Example: solve the following D.E? $y''-3y'+2y=e^{x}$

Solution: $y=y_h+y_p$

$$\begin{aligned}
&to\ find\ y_h\\
&Let\quad y''-3y'+2y=0 \quad\Rightarrow\quad \text{homogeneous}\\
&m^2-3m+2=0 \quad\Rightarrow\quad (m-1)(m-2)=0 \quad\Rightarrow\quad m_1=1 \quad m_2=2\\
&y_h=C_1e^{x}+C_2e^{2x}\\
&to\ find\ y_p\\
&y_p=C\,e^{x} \otimes but\ there\ is\ a\ term\ in\ the\ eq.\ of\ \ y_h\ \ which\ is\ also\ C\,e^{x}\ \ so\ we\ multiply\ y_p\ by\ x\\
&\therefore\quad y_p=C\,x\,e^{x}\\
&y'_p=C\ xe^{x}+Ce^{x}=C\,(e^{x}+x\,e^{x})\\
&y''_p=C\ xe^{x}+Ce^{x}+Ce^{x}=C\,(2\,e^{x}+x\,e^{x})\\
&substitute\ in\ the\ original\ equation
\end{aligned}$$

$$\begin{aligned}
&C\,(2+x)\,e^{x}-3C\,(1+x)\,e^{x}+2C\,x\,e^{x}=e^{x}\\
&2C\,e^{x}+C\,x\,e^{x}-3C\,e^{x}-3C\,x\,e^{x}+2C\,x\,e^{x}=e^{x}\\
&-C\,e^{x}=e^{x} \quad\Rightarrow\quad C=-1\\
&\therefore\quad y_p=-x\,e^{x} \quad\Rightarrow y=C_1\,e^{x}+C_2\,e^{2x}-x\,e^{x}
\end{aligned}$$

(1.2.2.2) Method of Variation of Parameters:

In general if *f(x)* is not one of the types of functions considered in the (undetermined coefficients method), or if the differential equation ***does not have constant coefficient***, then this method is preferred.

Variation of parameters is another method for finding a particular solution of the n^{th}-order linear differential equation. It can be applied to all linear D.E's. It is therefore more powerful than the undetermined coefficients which is restricted to linear D.E's with constant coefficients and particular forms of *f(x)*.

The general form f the linear D.E is

$$\frac{d^{n}y}{dx^{n}}+a_1\,\frac{d^{n-1}y}{dx^{n-1}}+a_2\,\frac{d^{n-2}y}{dx^{n-2}}+.................+a_n\ y=f(x) \quad or \quad L(y)=f(x)$$

The solution as we know is $y = y_h + y_p$ where y_h is the general solution of the corresponding homogeneous equation L(y)=0 which is expressed as:

$$y_h = c_1\, y_1(x) + c_2\, y_2(x) + \dots\dots\dots + c_n\, y_n(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

and y_p is the particular solution we need to obtain which can be expressed as:

$$y_p = v_1\, y_1 + v_2\, y_2 + \dots\dots\dots + v_n\, y_n$$

Where $y_i = y_i(x)\ (i = 1,2,\dots\dots,n)$ is given in eq. (1) and,

$v_i\ (i = 1,2,\dots\dots,n)$ is un unknown functions of x that still must be determined.

To find v_i , first solve the following linear equations simultaneously for v'_i :

$$\begin{aligned}
v'_1\, y_1 + v'_2\, y_2 + \dots\dots + v'_n\, y_n &= 0 \\
v'_1\, y'_1 + v'_2\, y'_2 + \dots\dots + v'_n\, y'_n &= 0 \\
&\vdots \\
v'_1\, y_1^{(n-2)} + v'_2\, y_2^{(n-2)} + \dots\dots + v'_n\, y_n^{(n-2)} &= 0 \\
v'_1\, y_1^{(n-1)} + v'_2\, y_2^{(n-1)} + \dots\dots + v'_n\, y_n^{(n-1)} &= f(x)
\end{aligned}$$

Then integrate each v'_i to obtain v_i , disregarding all constants of integration. This is permissible because we are seeking only one particular solution.

Example: for the special case *n=3*,

$$\begin{aligned}
&v'_1\, y_1 + v'_2\, y_2 + v'_3\, y_3 = 0 \\
&v'_1\, y'_1 + v'_2\, y'_2 + v'_3\, y'_3 = 0 \\
&v'_1\, y''_1 + v'_2\, y''_2 + v'_3\, y''_3 = f(x)
\end{aligned}$$

for the special case *n=2*,

$$\begin{aligned}
&v'_1\, y_1 + v'_2\, y_2 = 0 \\
&v'_1\, y'_1 + v'_2\, y'_2 = f(x)
\end{aligned}$$

for the special case *n=1*,

$$v'_1\, y_1 = f(x)$$

Example: solve $y'''+y'=\sec x$.

Solution:

This is a 3rd order equation with

$$y_h = c_1 + c_2\cos x + c_3\sin x \quad \Rightarrow \quad y_p = v_1 + v_2\cos x + v_3\sin x$$

Here $y_1 = 1$, $y_2 = \cos x$, $y_3 = \sin x$, and $f(x) = \sec x \Rightarrow$

$$\begin{aligned}
v'_1(1) + v'_2(\cos x) + v'_3(\sin x) &= 0\\
v'_1(0) + v'_2(-\sin x) + v'_3(\cos) &= 0\\
v'_1(0) + v'_2(-\cos x) + v'_3(-\sin x) &= \sec x
\end{aligned}$$

Solving this set of equations simultaneously, we obtain :

$v'_1 = \sec x \qquad v'_2 = -1 \qquad v'_3 = -\tan x$. Thus,

$$\begin{aligned}
v_1 &= \int v'_1\, dx = \int \sec x\, dx = \ln|\sec x + \tan x|\\
v_2 &= \int v'_2 dx = \int -1\, dx = -x\\
v_3 &= \int v'_3 dx = \int -\tan x\, dx = -\int \frac{\sin x}{\cos x} dx = \ln|\cos x|
\end{aligned}$$

Substituting these values into $y_p \Rightarrow$

$$y_p = \ln|\sec x + \tan x| - x\cos x + (\sin x)\ln|\cos x|$$

The general solution is therefore:

$$\begin{aligned}
y &= y_h + y_p\\
&= c_1 + c_2\cos x + c_3\sin x + \ln|\sec x + \tan x| - x\cos x + (\sin x)\ln|\cos x|
\end{aligned}$$

(1.2.2.3) the Euler-Cauchy Differential Equations:

So far, this section of the first chapter has been devoted to a study of linear differential equations with constant coefficients. However, there is one type of linear equation with ***variable*** coefficients which it is appropriate to discuss at this point because by a simple change of independent variable it can always be transformed into a linear equation with constant coefficients. This is the so-called ***Euler-Cauchy***$^{(1789\text{-}1857)}$ ***equation***.

$$a_o\, x^n\, y^{(n)} + a_1\, x^{n-1}\, y^{(n-1)} + \ldots\ldots + a_{n-1}\, x\, y' + a_n\, y = f(x)$$

in which the coefficient of each derivative is a constant multiple of the corresponding power of the independent variable. As we shall soon see, the change of the independent variable defined by

$$|x| = e^z \qquad or \qquad z = \ln|x| \qquad x \neq 0$$

will always convert this equation into a linear equation with constant coefficients.

Example: solve the D.E $x^3\, y''' + 2\, x^2\, y'' + x\, y' + y = 0$

Solution: $x = e^z \qquad z = \ln|x| \qquad \dfrac{dz}{dx} = \dfrac{1}{x}$,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\right)$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \frac{1}{x^2}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^2}\frac{d^3y}{dz^3}\frac{dz}{dx}$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \frac{1}{x^3}\frac{d^2y}{dz^2} - \frac{2}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^3}\frac{d^3y}{dz^3}$$

$$= \frac{2}{x^3}\frac{dy}{dz} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{1}{x^3}\frac{d^3y}{dz^3}$$

$$x^3\,y'''+2x^2\,y''+x\,y'+y=0$$

$$x^3\left(\frac{2}{x^3}\frac{dy}{dz}-\frac{3}{x^3}\frac{d^2y}{dz^2}+\frac{1}{x^3}\frac{d^3y}{dz^3}\right)+2x^2\left(-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}\right)+x\left(\frac{1}{x}\frac{dy}{dz}\right)+y=0$$

$$2\frac{dy}{dz}-3\frac{d^2y}{dz^2}+\frac{d^3y}{dz^3}-2\frac{dy}{dz}+2\frac{d^2y}{dz^2}-\frac{dy}{dz}+y=0$$

$$\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}-\frac{dy}{dz}+y=0$$

$$m^3-m^2-m+1=0$$

$$(m-1)(m^2-1)=0$$

$$m_1=1 \qquad m_{2,3}=\mp 1$$

$$y(z)=c_1e^z+c_2\,z\,e^z+c_3e^{-z}$$

$$z=\ln x \qquad x=e^z$$

$$y(x)=c_1x+c_2\,x\ln x+\frac{c_3}{x}$$

Leonhard Euler

(1707 – 1783)

Applications of Linear Differential Equations with constant coefficients:

Free Oscillation:

Static Case:

$$\sum fy=0$$

$$m\,g-k\,s_o=0$$

Dynamic Case:

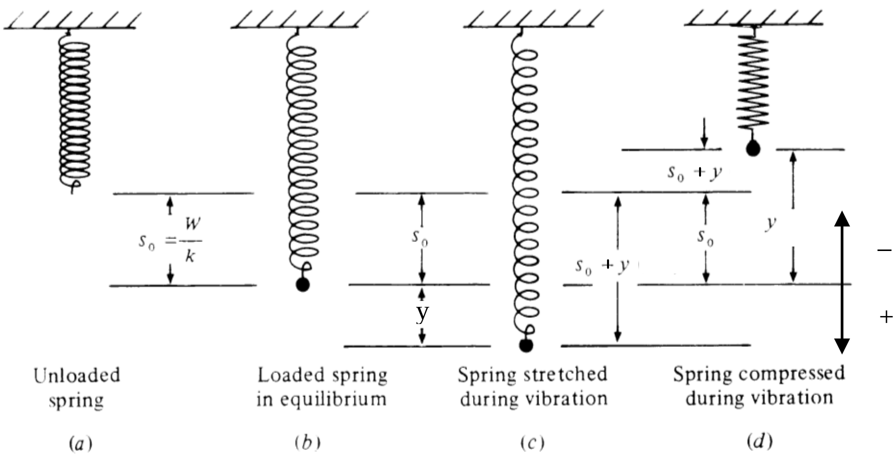
$$F=m\frac{d^2y}{dt^2}$$

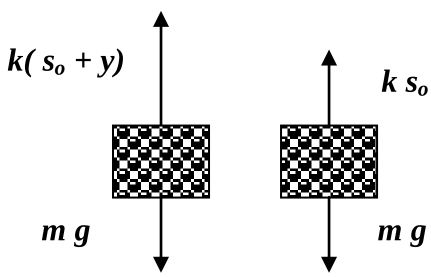
$$m\,g-k\,(s_o+y)=m\,y''$$

$$but\ m\,g=k\,s_o \quad \Rightarrow$$

$$-k\,y=m\,y'' \quad \Rightarrow$$

$$\boxed{m\,y''+k\,y=0}$$





$$\frac{d^2y}{dt^2}+\frac{k}{m}y=0 \;\Rightarrow\; \frac{d^2y}{dt^2}+\omega^2 y=0 \text{ where}$$

$$\omega^2=\frac{k}{m} \quad or \quad \omega=\sqrt{\frac{k}{m}}$$

$$m^2+\omega^2=0$$

$$m^2=-\omega^2$$

$$m=\mp\omega i$$

$$\boxed{y=A\cos\omega t+B\sin\omega t}$$

Case I (General case):

$$y(0)=y_o \quad \Leftrightarrow \quad y'(0)=0$$

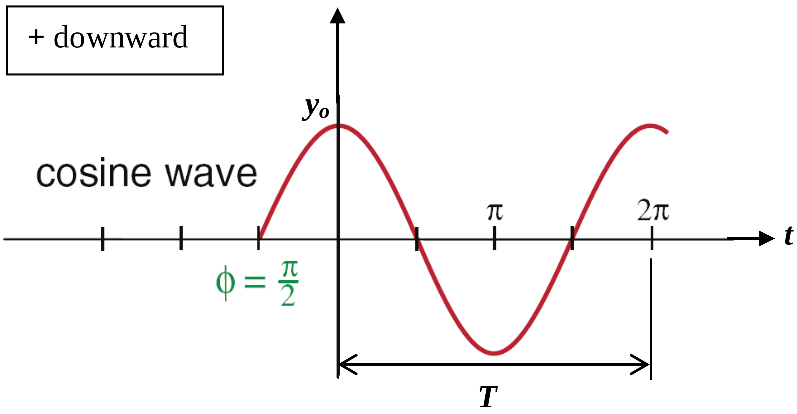
$$A=y_o \quad B=0 \quad \Rightarrow$$

$$\boxed{y=y_o\cos\omega t}$$

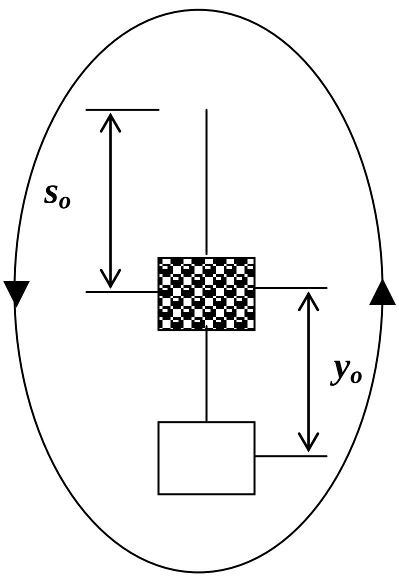
for a complete cycle

$$\omega t=2\pi \quad \Rightarrow \quad t=\frac{2\pi}{\omega}$$

$$\therefore T=\frac{2\pi}{\omega} \quad \text{cycle period}$$



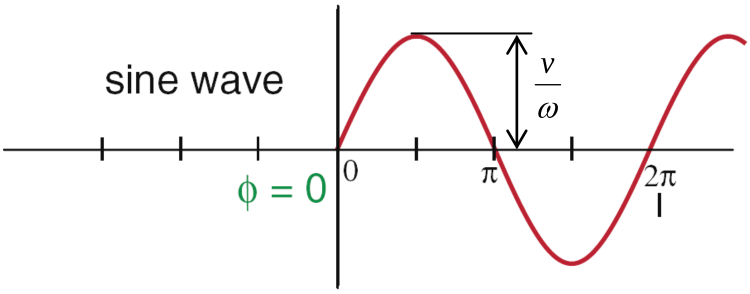
$$\boxed{\cos\phi=\cos(\phi+2\pi) \quad \Rightarrow \quad \cos\omega t=\cos(\omega t+2\pi)}$$



Complete Cycle

Case II:

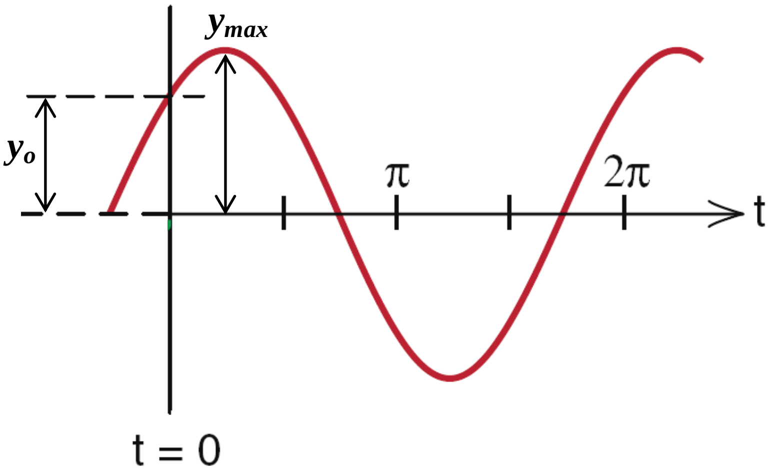
$$y(0) = 0 \quad \Leftrightarrow \quad y'(0) = v$$
$$0 = A + 0 \quad \Rightarrow \quad A = 0$$
$$\frac{dy}{dt} = \omega B \cos \omega t$$
$$\left.\frac{dy}{dt}\right|_{t=o} = \omega B = v$$
$$\boxed{y = \frac{v}{\omega} \sin \omega t}$$

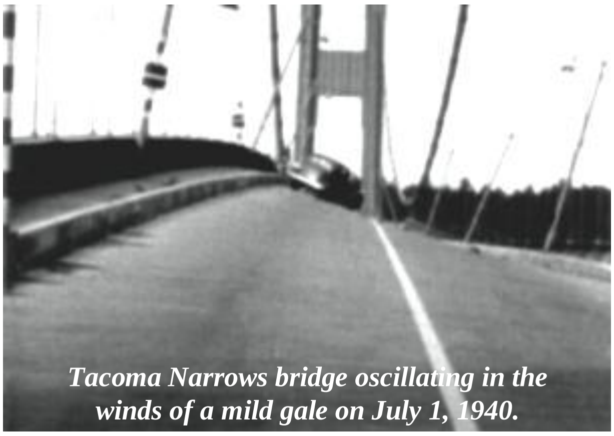


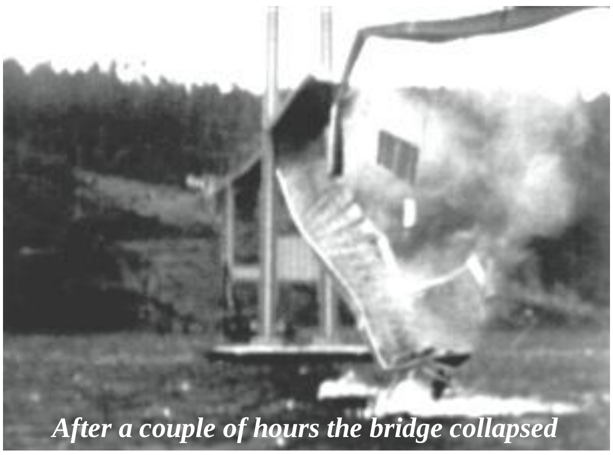
$$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T \propto \frac{1}{\omega}$$

Case III:

$$y(0) = y_o \quad y'(0) = v$$
$$y_o = A + 0 \quad \Rightarrow \quad A = y_o$$
$$\frac{dy}{dt} = -\omega A \sin \omega t + \omega B \cos \omega t$$
$$v = 0 + \omega B$$
$$B = \frac{v}{w}$$
$$\boxed{y = y_o \cos \omega t + \frac{v}{\omega} \sin \omega t}$$




Tacoma Narrows bridge oscillating in the winds of a mild gale on July 1, 1940.


After a couple of hours the bridge collapsed

Example: A weight of (***7 N***) is suspended from a spring of modulus (***k=36/35 N/cm***). At ***t = 0***, while the weight in static equilibrium it is given suddenly an initial velocity of (***48 cm/sec***) in downward.

a. Find the vertical displacement as a function of ***t***.
b. What are the period and frequency of motion?
c. Through what amplitude does the weight moves.
d. At what time does the load reach its extreme displacement above and below its equilibrium position?

Solution:

Heirrich Rudolf Hertz

(1857 – 1894)

A.

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{36/35}{980}}=12$$

$$y=A\cos 12t+B\sin 12t$$

$$y(0)=A+0 \quad\Rightarrow\quad A=0$$

$$y=B\sin 12t$$

$$y'=12\,B\cos 12t$$

$$y'\big|_{t=o}=12\,B \quad\Rightarrow\quad 48=12\,B \quad\Rightarrow\quad B=4$$

$$\boxed{y=4\sin 12t}$$

B.

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{12}=\frac{\pi}{6}\quad \sec/cycle$$

$$f=\frac{6}{\pi}\quad cycle/\sec\quad (Hertz)$$

C.

$$y=4\sin 12\,t \qquad -1\le \sin 12\,t\le 1$$

$$y_{\max}=4 \qquad y_{\min}=-4$$

$$Amplitude=4+\left|-4\right|=8$$

D.

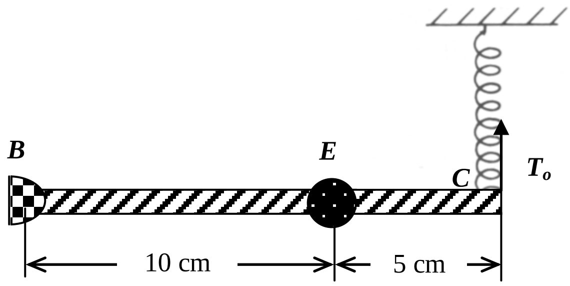
$$\sin 12\,t=\mp 1$$

$$12t=(2n+1)\frac{\pi}{2}\quad\Leftrightarrow\quad t=\left(\frac{1+2n}{24}\right)\pi \qquad n=0,1,2,\quad Multiplica\,tion\ of\ \frac{\pi}{2}$$

Example: the slender homogeneous rod ***BC*** shown below weighs (***36 N***) and the small (***20 N***) body ***E*** is welded to the rod. The spring has a modulus of (***12.25 N/cm***). When it is in equilibrium the rod is displaced an angle of ***(8°)*** clockwise and released from rest. Determine the natural frequency and the maximum velocity of vibration?

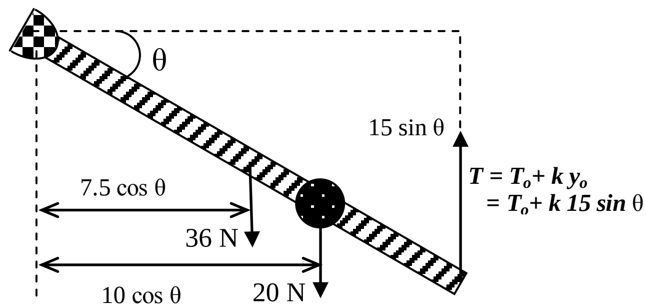
Solution:

Equilibrium phase



$$\sum M_B = 0$$
$$T_o \times 15 = 20 \times 10 + 36 \times 7.5$$
$$T_o = 31.33\,N$$

Vibration phase (Motion)



$$T = T_o + k \times 15\sin\theta$$
$$T = 31.33 + 12.25 \times 15\sin\theta$$
$$T = 31.33 + 183.75\sin\theta$$
$$\sum M_B = I_B \times \theta^{\bullet\bullet}$$

$$where \quad I_B = moment\ of\ inertia = \frac{1}{3} M\,L^2$$

$$\theta^{\bullet\bullet} = \text{second}\ derivative\ of\ the\ angle\ \frac{d^2\theta}{dt^2}$$

$$[36(7.5\cos\theta) + 20(10\cos\theta) - (31.33 + 183.75\sin\theta)15\cos\theta] \times \frac{1\,m}{100\,cm} = \hookleftarrow$$

$$\left[\frac{1}{3} \times \frac{36}{9.8}(15)^2 + \frac{20}{9.8}(10)^2\right]\theta^{\bullet\bullet} \times \frac{1\,m^2}{10000\,cm^2}$$

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1$$

$$270 + 200 - 470 - 2756\theta = 4.8\,\theta^{\bullet\bullet}$$
$$4.8\,\theta^{\bullet\bullet} + 2756\theta = 0$$
$$\theta^{\bullet\bullet} + 574\theta = 0$$
$$m^2 + 574 = 0 \quad \Rightarrow \quad m_{1,2} = \mp\sqrt{574}\,i$$
$$\theta = A\cos\sqrt{574}\,t + B\sin\sqrt{574}\,t$$

$$T = \frac{2\pi}{\sqrt{574}} = cycle\ \ period \quad \Leftrightarrow \quad f = \frac{\sqrt{574}}{2\pi} = frequancy$$

$$at \quad t=0 \quad \Rightarrow \quad \theta = 8\times\frac{\pi}{180}$$

$$\frac{8\pi}{180} = A\times 1 + 0 \quad \Rightarrow \quad A = \frac{8\pi}{180}$$

$$at \quad t=0 \quad \Rightarrow \quad \theta^{\bullet} = 0$$

$$0 = 0 + B(\cos\sqrt{574}\times 0) \quad \Rightarrow \quad B = 0$$

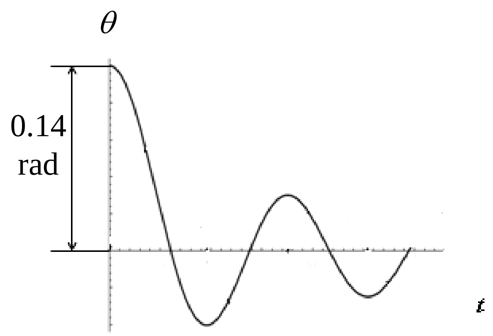
$$\theta = \frac{8\pi}{180}\cos\sqrt{574}\,t + 0$$

$$\theta = 0.14\cos\sqrt{574}\,t$$

$$\theta^{\bullet} = -0.14\sin\sqrt{574}\,t\times\sqrt{574}$$

$$\theta^{\bullet} = -3.36\sin\sqrt{574}\,t$$

$$\theta^{\bullet}_{MAX} = \mp 3.36\,\frac{rad}{\sec}$$



Hint

$$I_B = I_o + M\,d^2$$

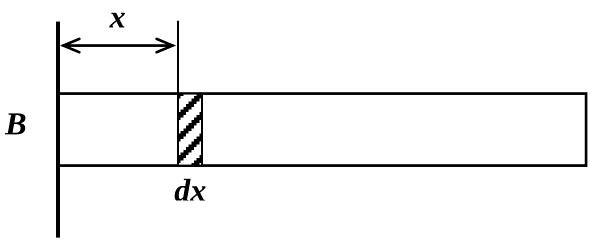
$$I_o \approx \quad too\ small\ (negligible)$$

$$I_B = \int_0^L \underline{A\,dx\,\rho}\,x^2 = A\rho\left[\frac{x^3}{3}\right]_0^L$$

$$I_B = A\rho\frac{L^3}{3}$$

$$I_B = \underline{A\,L\,\rho}\,\frac{L^2}{3}$$

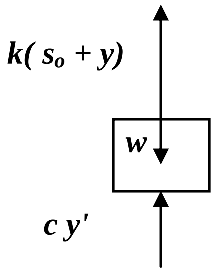
$$I_B = M\,\frac{L^2}{3}$$

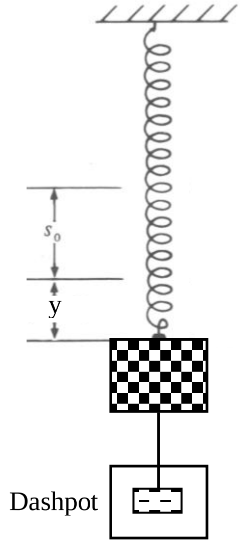


$$\boxed{d\,I_B = \underline{A\,dx\,\rho}\,x^2}$$

Damped Oscilation:

Damping $\propto \dfrac{dy}{dt} = c\,y'$





$$\oplus \downarrow \sum f\,y = M\,y''$$
$$w - k\,(s_o + y) - c\,y' = M\,y''$$
$$k\,y + M\,y'' + cy' = 0$$

$\boxed{M\,y'' + c\,y' + k\,y = 0}$ Arithmetic Model

$M\,m^2 + c\,m + k = 0$ Characteristic Equation

$$m_{1,2} = \frac{-c \mp \sqrt{c^2 - 4M\,k}}{2M} = \frac{-c}{2M} \mp \frac{1}{2M}\sqrt{c^2 - 4M\,k}$$
$$m_{1,2} = -\alpha \mp \beta$$
$$\alpha = \frac{c}{2M} \qquad \beta = \frac{1}{2M}\sqrt{c^2 - 4M\,k}$$

Critical Damping Coefficient:

The Critical Damping Coefficient is the value of ***c*** which makes:

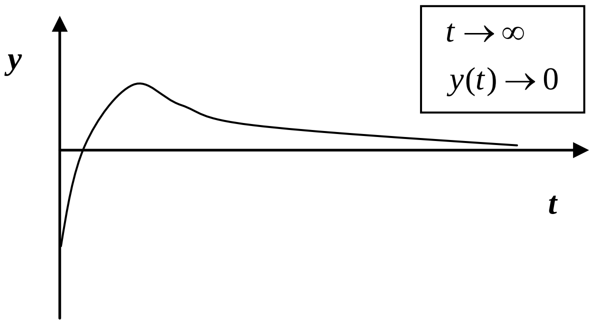
$$\sqrt{c^2 - 4M\,k} = 0$$

$$\boxed{C_{cr} = 2\sqrt{k\,M}}$$


Tacoma Narrows bridge in an n = 2 second harmonic standing wave pattern.

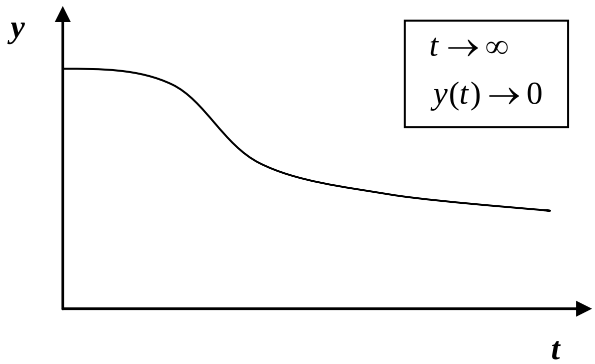
Case 1: $c \succ C_{cr}$ i.e $\sqrt{c^2 - 4Mk} \succ 0 \Rightarrow$ *Over Damping* $\Leftrightarrow \alpha$ *and* β *are real*

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$



Case 2: $c = C_{cr}$ i.e $m_1 = m_2 = -\alpha \Rightarrow$ *Ciritical Damping*

$$y(t) = (c_1 + c_2\, t)\, e^{-\alpha t}$$



Case 3: $c \prec C_{cr} \Leftrightarrow$ *Under Damping*

$$m_{1,2} = \frac{-c}{2M} \mp \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

$$= -\alpha \mp \beta$$

$$\alpha = \frac{c}{2M} \quad \beta = \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

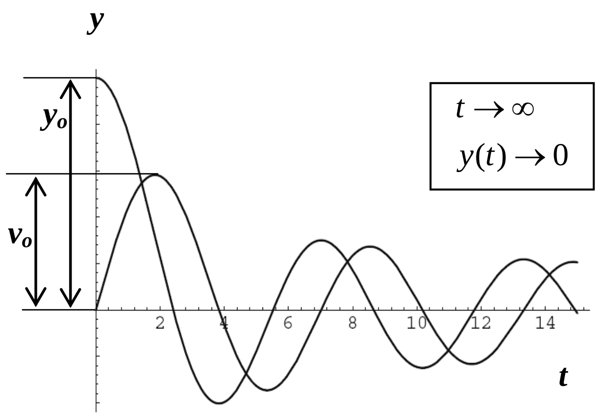
$$= \frac{1}{2M}\sqrt{4Mk - c^2} \times i$$

$$= \omega^* i$$

$$m_{1,2} = -\alpha \mp \omega^* i$$

$$y(t) = e^{pt}(A\cos qt + B\sin qt)$$

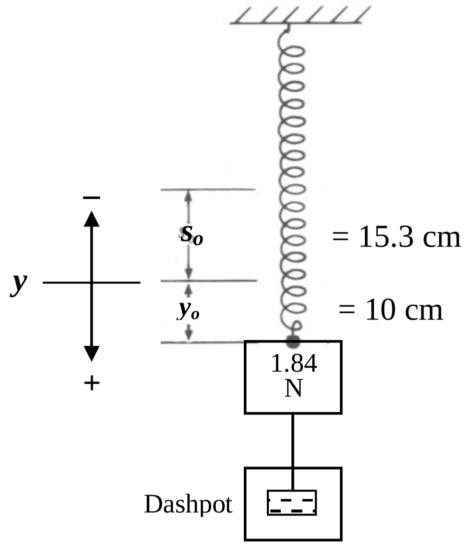
$$y = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$



Example: A ***(1.84 N)*** body is suspended by a spring which is stretched ***(15.3 cm)*** when it is loaded. If the body is drawn down ***(10 cm)*** from the position of equilibrium; find the position of the spring as a function of time ***(t)*** if:

1. c = 1.5
2. c = 3.75
3. c = 3

Solution:



$$M\,y''+c\,y'+k\,y=0$$

$$M=\frac{w}{g}=\frac{1.84}{9.8}=0.188\ kg$$

$$k=\frac{w}{s_o}=\frac{1.84}{0.153}=12\ \frac{N}{m}$$

1) for c = 1.5

$$0.188\,y''+1.5\,y'+12\,y=0$$

$$y''+8\,y'+64\,y=0$$

$$m^2+8m+64=0$$

$$m_{1,2}=\frac{-8\mp\sqrt{64-4\times 64}}{2}=-4\mp 4\sqrt{3}\,i \quad \Leftrightarrow \quad \textit{Under Damping}$$

$$y(t)=e^{-4t}(A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t)$$

$\underline{\textit{Initial Conditions}}$:

$$at\quad t=0 \quad\Rightarrow\quad y'=o \quad\Rightarrow y=0.1m$$

$$y(0)=A+0=0.1 \quad\Rightarrow\quad A=0.1$$

$$y'(t)=\left[-4\sqrt{3}\ A\sin 4\sqrt{3}\,t+4\sqrt{3}\ B\cos 4\sqrt{3}\,t\right]+\left[A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t\right](-4e^{-4t})$$

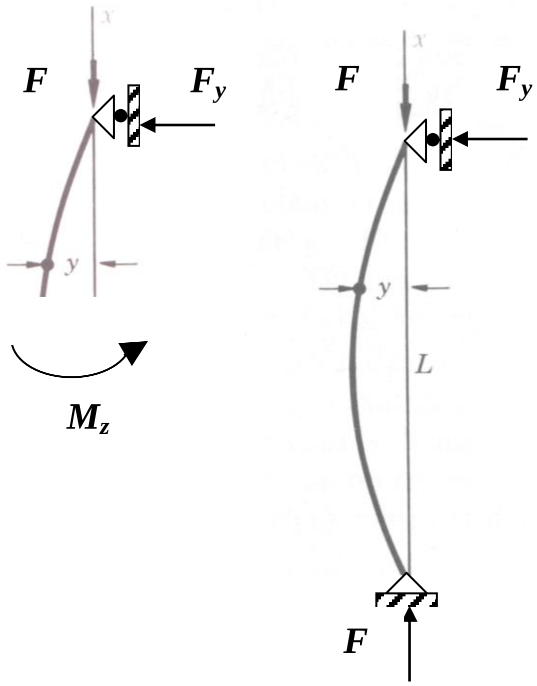
$$y'(0)=4\sqrt{3}\ B-4A=4\sqrt{3}\ B-4\times 0.1=0 \quad\Rightarrow\quad B=\frac{1}{10\sqrt{3}}$$

$$y=e^{-4t}(0.1\cos 4\sqrt{3}\,t+\frac{1}{10\sqrt{3}}\sin 4\sqrt{3}\,t)$$

2) for c = 3.75

3) for c = 3

Column Buckling:



$$\sum f_y = 0 \quad \Rightarrow \quad F_y = 0$$

$$M = -F\,y$$

$$\frac{d^2 y}{d x^2} = \frac{M}{E\,I}$$

$$y'' = -\frac{F}{E\,I}\,y \quad \Rightarrow \quad y'' + \frac{F}{E\,I}\,y = 0$$

$$m^2 + \frac{F}{E\,I} = 0 \quad \Rightarrow \quad m^2 = -\frac{F}{E\,I}$$

$$m_{1,2} = \mp\sqrt{\frac{F}{E\,I}}\;i$$

$$y = A\cos\sqrt{\frac{F}{E\,I}}\,x + B\sin\sqrt{\frac{F}{E\,I}}\,x$$

$$at \quad x = 0 \qquad y = 0$$

$$0 = A + B \times 0 \quad \Rightarrow \quad A = 0$$

$$\therefore\; y = B\sin\sqrt{\frac{F}{E\,I}}\,x$$

$$at \quad x = L \qquad y = o \quad also$$

but this means that $B = 0$ *which results in zero equation. However,*

If the load F *have just the right value to make:*

$\sqrt{\frac{F}{E\,I}}\,L = n\,\pi$ *then the last equation will be satisfied without* B *being* 0 *and*

the equilibrium is possible in a deflected position defined by:

$$\sqrt{\frac{F}{E\,I}} = \frac{n\,\pi}{L} \qquad n = 1, 2, 3, \;\ldots\ldots\ldots \quad \Rightarrow \quad y = B\sin\frac{n\,\pi\,x}{L}$$

$$\sqrt{\frac{F_n}{E\,I}} = \frac{n\,\pi}{L} \quad \Rightarrow \quad \frac{F_n}{E\,I} = \left(\frac{n\pi}{L}\right)^2 \quad \Rightarrow \quad F_n = \left(\frac{n\pi}{L}\right)^2 E\,I$$

$$for\; n = 1 \quad \Rightarrow \quad F_1 = \frac{\pi^2\,E\,I}{L^2}$$

For values of F below the lowest critical load, the column will remain in its undeflected vertical position, or if displaced slightly from it, will return to it as an equilibrium configuration. For values of F above the lowest critical load and different from the higher critical loads, the column can theoretically remain in a vertical position, but the equilibrium is unstable, and if the column is deflected slightly, it will not return to a vertical position but will continue to deflect until it collapses. Thus, only the lowest critical load is of much practical significance.