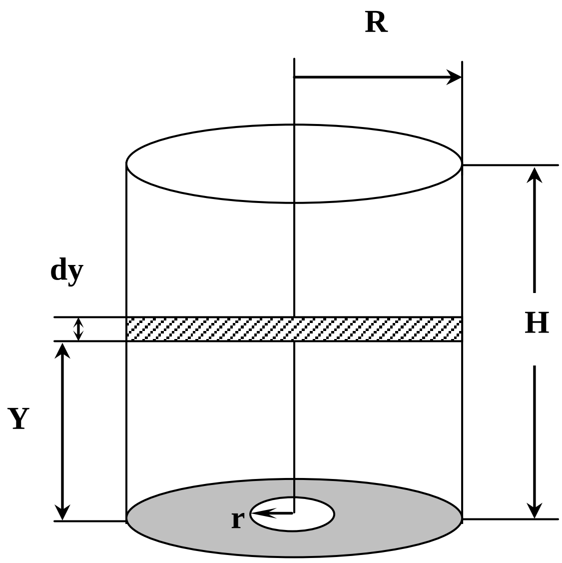
Firs Order

Ordinary

Differential Equations

Example: A cylindrical tank of radius R and height H initially filled with water. At the bottom of the tank there is a hole of radius r , through which water drains under the influence of gravity. Find the depth of water at any time t , and determine how long it takes the tank to drain completely?

Solution: $dV = -\Pi . R^2 . dy$



$$v\,(velocity) = \sqrt{2\,g\,h} = \sqrt{2\,g\,y}$$

$$\frac{dV}{dt} = Q = A \times v \quad \Rightarrow \quad Q = \Pi\, r^2 \sqrt{2\,g\,y}$$

$$dV = \Pi\, r^2 \sqrt{2\,g\,y} . dt$$

$$\Pi\, r^2 \sqrt{2\,g\,y} . dt = -\Pi . R^2 . dy$$

$$r^2 \sqrt{2\,g\,y} . dt = -R^2 . dy$$

$$\frac{dy}{\sqrt{2\,g\,y}} = -\frac{r^2}{R^2} . dt \quad \Rightarrow \quad -\frac{dy}{\sqrt{2\,g\,y}} = \frac{r^2}{R^2} . dt$$

$$-(2\,g\,y)^{-1/2} . dy = \frac{r^2}{R^2} . dt$$

$$\frac{2g}{2g} \int -(2\,g\,y)^{-1/2} . dy = \int \frac{r^2}{R^2} . dt$$

$$-\frac{(2\,g\,y)^{1/2}}{1/2 \times 2\,g} = \frac{r^2}{R^2} . t + c$$

$$-\frac{\sqrt{2\,g\,y}}{g} = \frac{r^2}{R^2} . t + c$$

$$+\sqrt{\frac{2\,y}{g}} = -\frac{r^2}{R^2} . t + c \quad initial\ conditions, \quad y = H \quad at \quad t = 0$$

$$\sqrt{\frac{2\,H}{g}} = 0 + c \quad \Rightarrow \quad c = \sqrt{\frac{2\,H}{g}}$$

$$\sqrt{\frac{2\,y}{g}} = -\frac{r^2}{R^2} . t + \sqrt{\frac{2\,H}{g}}$$

To find the required time for the tank to drain completely (t_o), we substitute ***y = 0***,

$$\sqrt{\frac{2y}{g}} = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

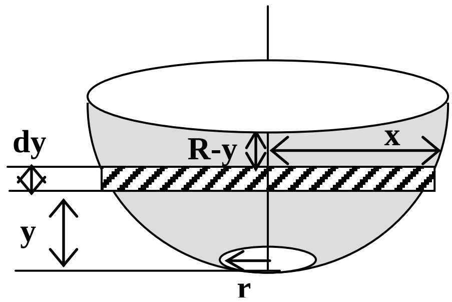
$$0 = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

$$t_o = \sqrt{\frac{2H}{g}}.\frac{R^2}{r^2}$$

Example: A spherical (half ball) tank of radius R initially filled with water. At the bottom of the tank there is a hole of radius r, through which water drains under the influence of gravity. Find the depth of water at any time t, and determine how long it takes the tank to drain completely?

Solution: $dV = -\Pi x^2\, dy$ water loose

$dV = \sqrt{2gy}.\Pi r^2\, dt$ water outlet



$$R^2 = x^2 + (R-y)^2$$
$$x^2 = R^2 - (R-y)^2$$
$$x^2 = R^2 - (R^2 - 2Ry + y^2)$$
$$x^2 = 2Ry - y^2$$

$$\sqrt{2gy}.\Pi r^2\, dt = -\Pi x^2\, dy$$

$$\sqrt{2gy}.r^2\, dt = -x^2\, dy$$

$$\sqrt{2gy}.\ r^2\, dt = (y^2 - 2Ry)\,dy$$

$$\sqrt{2g}.\ r^2\, dt = \frac{(y^2 - 2Ry)\,dy}{\sqrt{y}}$$

$$\int \sqrt{2g}.\ r^2\, dt = \int (y^{3/2} - 2Ry^{1/2})\,dy$$

$$\sqrt{2g}.\ r^2.t = \frac{2}{3}y^{5/2} - 2Ry^{3/2} \times \frac{2}{3} + c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{2/3} = \sqrt{2g}.\ r^2.t + c$$

$initial\ condition: at \quad t = 0 \quad y = R$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}RR^{2/3} = 0 + c$$

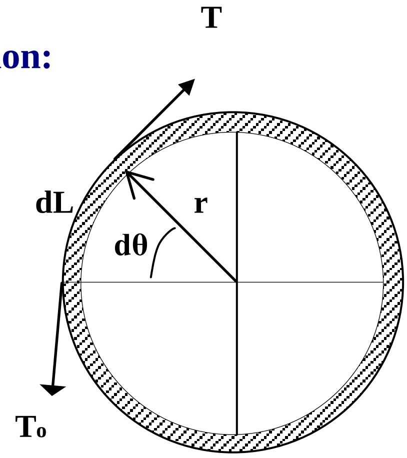
$$(\frac{2}{5} - \frac{4}{3})R^{5/2} = c$$

$$c = -\frac{14}{15}R^{5/2} = c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{3/2} = \sqrt{2g}.\ r^2.t + \frac{14}{15}R^{5/2}$$

Example: It is a fact of common experience that when a rope is wound around a rough cylinder, a small force at an end can resist a much larger force at the other. It is found that throughout the portion of the rope that is in contact with cylinder the change in tension per unit length is proportional.
The constant of proportionality is the coefficient of friction between the rope and the cylinder divided by radius of the cylinder.
Assuming a coefficient of friction of 0.35, how many times must a rope be snubbed around a cylinder of 1 foot diameter for a man holding one end to be able to resist a force 200 times greater than he exert?

Solution:



$$dL = r\,d\theta \quad \Rightarrow \quad \frac{dT}{r\,d\theta} = \frac{0.35}{r}T$$

$$\frac{dT}{T} = 0.35\,d\theta \quad \Rightarrow \quad \ln T = 0.35\theta + c$$

$$at \quad \theta = 0 \quad \Rightarrow \quad T = T_o$$

$$\ln T_o = 0 + c \quad \Rightarrow \quad c = \ln T_o$$

$$\ln T = 0.35\theta + \ln T_o$$

$$0.35\theta = \ln T - \ln T_o = \frac{\ln T}{\ln T_o}$$

$$\theta = \ln\frac{T}{T_o} \times \frac{1}{0.35} \quad \Rightarrow \quad \theta = \ln\frac{200T_o}{T_o} \times \frac{1}{0.35}$$

$$\theta = \frac{\ln 200}{0.35} = 15.13\,Rad$$

$$No.of\ revolution = \frac{15.13\,Rad}{2\Pi\,Rad} = 2.4\,\mathrm{Re}volution$$

Example: According to **Fourier's**(1768-1830) **law of heat conduction**. The amount of heat in ***Btu*** per unit time flowing through an area is proportional to the area and to the temperature gradient, in degrees per unit length, in the direction of the perpendicular to the area. On the basis of this law, obtain a formula for the steady-state heat loss per unit time from a unit length of pipe of radius r_o carrying steam at temperature T_o if the pipe is covered with insulation of thickness w, the outer surface of which remain at the constant temperature T_1. What is the temperature distribution through the insulation, i.e., what is the temperature in the insulation as a function of radius?

Solution: Since the problem tells us that steady-state conditions have been reached, it follows that the heat loss per unit time from a unit length of the pipe is a constant independent of time, say Q. furthermore, it is reasonable to suppose that heat conduction through the insulation in the direction of the length of the pipe is negligible in comparison with the heat flow in the radial direction; and this we shall assume to be the case. We shall also make the obvious assumption that the heat flow through the insulation has circular symmetry, i.e., we shall assume that the temperature in the insulation depends only on the radial distance r. Let as now consider a typical cross section of the pipe and insulation, as suggested in the figure below. Clearly under the assumption that all heat flow through the insulation is radial, it follows that for the unit length of pipe we are considering, all the heat that passes into the insulation through its inner surface will eventually pass into the air through its outer surface. Moreover, on the way, this same amount of heat Q will also pass through every coaxial cylindrical area between r_o and $r_1 = r_o + w$. Now if we let T denote the temperature in the insulation at the radius r, it follows that dT/dr is the temperature gradient, or temperature change per unit length, in the direction perpendicular to the cylindrical area of radius r. Hence, by Fourier's law, we have for the (as yet unknown) amount of heat Q flowing through this general area per unit time.

Q= thermal conductivity × area × temperature gradient

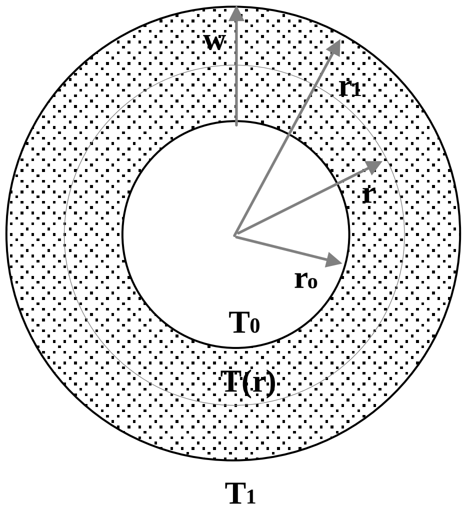
$$Q = k\,(1 \times 2\Pi\, r)\frac{dT}{dr}$$

We thus have the exceedingly simple separable equation

$$dT = \frac{Q}{2\Pi k}.\frac{dr}{r}$$

Hence,

$$T = \frac{Q}{2\Pi k}\ln r + c$$



To determine the integration constant c, we use the fact that $T=T_o$ when $r=r_o$; hence

$$T_o = \frac{Q}{2\Pi k}.\ln r_o + c \quad or \quad c = T_o - \frac{Q}{2\Pi k}.\ln r_o$$

$$T = T_o + \frac{Q}{2\Pi k}.(\ln r - \ln r_o)$$

$$furthermore \quad T = T_1 \quad when \quad r = r_o + w = r_1. \quad Hence,$$

$$T_1 = T_o + \frac{Q}{2\Pi k}.(\ln r_1 - \ln r_o)$$

from which we find easily that

$$Q = \frac{(T_1 - T_o).2\Pi k}{\ln r_1 - \ln r_o}$$

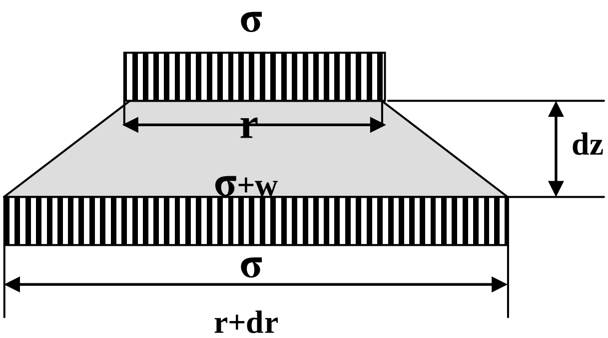
Since k is the (presumably) known thermal conductivity of the insulation, this formula gives the heat loss per unit time, as required.

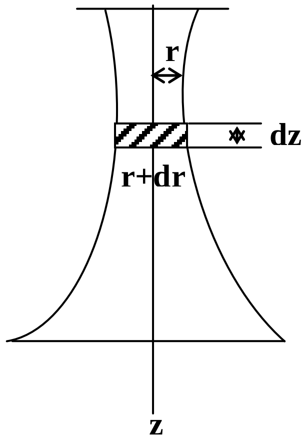
To find the temperature distribution, we merely substitute for $Q/2\Pi k$, getting

$$T = T_o + (T_1 - T_o).\frac{\ln r - \ln r_o}{\ln r_1 - \ln r_o}$$

Example: A weight W (kN), is to be supported by a column having a shape of solid of revolution. If the material of column weights ρ (kN/m^3) and if the radius of the upper base of the column is to be r_0 (m) determine how the radius of the column should vary if at all cross sections the stress is to be the same?

Solution:





$$\oplus \downarrow \sum fy = 0$$

$$\sigma A_1 + w - \sigma A_2 = 0$$

$$\sigma \Pi r^2 + \rho \Pi (r + \frac{dr}{2})^2 \, dz - \sigma \Pi (r + dr)^2 = 0$$

$$\sigma \, r^2 + \rho \left(r^2 + r\,dr + \frac{dr^2}{4} \right) . dz - \sigma \left(r^2 + 2\,r\,dr + dr^2 \right) = 0$$

$$Approximately \quad \frac{dr^2}{4} \quad and \quad dr^2 = 0$$

$$\rho \, (r^2 + r\,dr)\,dz - 2\sigma \, r\,dr = 0$$

$$\rho \, r^2 \, dz - 2\sigma \, r\,dr = 0$$

$$\rho \, r^2 \, dz = 2\sigma \, r\,dr$$

$$\int dz = \int \frac{2\sigma \, r\,dr}{\rho \, r^2} \quad \Rightarrow z = \frac{\sigma}{\rho} \ln r^2 + c$$

$$at \quad z = 0 \quad r = r_o$$

$$0 = \frac{\sigma}{\rho} \ln r_o^{\,2} + c \quad \Rightarrow \quad c = -\frac{\sigma}{\rho} \ln r_o^{\,2}$$

$$z = \frac{\sigma}{\rho} \ln r^2 - \frac{\sigma}{\rho} \ln r_o^{\,2}$$

$$z = \frac{\sigma}{\rho} \ln \left(\frac{r}{r_o} \right)^2 = 2 \frac{\sigma}{\rho} \, \ln \frac{r}{r_o}$$

$$\frac{\rho \, z}{2\sigma} = \ln \frac{r}{r_o}$$

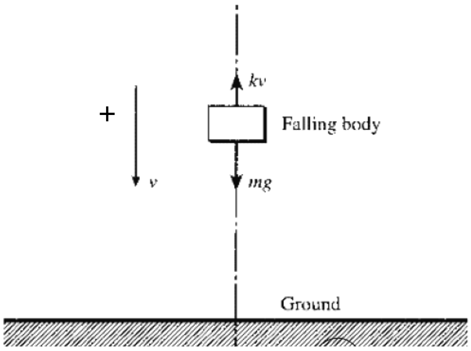
$$\frac{r}{r_o} = e^{\frac{\rho z}{2\sigma}}$$

$$\frac{\rho z}{2\sigma} = \frac{\rho z}{2\dfrac{W}{\Pi r_o^2}} = \frac{\rho z \Pi r_o^2}{2W} \quad \Leftrightarrow \quad \sigma = \frac{W}{A} = \frac{W}{\pi r_o^2}$$

$$r = r_o e^{\frac{\rho z \Pi r_o^2}{2W}}$$

Example: a body falls in a medium with resistance proportional to speed at any instant. If the limiting speed is 50 *ft/sec* and the speed of the body decrease to half (25 *ft/sec*) after (1 *sec*), what was the *initial velocity*?

Solution:



$$Force = mass \times acceleration$$

$$F = m\frac{dv}{dt}, \quad Newton's\ second\ law$$

$$m g - k v = m\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{k}{m}v = g \quad \Rightarrow \quad linear\ differential\ equation$$

$$Q = g \quad , \quad P = \frac{k}{m}$$

$$I = e^{\int p\,dt} \quad \Rightarrow \quad I = e^{\int \frac{k}{m}dt} \quad \Rightarrow \quad I = e^{\frac{k}{m}.t}$$

$$I.y = \int I.Q.dt \quad \Rightarrow \quad I.v = \int I.Q.dt$$

$$e^{\frac{k}{m}.t}.v = \int e^{\frac{k}{m}.t}.g.dt + c$$

$$e^{\frac{k}{m}.t}.v = g.\frac{m}{k}.e^{\frac{k}{m}.t} + c \qquad and\ dividing\ by\ e^{\frac{k}{m}.t}$$

$$v = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.t}}$$

Sir Isaac Newton

1643 - 1727

Initial conditions :

$at\ t = \infty \quad \Rightarrow \quad v = 50\ ft/\sec$

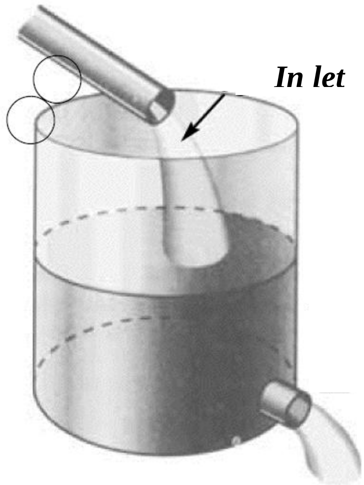
$$50 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.\infty}} \quad \Rightarrow \quad 50 = \frac{g.m}{k} + \frac{c}{\infty} \quad \Rightarrow \quad k = \frac{g.m}{50}$$

$at\ t = 1 \quad \Rightarrow \quad v = 25\ ft/\sec$

$$25 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.1}} \quad \Rightarrow \quad 25 = \frac{g.m}{\frac{g.m}{50}} + \frac{c}{e^{\frac{g.m}{50.m}}}$$

$$c = -25 \times e^{\frac{g}{50}}$$

Example: The tank in the Fig. below, contains 200 *gal* of water in which 40 *lb* of salt are dissolved, five gallon of brine each containing 2 *lb* of dissolved salt, run into the tank per *minute*, and the mixture kept uniform by stirring, runs out at the same rate. Find the amount of salt *y(t)* in the tank at any time?



Solution:

The rate of change $y' = \frac{dy}{dt}$

Where *y(t)* is the amount of salt.

Salt inflow rate = $5\,gal \times 2\frac{lb}{gal}.\frac{1}{\min} = 10\frac{lb}{\min}$

Salt Outflow rate = (the tank is always 200 *gal* because 5 *gal* flow in and 5 *gal* flow out every per minute), thus, 1 *gal* contain *y(t)/200*. Hence 5 out flowing *gal* contain $5 \times y(t)/200 = 0.025\, y(t)$.

y' = Salt inflow rate – salt outflow rate

y' = 10 – 0.025 y Variable separable,

$$\frac{dy}{dt}=10-0.025\,y \quad\Rightarrow\quad \frac{dy}{dt}=-0.025\,y+10$$

$$\frac{dy}{-0.025\,dt}=y-400 \quad\Rightarrow\quad \frac{dy}{y-400}=-0.025\,dt$$

$$\ln|y-400|=-0.025\,t+c$$

$$y-400=c\,e^{-0.025t}$$

$$at\ t=0 \quad y=40\,lb \quad\Rightarrow\quad y(0)=40$$

$$40-400=c\times 1 \quad\Rightarrow\quad c=-360$$

$$\therefore \quad y(t)=-360\,e^{-0.025t}+400 \quad \textit{the amount of salt at any time}\quad t.$$

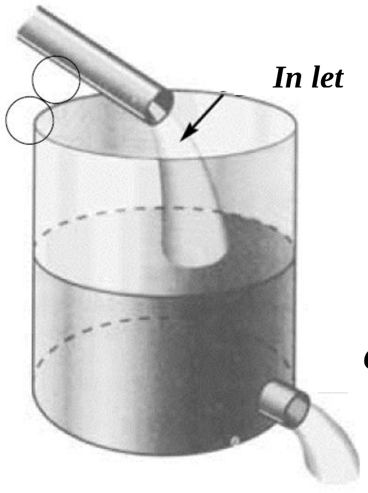
Example: the tank in the fig. below contain 1000 *gal* 0f water in which 200 *lb* of salt are dissolved, *fifty gallon*, of brine each containing $1+\cos t$ *lb* of dissolved salt run into the tank per *minute*, the mixture kept uniform by stirring, run out at the same rate. Find the amount of salt *y(t)* in the tank at any time?

Solution:

Salt inflow rate = *50 (1+ cos t)*

Salt outflow rate = (the tank always contain 1000 *gal*, thus, 1 *gal* contain *y(t)/1000*, hence *50 y(t)/1000 = 0.05 y(t)* is the salt content in outflow per *minute*.

The rate of change $y'=\frac{dy}{dt}$ of y equal the balance:



y' = Inflow – Outflow = 50 (1+cos t) - 0.05 y

$$y'+0.05\,y=50(1+\cos t) \quad\Rightarrow\quad \textit{linear differential equation}$$

$$P=0.05 \qquad Q=50(1+\cos t)$$

$$I=e^{\int P\,dt}=e^{\int 0.05\,dt}=e^{0.05t}$$

$$I.y=\int I.Q.\,dt+c$$

$$e^{0.05t}.y=(\int e^{0.05t}.50\,(1+\cos t)\,dt+c)$$

$$y=e^{-0.05t}\left[\left\{\int 50\,e^{0.05t}\,dt+50\int e^{0.05t}.\cos t\,dt\right\}+c\right]$$

$$y=e^{-0.05t}\left[\frac{50}{0.05}e^{0.05t}+50\int e^{0.05t}.\cos t\,dt+c\right]$$

We first solve the following integration:

$$\int e^{0.05t}.\cos t.\,dt$$
$$Let\ u = e^{0.05t} \quad \Rightarrow \quad du = 0.05\, e^{0.05t}\ dt$$
$$Let\ dv = \cos t\ dt \quad \Rightarrow \quad v = \sin t$$
$$\int e^{0.05t}.\cos t.\,dt = e^{0.05t}.\sin t - 0.05\int e^{0.05t}.\sin t.\,dt$$

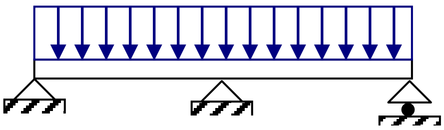
now we have to integrate another partial integration:

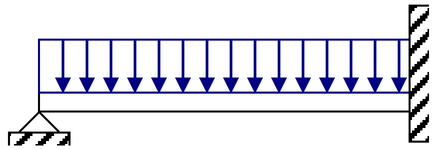
$$\int e^{0.05t}.\sin t.\,dt$$
$$Let \quad u = e^{0.05t} \quad \Rightarrow \quad du = 0.05\, e^{0.05t}\ dt$$
$$Let \quad dv = \sin t \quad \Rightarrow \quad v = -\cos t$$
$$\int e^{0.05t}.\cos t.\,dt = e^{0.05t}.\sin t - 0.05\left[-e^{0.05t}.\cos t + 0.05\int e^{0.05t}.\cos t.dt\right]$$
$$\int e^{0.05t}.\cos t.\,dt = e^{0.05t}.\sin t + 0.05\, e^{0.05t}.\cos t - 0.0025\int e^{0.05t}.\cos t\ dt$$
$$(1-0.0025)\int e^{0.05t}.\cos t.\,dt = e^{0.05t}.\sin t + 0.05\, e^{0.05t}.\cos t$$
$$\int e^{0.05t}.\cos t.\,dt = 0.9975\, e^{0.05t}.\sin t + 0.0498\, e^{0.05t}.\cos t$$
$$y(t) = e^{-0.05t}\left[1000\, e^{0.05t} + 49.88\, e^{0.05t}\ \sin t + 2.494\, e^{0.05t}\ \cos t + c\right]$$
$$y(t) = 1000 + 49.88\,\sin t + 2.494\,\cos t + c\, e^{-0.05t}$$

$\underline{Initial\ \ conditions}$

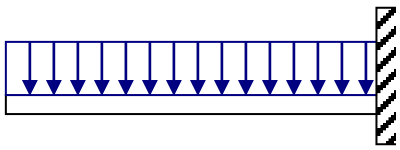
$$at \quad t = 0 \quad \Rightarrow \quad y = 200$$
$$y(0) = 1000 + 0 + 2.494\,(1) + c = 200$$
$$c = -802.494$$
$$\therefore\ y(t) = y(t) = 1000 + 49.88\,\sin t + 2.494\,\cos t - 802.494\, e^{-0.05t}$$
$$the\ amount\ of\ \ salt\ at\ any\ time\ t$$

Structural Applications:





$Y = 0$	$Y = 0$	$Y = 0$	$Y = 0$
$Y' \neq 0$	$Y' \neq 0$	$Y' \neq 0$	$Y' = 0$
$M \neq 0$	$M = 0$	$M = 0$	$M \neq 0$
$V \neq 0$	$V \neq 0$	$V \neq 0$	$V \neq 0$



$Y \neq 0$	$Y = 0$
$Y' \neq 0$	$Y' = 0$
$M = 0$	$M \neq 0$
$V = 0$	$V \neq 0$

EI y = Deflection
EI y' = Rotation
EI y'' = Moment
EI y''' =Shear
EI y^{IV} = Load