Example1: Solve the following linear program using the simplex method.

Maximize $z = 5x_1 + 4x_2$
subject to

$$
\begin{aligned}
6x_1 + 4x_2 &\leq 24 \\
x_1 + 2x_2 &\leq 6 \\
-x_1 + x_2 &\leq 1 \\
x_2 &\leq 2 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The inequality of each constraint should convert to equality; thus, the canonical form of the model is converted to standard form as follows:

Maximize $z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$
subject to

$$
\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \\
x_1 + 2x_2 + s_2 &= 6 \\
-x_1 + x_2 + s_3 &= 1 \\
x_2 + s_4 &= 2 \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}
$$

The variables s_1, s_2, s_3, and s_4 are the slacks associated with the respective constraints. Next, we write the objective equation as $z - 5x_1 - 4x_2 = 0$

In this manner, the starting simplex tableau can be represented as follows:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	**solution**
z	-5	-4	0	0	0	0	0
s_1	6	4	1	0	0	0	24
s_2	1	2	0	1	0	0	6
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2

The result can be seen by setting the non-basic variables (x_1, x_2) equal to zero in all the equations, and also by noting the special identity-matrix arrangement of the constraint coefficients of the basic variables (all diagonal elements are 1, and all off-diagonal elements are 0).

In the simplex tableau where the objective function is written as $z - 5x_1 - 4x_2 = 0$, the selected variable is the non-basic variable with the most negative coefficient in the objective equation. In the terminology of the simplex algorithm, x_1 is known as the entering variable because it enters the basic solution.

If x_1 is the entering variable, one of the current basic variables must leave; that is, it becomes non-basic at zero level (recall that the number of non-basic variable must always be $n - m$).

The mechanics for determining the ***leaving variable*** calls for computing the ***ratios*** of the right-hand side of the equations (Solution column) to the corresponding (strictly) positive constraint coefficients under the entering variable, x_1, as the following table shows.

Basic	x_1	solution	
s_1	6	24	x_1=24/6 = 4
s_2	1	6	x_1=6/1= 6
s_3	-1	1	x_1=1/-1= -1 (non-negative denominator, ignore)
s_4	0	2	x_1=2/0= ∞ (zero denominator, ignore)

x_1 enters (at level 4) and s_1 leaves (at level zero).

The new solution is determined by "swapping" the entering variable x_1 and the leaving variable s_1 in the simplex tableau to yield non-basic variables which are s_1, x_2.
Basic variables are x_1, s_2, s_3, s_4

The swapping process is based on the Gauss-Jordan row operations. It identifies the entering variable column as the ***pivot column*** and the leaving variable row as the pivot row with their intersection being the ***pivot element***. The following tableau is a restatement of the starting tableau with its pivot row and column highlighted.

Basic	x_1 (Enter)	x_2	s_1	s_2	s_3	s_4	**solution**
z	-5	-4	0	0	0	0	0
s_1 Leave	6	4	1	0	0	0	24 Pivot row
s_2	1	2	0	1	0	0	6
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2
	Pivot column						

The Gauss-Jordan computations needed to produce the new basic solution include two types.

1. Pivot row
   a. Replace the leaving variable in the Basic column with the entering variable.
   b. New pivot row = Current pivot row ÷ Pivot element.

2. All other rows, including z

New row = (Current row) – (Pivot column coefficient) × (New pivot row)

These computations are applied to the preceding tableau in the following manner:

1. Replace s_1 in the Basic column with x_1:
New x_1-row = Current s_1-row ÷ 6
= 1/6 (6 4 1 0 0 0 24) = (1 2/3 1/6 0 0 0 4)

2. New z-row = Current z-row – (-5) × New x_1-row
= (-5 -4 0 0 0 0 0) – (-5) × (1 2/3 1/6 0 0 0 4) = (0 -2/3 5/6 0 0 0 20)

3. New s_2-row = Current s_2-row – (1) × New x_1-row
= (1 2 0 1 0 0 6) – (1) ×(1 2/3 1/6 0 0 0 4) = (0 4/3 -1/6 1 0 0 2)

4. New s_3-row = Current s_3-row – (-1) × New x_1-row
= (-1 1 0 0 1 0 1) - (-1) × (1 2/3 1/6 0 0 0 4) = (0 5/6 1/6 0 1 0 5)

5. New s_4-row = Current s_4-row – (0) × New x_1-row
= (0 1 0 0 0 1 2) – (0) × (1 2/3 1/6 0 0 0 4) = (0 1 0 0 0 1 2)

The new basic solution is (x_1, s_2, s_3, s_4), and the new tableau becomes

Enter
Pivot column

Basic	x_1	x_2	s_1	s_2	s_3	s_4	**solution**
z	0	-2/3	5/6	0	0	0	20
x_1	1	2/3	1/6	0	0	0	4
s_2 Leave	0	4/3	-1/6	1	0	0	2 Pivot row
s_3	0	5/3	1/6	0	1	0	5
s_4	0	1	0	0	0	1	2

As a result, when we set the new non-basic variables x_2 and s_1 to zero, the Solution-column automatically yields the new basic solution (x_1 = 4, s_2 = 2, s_3 = 5, s_4 = 2) This "conditioning" of the tableau is the result of the application of the Gauss-Jordan row operations. The corresponding new objective value is z = 20.

In the last tableau, the optimality condition shows that x_2 (with the most negative z-row coefficient) is the entering variable. The feasibility condition produces the following information:

Basic	x_2	solution	
x_1	2/3	4	$x_2 = 4 \div 2/3 = 6$
s_2	4/3	2	$x_2 = 2 \div 4/3 = 1.5$ minimum
s_3	5/3	5	$x_2 = 5 \div 5/3 = 3$
s_4	1	2	$x_2 = 2 \div 1 = 2$

Thus, s_2 leaves the basic solution, and the new value of x_2 is 1.5. The corresponding increase in z is 2/3 x_2 = 2/3 × 1.5 = 1, which yields new z = 20 + 1 = 21, as the tableau below confirms. Replacing s_2 in the Basic column with entering x_2, the following Gauss-Jordan row operations are applied:

1. New pivot x_2-row = Current s_2-row ÷ 4/3
2. New z-row = Current z-row – (-2/3) × New x_2-row
3. New x_1-row = Current x_1-row – (2/3) × New x_2-row
4. New s_3-row = Current s_3-row – (5/3) × New x_2-row
5. New s_4-row = Current s_4-row – (1) × New x_2-row

The operations above produce the following tableau:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	**solution**
z	0	0	3/4	1/2	0	0	21
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

Based on the optimality condition, none of the z-row coefficients are negative. Hence, the last tableau is optimal.

The optimal values of the variables in the *Basic* column are given in the right-hand-side Solution column and can be interpreted as

Decision variables	**Optimum value**	***Recommendations***
x_1	3	Produce 3 tons of product 1
x_2	3/2	Produce 1.5 tons of product 2
z	21	Maximum profit

The solution also gives the status of the resources. A resource is designated as scarce if its associated slack variable is zero; that is, the activities (variables) of the model have used the resource completely. Otherwise, if the slack is positive, then the resource is ***abundant***. The following table classifies the constraints of the model:

Resource	**Slack value**	**Status**
Constraint 1	$s_1 = 0$	Scarce
Constraint 2	$s_2 = 0$	Scarce
Constraint 3	$s_3 = 5/2$	Abundant
Constraint 4	$s_4 = 1/2$	Abundant