

**Example:** Find the equation of the deflection curve of the beam shown below?

**Solution:**

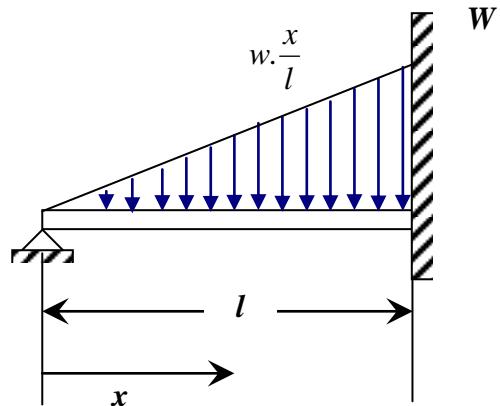
$$EI y^{IV} = w \cdot \frac{x}{l} \quad \text{Load equation}$$

$$EI y''' = \frac{w \cdot x^2}{2l} + c_1$$

$$EI y'' = \frac{w \cdot x^3}{6l} + c_1 x + c_2$$

$$EI y' = \frac{w \cdot x^4}{24l} + c_1 \frac{x^2}{2} + c_2 x + c_3$$

$$EI y = \frac{w \cdot x^5}{120l} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$$



**Initial Conditions:**

$$\text{at } x = 0 \quad y = 0$$

$$0 = \frac{w(0)}{120l} + \frac{c_1(0)}{6} + \frac{c_2(0)}{2} + c_3(0) + c_4 \Rightarrow c_4 = 0$$

$$\text{at } x = 0 \quad y'' = 0$$

$$0 = \frac{w(0)}{6l} + c_1(0) + c_2 \Rightarrow c_2 = 0$$

$$\text{at } x = l \quad y = 0$$

$$0 = \frac{w(l^5)}{120l} + \frac{c_1(l^3)}{6} + \frac{0(l^2)}{2} + c_3(l) + 0 \Rightarrow \frac{wl^4}{120} + c_1 \frac{l^3}{6} + c_3 l = 0 \quad \dots \quad (1)$$

$$\text{at } x = l \quad y' = 0$$

$$0 = \frac{w(l^4)}{24l} + \frac{c_1(l^2)}{2} + (0)l + c_3 \Rightarrow \frac{wl^3}{24} + c_1 \frac{l^2}{2} + c_3 = 0 \quad \dots \quad (2)$$

**Solve equation (1) and (2) to find  $c_1$  and  $c_3$**

$$\frac{wl^3}{120} + c_1 \frac{l^2}{6} + c_3 = 0 \quad \dots \quad (1') \quad \text{dividing eq (1) by } l$$

$$\frac{wl^3}{24} + c_1 \frac{l^2}{2} + c_3 = 0 \quad \dots \quad (2)$$

*Another way to solve the previous example is by starting with the moment equation:*

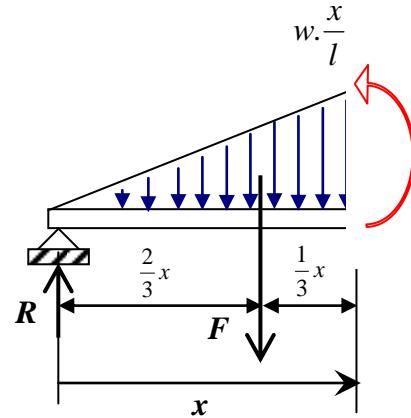
$$EI y'' = -M$$

$$EI y'' = - \left[ \left( Rx - \frac{1}{2} w \frac{x^2}{l} \left( \frac{1}{3} x \right) \right) \right]$$

$$EI y'' = -Rx + \frac{wx^3}{6l}$$

$$EI y' = -R \frac{x^2}{2} + \frac{wx^4}{24l} + c_1$$

$$EI y = -R \frac{x^3}{6} + \frac{wx^5}{120l} + c_1 x + c_2$$



$$\text{at } x = 0 \quad y = 0$$

$$0 = 0 + 0 + 0 + c_2 \Rightarrow c_2 = 0$$

$$\text{at } x = l \quad y = 0$$

$$0 = -\frac{Rl^3}{6} + \frac{wl^5}{120l} + c_1 l + 0 \Rightarrow -\frac{Rl^3}{6} + \frac{wl^4}{120} + c_1 l = 0 \quad \dots \dots \dots \quad (1)$$

$$\text{at } x = l \quad y' = 0$$

$$0 = -R \frac{l^2}{2} + \frac{wl^4}{24l} + c_1 \Rightarrow -R \frac{l^2}{2} + \frac{wl^3}{24} + c_1 = 0 \quad \dots \dots \dots \quad (2)$$

*Solve Equation (1) and (2) to find  $c_1$  and substitute in the deflection equation.*

**Example:** For the cantilever beam shown below find the deflection curve?

**Solution:**

$$EI y'' = -wx \left( \frac{x}{2} \right)$$

$$EI y'' = -\frac{wx^2}{2}$$

$$y'' = -\frac{wx^2}{2EI}$$

$$y' = -\frac{wx^3}{6EI} + A$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

$$\text{at } x = l \quad y = 0$$

$$0 = -\frac{wl^4}{24EI} + Al + B \quad \dots \dots \dots \quad (1)$$

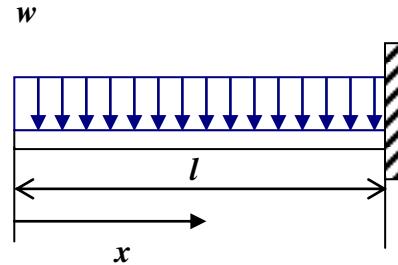
$$\text{at } x = l \quad y' = 0$$

$$0 = -\frac{wl^3}{6EI} + A \quad \Rightarrow \quad A = \frac{wl^3}{6EI}$$

**Substitute in eq. (1)**

$$0 = -\frac{wl^4}{24EI} + \frac{wl^3}{6EI}(l) + B \quad \Rightarrow \quad B = \frac{wl^4}{24EI} - \frac{wl^4}{6EI} = \frac{3}{24} \cdot \frac{wl^4}{EI}$$

$$\therefore y = -\frac{wx^4}{24EI} + \frac{wl^3}{6EI}(x) - \frac{3}{24} \cdot \frac{wl^4}{EI} \quad \text{complete solution}$$

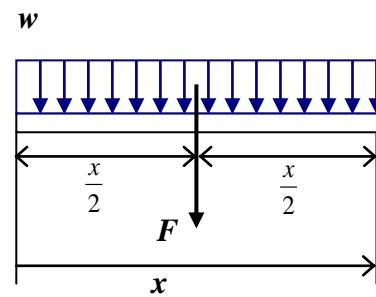


$$\mathbf{Y}'' = \mathbf{0}$$

$$\mathbf{Y}''' = \mathbf{0}$$

$$\mathbf{Y} = \mathbf{0}$$

$$\mathbf{Y}' = \mathbf{0}$$



## **1.2 Linear Differential Equations with constant coefficients.**

**The General formula of a linear D.E is:**

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = f(x)$$

where  $P_1, P_2, \dots, P_n, f(x)$  are functions of  $(x)$

**The General formula of a linear D.E with constant coefficients is:**

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x)$$

If  $f(x) = 0$  then the equation is **Homogeneous**.

If  $f(x) \neq 0$  then the equation is **Inhomogeneous**.

**Example:**

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 5 y = x \Rightarrow \text{Inhomogeneous}$$

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 5 y = 0 \Rightarrow \text{Homogeneous}$$

**Theorem 1:** If  $y_1$  and  $y_2$  are any solution of homogeneous equation then  $y = c_1 y_1 + c_2 y_2$  where  $c_1$  and  $c_2$  are arbitrary constants is also a solution.

The equation:  $\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6 y = 0$  has two solutions,

$$y_1 = e^{2x} \quad y_2 = e^{3x} \quad \text{prove that}$$

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{3x} \quad \text{is also a solution}$$

$$y = c_1 e^{2x} + c_2 e^{3x}$$

$$\frac{dy}{dx} = 2 c_1 e^{2x} + 3 c_2 e^{3x}$$

$$\frac{d^2 y}{dx^2} = 4 c_1 e^{2x} + 9 c_2 e^{3x} \quad \text{substitute in the D.E}$$

$$4 c_1 e^{2x} + 9 c_2 e^{3x} - 5(2 c_1 e^{2x} + 3 c_2 e^{3x}) + 6(c_1 e^{2x} + c_2 e^{3x}) = 0$$

$\therefore y$  is a solution too.

**Theorem 2:** If  $Y$  is any specific solution of Inhomogeneous equation and if  $y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$  is a complete solution of the related Homogeneous equation then a complete solution of the Inhomogeneous is:

$$y = y_c + Y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + Y$$

$y_c$  = complementary function

$Y$  = particular integral

### Using Operator:

$$\frac{d}{dx} = \text{Operator} = D \Rightarrow \frac{dy}{dx} = D y$$

$$\frac{d^2 y}{d x^2} + 5 \frac{dy}{dx} + 6 y = 0$$

$$(D^2 + 5D + 6) y = 0 \Leftrightarrow \because y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2 y}{d x^2} = -\sin x$$

### (1.2.1) Solutions of Homogeneous Linear D.E with constant coefficients:

#### The Characteristic Equation:

$$\frac{d^n y}{d x^n} + a_1 \frac{d^{n-1} y}{d x^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

$$f(D) y = (D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = 0$$

$$f(D) y = ((D - m_1)(D - m_2) + \dots + (D - m_n)) y = 0$$

$$f(m) = (m - m_1)(m - m_2) + \dots + (m - m_n) = 0$$

#### Example:

$$\frac{d^2 y}{d x^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \Rightarrow (D^2 + a_1 D + a_2) y = 0$$

$$\therefore m^2 + a_1 m + a_2 = 0$$

**Caution !**

Characteristic equations are only defined for linear homogeneous differential equations with constant coefficients.

**General solution for second-order homogeneous linear D.F:**

**Case 1: If  $m_1$  and  $m_2$  are real and distinct  $m_1 \neq m_2$ .**

$$(D^2 + a_1 D + a_2) y = 0$$

$$(D - m_1)(D - m_2) y = 0$$

$$\text{let } u = (D - m_2) y \Rightarrow (D - m_1)u = 0$$

$$Du = m_1 u \Rightarrow \frac{du}{dx} = m_1 u \Rightarrow \int \frac{du}{u} = \int m_1 dx + c$$

$$\ln u = m_1 x + c \Rightarrow u = e^{m_1 x + c} = c_1 e^{m_1 x}$$

$$u = (D - m_2) y \Rightarrow (D - m_2) y = c_1 e^{m_1 x}$$

$$\frac{dy}{dx} - m_2 y = c_1 e^{m_1 x} \Leftrightarrow \text{linear D.E}$$

$$\frac{dy}{dx} + P y = Q \Rightarrow P = -m_2 \quad Q = c_1 e^{m_1 x}$$

$$I(x) = \int e^{P(x) dx} = \int e^{-m_2 x} = e^{-m_2 x}$$

$$I(x) y = \int I(x).Q dx + c_2$$

$$e^{-m_2 x}.y = \int e^{-m_2 x}.c_1 e^{m_1 x}.dx + c_2$$

$$e^{-m_2 x}.y = \int c_1 e^{(m_1 - m_2)x}.dx + c_2$$

$$e^{-m_2 x}.y = \frac{1}{m_1 - m_2} \cdot \int c_1 e^{(m_1 - m_2)x}.dx + c_2$$

$$e^{-m_2 x}.y = \frac{c_1 e^{(m_1 - m_2)x}}{m_1 - m_2} + c_2$$

$$y = \frac{c_1 e^{(m_1 - m_2)x}}{m_1 - m_2} \cdot e^{m_2 x} + c_2 \cdot e^{m_2 x} \Rightarrow y = \frac{c_1 e^{m_1 x - m_2 x + m_2 x}}{m_1 - m_2} + c_2 \cdot e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

**Case 2: If  $m_1 = m_2$ ,**

$$(D - m)^2 y = 0$$

$$(D - m)(D - m) y = 0$$

$$\text{Let } z = (D - m) y \Rightarrow (D - m) z = 0$$

$$\frac{dz}{dx} - mz = 0 \Rightarrow \frac{dz}{dx} = mz \Rightarrow \frac{dz}{z} = m dx \Rightarrow \ln z = mx + c_1$$

$$z = e^{mx} \cdot e^{c_1} = C_1 e^{mx}$$

$$\because (D - m) y = z \Rightarrow (D - m) y = C_1 e^{mx} \Rightarrow \frac{dy}{dx} - my = C_1 e^{mx} \Leftrightarrow \text{linear}$$

$$P = -m \quad Q = C_1 e^{mx}$$

$$I(x) = \int e^{P(x)dx} = \int e^{-m dx} = e^{-mx}$$

$$I(x) y = \int I(x) \cdot Q \cdot dx + c_2$$

$$e^{-mx} \cdot y = \int e^{-mx} \cdot C_1 e^{mx} \cdot dx + c_2$$

$$e^{-mx} \cdot y = (C_1 x + c_2) \Rightarrow$$

$$y = (C_1 x + c_2) \cdot e^{mx}$$

**Case 3: If  $m_1 = p + i q$  and  $m_2 = p - i q$  the characteristic equation has a conjugate complex root.**

Let the general homogeneous Linear D.E form be:

$$y'' + p_1 y' + p_2 y = 0$$

$$m_{1,2} = \frac{-p_1 \mp \sqrt{p_1^2 - 4p_2}}{2}$$

$$\text{Let } p_1^2 - 4p_2 = -4q^2$$

$$m_{1,2} = -\frac{p_1}{2} \mp i q$$

$$y = c_1 e^{\left(-\frac{p_1}{2} - iq\right)x} + c_2 e^{\left(-\frac{p_1}{2} + iq\right)x}$$

$$y = e^{-\frac{p_1}{2}x} \left[ c_1 e^{-iqx} + c_2 e^{iqx} \right]$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2i} \quad \text{Euler's formula}$$

$$e^{ix} = \cos x + i \cdot \sin x$$

$$e^{-ix} = \cos x - i \cdot \sin x$$

$$y = e^{-\frac{p_1}{2}x} [c_1(\cos qx - i \sin qx) + c_2(\cos qx + i \sin qx)]$$

$$y = e^{-\frac{p_1}{2}x} [(c_1 + c_2) \cos qx + (c_1 - c_2) \sin qx]$$

$$\therefore y = e^{-\frac{p_1}{2}x} (A \cos qx + B \sin qx)$$

$$y = e^{px} (A \cos qx + B \sin qx) \quad \text{where} \quad m_{1,2} = p + iq$$

**Example:** Find the general solution of the equation:  $y''+7y'+12y=0$  ?

**Solution:**  $y''+7y'+12y=0 \Rightarrow m^2 + 7m + 12 = 0$

$$m_{1,2} = \frac{-7 \mp \sqrt{49-48}}{2} \Rightarrow m_1 = -4 \quad m_2 = -3$$

$$y = C_1 e^{m_1 x} + c_2 \cdot e^{m_2 x} = C_1 e^{-4x} + c_2 \cdot e^{-3x}$$

**Example:** Solve the equation:  $\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$  ?

**Solution:**  $m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \Rightarrow m_1 = 3 \quad m_2 = 2$

$$y = C_1 e^{m_1 x} + c_2 \cdot e^{m_2 x} = C_1 e^{3x} + c_2 \cdot e^{2x}$$

**Example:** Solve the equation:  $4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = 0$  ?

**Solution:**

$$4m^2 + 4m + 1 = 0 \Rightarrow m^2 + m + \frac{1}{4} = 0 \Rightarrow (2m+1)^2 = 0 \Rightarrow m_1 = m_2 = -\frac{1}{2}$$

$$y = (C_1 x + c_2) \cdot e^{mx} \Rightarrow y = (c_1 x + c_2) e^{-\frac{1}{2}x}$$

**Example:** Solve the equation:  $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$  ?

**Solution:**

$$2m^2 + 3m + 4 = 0$$

$$m_{1,2} = \frac{-3 \mp \sqrt{9 - (4 \times 2 \times 4)}}{2 \times 2} = \frac{-3 \mp \sqrt{-23}}{4} = -\frac{3}{4} \mp \frac{\sqrt{23}}{4}i$$

$$y = e^{-\frac{3}{4}x} \left( A \cos \frac{\sqrt{23}}{4}x + B \sin \frac{\sqrt{23}}{4}x \right)$$

**Example:** Find the special solution of the equation:  $4\frac{d^2y}{dx^2} + 16\frac{dy}{dx} + 17y = 0$

If  $y = I$  at  $x = 0$  and  $y = 0$  at  $x = \pi$  ?

**Solution:**

$$4m^2 + 16m + 17 = 0 \Rightarrow m_{1,2} = \frac{-16 \mp \sqrt{256 - 272}}{8} = \frac{-16 \mp \sqrt{-16}}{8} = \frac{-16 \mp 4i}{8} = -2 \mp \frac{1}{2}i$$

$$y = e^{-2x} \left( A \cos \frac{x}{2} + B \sin \frac{x}{2} \right) \Leftrightarrow \text{the general solution}$$

$$\text{at } x = 0 \quad y = 1 \Rightarrow 1 = e^0 (A \cos 0 + B \sin 0) \Rightarrow A = 1$$

$$\text{at } x = \pi \quad y = 0 \Rightarrow 0 = e^{-2\pi} (1 \times 0 + B \times 1) \Rightarrow B = 0$$

$$\therefore y = e^{-2x} \left( \cos \frac{x}{2} \right)$$

### (1.2.1.1) Homogeneous equation of higher order:

Consider the general equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0 \quad \text{where } n \text{ is the differentiation order}$$

The characteristic equation will be as follows:

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \quad \text{which has } n \text{ roots}$$

$$(m - m_1)(m - m_2) \dots (m - m_n) = 0$$

**Case 1: If the roots are real and unequal.**

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

**Case 2: If the roots are real and equal (repeated roots).**

$$y = (c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n) e^{m x}$$

**Case 3: Imaginary roots.**

$$y = e^{p_1 x} (A_1 \cos q_1 x + B_1 \sin q_1 x) + e^{p_2 x} (A_2 \cos q_2 x + B_2 \sin q_2 x) + \dots + e^{p_r x} (A_r \cos q_r x + B_r \sin q_r x)$$

$$\text{where } r = \frac{n}{2}$$

**Example:** Consider the following D.E?

$$y''' + 5y'' + 9y' + 5y = 0 \Leftrightarrow \text{Homogeneous}$$

the corresponding characteristic eq. is

$$m^3 + 5m^2 + 9m + 5 = 0$$

by trial  $m = -1$  is a root  $\Rightarrow (m+1)(m^2 + 4m + 5) = 0$

$$m_{1,2} = \frac{-4 \mp \sqrt{16-20}}{2} \Rightarrow m_{1,2} = -\frac{4}{2} \mp \frac{\sqrt{-4}}{2} = -2 \mp \frac{\sqrt{4}}{2}i = -2 \mp i$$

$$y = \underline{c_1 e^{-x}} + e^{-2x} (\underline{c_2 \cos x} + \underline{c_3 \sin x})$$

1<sup>st</sup> root      2<sup>nd</sup> root    3<sup>rd</sup> root

**Hint: Algebraic Division**

$$\begin{array}{r} (m^3 + 5m^2 + 9m + 5) \quad | \quad (m+1) \\ \hline \quad \quad \quad (m^2 + 4m + 5) \\ \underline{-m^3 - m^2} \\ \quad \quad \quad 4m^2 + 9m + 5 \\ \underline{-4m^2 - 4m} \\ \quad \quad \quad 5m + 5 \\ \underline{-5m - 5} \\ \quad \quad \quad 0 + 0 \end{array}$$

**Example:** consider the D.E  $(D^4 + 8D^2 + 16)y = 0$  ?

**Solution:**  $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

$$m^4 + 8m^2 + 16 = 0$$

$$\text{Let } x = m^2 \Rightarrow x^2 + 8x + 16 = 0$$

$$(x+4)^2 = 0$$

$$x_1 = -4 \Rightarrow m^2 = -4$$

$$x_2 = -4 \Rightarrow m^2 = -4$$

$$m_{1,2} = \mp 2i$$

$$m_{3,4} = \mp 2i$$

$$y = e^0 [c_1 \cos 2x + c_2 \sin 2x] + e^0 [c_3 \cos 2x + c_4 \sin 2x]$$

But the first and second term are equal so we multiply one of them by  $x$

$$y = e^0 [c_1 x \cos 2x + c_2 x \sin 2x] + e^0 [c_3 \cos 2x + c_4 \sin 2x]$$

$$y = (c_1 x + c_3) \cos 2x + (c_2 x + c_4) \sin 2x$$

**Example:** consider the D.E  $(D^4 + 3D^3 + 3D^2 + D)y = 0$

**Solution:**

$$m^4 + 3m^3 + 3m^2 + m = 0$$

$$m(m^3 + 3m^2 + 3m + 1) = 0$$

by trial  $m = -1$

$$\begin{array}{r} m^2 + 2m + 1 \\ m+1 \overline{)m^3 + 3m^2 + 3m + 1} \\ \underline{m^3 + m^2} \\ 2m^2 + 3m + 1 \\ \underline{2m^2 + 2m} \\ m+1 \\ \underline{m+1} \\ 0+0 \end{array}$$

$$m(m+1)(m^2 + 2m + 1) = 0 \Rightarrow m(m+1)(m+1)^2 = 0 \Rightarrow m(m+1)^3 = 0$$

$$m_1 = 0 \quad m_2 = m_3 = m_4 = -1$$

$$y = c_1 + (c_2 x^2 + c_3 x + c_4) e^{-x}$$

### (1.2.2) Solutions of Inhomogeneous Linear D.E with constant coefficients:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x)$$

The general solution to the linear inhomogeneous differential equation is,

$y = y_h + y_p$  where  $y_p$  denotes one solution to the differential equation and  $y_h$  is the general solution to the associated homogeneous equation.

#### (1.2.2.1) The Method of Undetermined Coefficients:

The method is initiated by assuming a particular solution of the form:

$$y_p(x) = A_1 y_1(x) + A_2 y_2(x) + \dots + A_n y_n(x)$$

where  $A_1, A_2, \dots, A_n$  denote arbitrary multiplicative constants. These constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

**Use the following table to find  $y_p(x)$ :**

$f(x)$	$y_p(x)$
a	A
$a x^n$	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$
$a e^{px}$	$A e^{px}$
$a \cos \beta x$	$A \cos \beta x + B \sin \beta x$
$a \sin \beta x$	
$a e^{px} \cos \beta x$	$A e^{px} (B_1 \cos \beta x + B_2 \sin \beta x)$
$a e^{px} \sin \beta x$	
$a x^n e^{px} \cos \beta x$	$A e^{px} (A_1 x^n + A_2 x^{n-1} + \dots + A_n) (B_1 \cos \beta x + B_2 \sin \beta x)$
$a x^n e^{px} \sin \beta x$	

**Example:** Using the table to write  $y_p$  ?

$$1) \quad f(x) = 2x^3 \Rightarrow y_p = Ax^3 + Bx^2 + Cx + D$$

$$2) \quad f(x) = 3e^{2x} \Rightarrow y_p = Ce^{2x}$$

$$3) \quad f(x) = \frac{1}{2} \sin 2x \Rightarrow y_p = A \cos 2x + B \sin 2x$$

$$4) \quad f(x) = \sin(x) + \cos(x) \Rightarrow y_p = A \cos x + B \sin x$$

**Example:** solve the following D.E?  $y'' - 4y = 8x^2$

**Solution:**  $y = y_h + y_p$

to find  $y_h$

Let  $y'' - 4y = 0 \Rightarrow$  homogeneous

$$m^2 - 4 = 0 \Rightarrow m_1 = -2 \quad m_2 = 2$$

$$y_h = C_1 e^{-2x} + C_2 e^{2x}$$

to find  $y_p \Leftrightarrow$  since  $f(x) = 8x^2$

$$y_p = Ax^2 + Bx + C$$

$$y'_p = 2Ax + B$$

$$y''_p = 2A$$

Substitute in the original equation

$$2A - 4Ax^2 - 4Bx - 4C = 8x^2$$

$$-4Ax^2 - 4Bx - (4C - 2A) = 8x^2$$

and by equalizing the constants of the two sides of the equation  $\Rightarrow$

$$-4A = 8 \Rightarrow A = -2$$

$$-4B = 0 \Rightarrow B = 0$$

$$-4C + 2A = 0 \Rightarrow -4C + (2 \times (-2)) = 0 \Rightarrow C = -1$$

$$\therefore y_p = -2x^2 - 1$$

$$y = y_p + y_h \Rightarrow y = C_1 e^{-2x} + C_2 e^{2x} - 2x^2 - 1$$

**Example:** solve the following D.E?  $y'' - y' - 6y = 2e^{-2x}$

**Solution:**  $y = y_h + y_p$

to find  $y_h$

Let  $y'' - y' - 6y = 0 \Rightarrow$  homogeneous

$$m^2 - 4m - 6 = 0 \Rightarrow$$

$$m_{1,2} = \frac{-1 \mp \sqrt{1+24}}{2} \Rightarrow m_1 = -3 \quad m_2 = 2$$

$$y_h = C_1 e^{-3x} + C_2 e^{2x}$$

to find  $y_p$

$$y_p = C e^{-2x}$$

$$y'_p = -2C e^{-2x}$$