Chapter 3

Simultaneous Linear Differential Equations

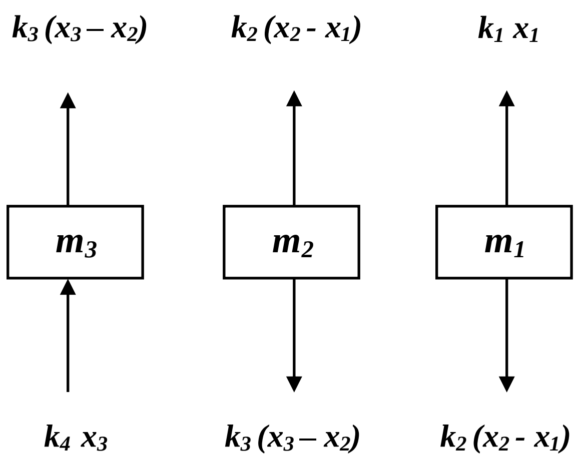
If two or more dependant variables are function of a single independent variable, the equation involving their derivatives is called simultaneous D.E.

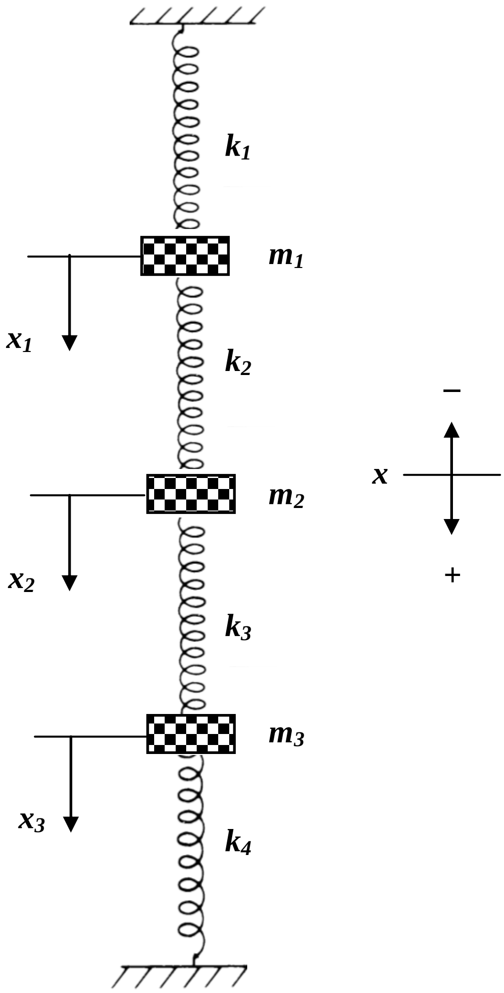
$$\left.\begin{aligned} x+y&=5 \\ 2x-y&=10 \end{aligned}\right]\quad \textit{simulteneous equations}$$

$$\left.\begin{aligned} \frac{dx}{dt}+4y&=t \\ \frac{dy}{dt}+2x&=e^{t} \end{aligned}\right]\quad \textit{simulteneous differential equations}$$

Newton's second law module:

$$F=ma=m\frac{d^2y}{dt^2}$$





Note: *The force of gravity (weight) can be neglected provided we also neglect the initial elongation of the springs and assume that each is unstretched when the system is in equilibrium.*

$$k_2(x_2-x_1)-k_1x_1=m_1\frac{d^2x_1}{dt^2}$$

$$k_3(x_3-x_2)-k_2(x_2-x_1)=m_2\frac{d^2x_2}{dt^2}$$

$$-k_4x_3-k_3(x_3-x_2)=m_3\frac{d^2x_3}{dt^2}$$

Methods of solution:

1- Elimination of dependant variables by differentiation.
2- Elimination of dependant variables using operator equation.
3- Solution by Cramer rule.

First Method: Elimination of dependant variables by differentiation

Example:

$$\frac{dx}{dt}+3\frac{dy}{dt}+y=e^{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{dy}{dt}-x=y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$from\ eq.(2)$$

$$x=\frac{dy}{dt}-y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\frac{dx}{dt}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$

$$substitute\ in\ eq.(1)$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+3\frac{dy}{dt}+y=e^{t}$$

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=e^{t}$$

$$y=c_1e^{-t}+c_2te^{-t}+\frac{1}{4}e^{t}$$

$$from\ eq.(3)$$

$$x=\frac{dy}{dt}-y=\frac{d}{dt}\left[c_1e^{-t}+c_2te^{-t}+\frac{1}{4}e^{t}\right]-\left[c_1e^{-t}+c_2te^{-t}+\frac{1}{4}e^{t}\right]$$

Second Method: Elimination of dependant variables using operator equation

Example:

$$\frac{d^2x}{dt^2}+\frac{dy}{dt}+x=y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{d^2y}{dt^2}+3\frac{dx}{dt}+2y=x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$D^2x+Dy+x-y=0$$
$$(D^2+1)x+(D-1)y=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$
$$D^2y+3Dx+2y-x=0$$
$$(D^2+2)y+(3D-1)x=0 \quad \ldots\ldots\ldots\ldots \quad (4)$$

To Eliminate y

$$eq.(3)\times(D^2+2)-eq.(4)\times(D-1) \quad \Rightarrow$$

$$(D^2+2)(D^2+1)x+(D^2+2)(D-1)y=0$$
$$-\underline{(D-1)(3D-1)x-(D^2+2)(D-1)y=0}$$

$$(D^2+2)(D^2+1)x-(D-1)(3D-1)x=0$$
$$(D^4+3D^2+2)x-(3D^2-4D+1)x=0$$
$$(D^4+4D+1)x=0 \quad where\ x=f(t)$$

solve for x

and using the same approach solve for y

Third Method: Solution by Cramer rule

Determinant:

For 2nd order matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\begin{vmatrix} (+) & & & \\ & \ddots & & \\ & & a_{11} & & a_{12} \\ & & & \ddots & \\ & & a_{21} & & a_{22} \\ & \therefore & & & \\ (-) & & & & \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

For 3rd order matrix:

A general third-order determinant can be expanded using the former equations:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,[a_{22}\, a_{33} - a_{23}\, a_{32}] - a_{12}\,[a_{21}\, a_{33} - a_{23}\, a_{31}] + a_{13}\,[a_{21}\, a_{32} - a_{22}\, a_{31}]$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31} - a_{11}\, a_{23}\, a_{32} - a_{12} a_{21}\, a_{33}$$

The expansion can also be obtained by diagonal multiplication, by repeating on the right the first two columns of the determinate and then adding the signed products of the elements on the various diagonals in the resulting array:

$$
\left|\begin{array}{ccccccc|cccc}
\ddots & (+) & \ddots & (+) & \ddots & (+) & & & & & \\
& \ddots & & \ddots & & \ddots & & & & & \\
& & a_{11} & & a_{12} & & a_{13} & & a_{11} & & a_{12} \\
& & & \ddots & & \ddots & & \ddots & & \iddots & \\
& & a_{21} & & a_{22} & & a_{23} & & a_{21} & & a_{22} \\
& & & \iddots & & \ddots & & \ddots & & \ddots & \\
& & a_{31} & & a_{32} & & a_{33} & & a_{31} & & a_{32} \\
& \iddots & & \iddots & & \iddots & & & & & \\
\iddots & (-) & \iddots & (-) & \iddots & (-) & & & & &
\end{array}\right.
$$

Cramer Rule:

Second Order:

$$
\left.\begin{matrix} a_1\,x + b_1\,y = d_1 \\ a_2\,x + b_2\,y = d_2 \end{matrix}\right] \Rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \Leftrightarrow a_{(i,j)}\; x_{(i)} = d_{(i)}
$$

$$
|D_s| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \;:\; |D_x| = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix} \;:\; |D_y| = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}
$$

$$
x = \frac{|D_x|}{|D_s|} \;:\; y = \frac{|D_y|}{|D_s|}
$$

Third Order:

$$
\left.\begin{matrix} a_1\,x + b_1\,y + c_1\,z = d_1 \\ a_2\,x + b_2\,y + c_2\,z = d_2 \\ a_3\,x + b_3\,y + c_3\,z = d_3 \end{matrix}\right] \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Leftrightarrow a_{(i,j)}\; x_{(i)} = d_{(i)}
$$

$$
|D_s| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \;:\; |D_x| = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \;:\; |D_y| = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \;:\; |D_z| = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}
$$

$$
x = \frac{|D_x|}{|D_s|} \;:\; y = \frac{|D_y|}{|D_s|} \;:\; z = \frac{|D_z|}{|D_s|}
$$

Example:

$$\frac{dx}{dt}+3\frac{dy}{dt}+y=e^{t} \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

$$\frac{dy}{dt}-x=y \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (2)$$

$$Dx+3Dy+y=e^{t}$$
$$Dy-x-y=0$$

$$Dx+(3D+1)y=e^{t} \quad \dots\dots\dots\dots\dots\dots\dots \quad (3)$$
$$-x+(D-1)y=0 \quad \dots\dots\dots\dots\dots\dots\dots \quad (4)$$

$$|Ds|=\begin{vmatrix} D & 3D+1 \\ -1 & D-1 \end{vmatrix} \quad : \quad |Dx|=\begin{vmatrix} e^{t} & 3D+1 \\ 0 & D-1 \end{vmatrix} \quad : \quad |Dy|=\begin{vmatrix} D & e^{t} \\ -1 & 0 \end{vmatrix}$$

$$|Ds|=D^{2}-D+3D+1=D^{2}+2D+1$$
$$|Dx|=(D-1)e^{t}-0=e^{t}-e^{t}=0$$
$$|Dy|=0+e^{t}=e^{t}$$

$$y=\frac{|Dy|}{|Ds|} \quad \Rightarrow \quad (Ds)y=Dy \quad \Rightarrow \quad (D^{2}+2D+1)y=e^{t}$$

$$\therefore \quad y=c_{1}e^{-t}+c_{2}\,t\,e^{-t}+\frac{e^{t}}{4}$$

$$x=\frac{|Dx|}{|Ds|} \quad \Rightarrow \quad (Ds)x=Dx \quad \Rightarrow \quad (D^{2}+2D+1)x=0$$

$$\therefore \quad x=$$

Example:

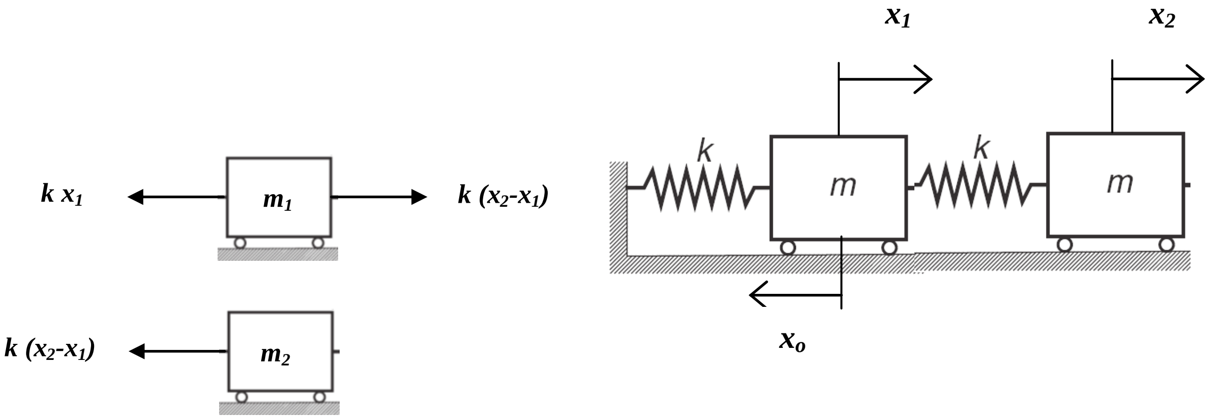
$$\frac{dx}{dt} - 3x - 6y = t^2$$
$$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$$

$$(D-3)x - 6y = t^2$$
$$Dx + (D-3)y = e^t$$

$$|Ds| = \begin{vmatrix} D-3 & -6 \\ D & D-3 \end{vmatrix} = (D-3)^2 + 6D = D^2 - 6D + 9 + 6D = D^2 + 9$$
$$|Dx| = \begin{vmatrix} t^2 & -6 \\ e^t & D-3 \end{vmatrix} = (D-3)t^2 + 6e^t = 2t - 3t^2 + 6e^t$$
$$|Dy| = \begin{vmatrix} D-3 & t^2 \\ D & e^t \end{vmatrix} = (D-3)e^t - Dt^2 = -2e^t - 2t$$
$$y = \frac{|Dy|}{|Ds|} \quad \Rightarrow \quad |Ds|\, y = |Dy|$$
$$(D^2 + 9)y = -2e^t - 2t$$
$$m^2 = -9$$
$$m_{1,2} = \mp 3i$$
$$y_h = (A\cos 3t + B\sin 3t)$$
$$y_p = -\frac{1}{5}e^t - \frac{2}{9}t$$
$$y = A\cos 3t + B\sin 3t - \frac{1}{5}e^t - \frac{2}{9}t$$
and
$$x = \frac{|Dx|}{|Ds|} \quad \Rightarrow \quad |Ds|\, x = |Dx|$$
$$(D^2 + 9)x = 2t - 3t^2 + 6e^t$$
$$m^2 + 9 = 0 \quad \Rightarrow \quad m_{1,2} = \mp 3i$$
$$y_h = A\cos 3t + B\sin 3t$$

find y_p
$$y = y_h + y_p$$

Application Example: Assuming the friction to be neglected, find the position of masses m_1 , m_2 of the system shown below w.r.t. time. And find the frequency and position if the motion is to right?
If k = constant, $m_1 = m_2$ and at $t=0$, $x_1 = -x_o$ (to left).



$$\sum fx_1 = k\,(x_2 - x_1) - k\,x_1 = k\,(x_2 - 2x_1)$$

$$m\frac{d^2x_1}{dt^2} = k\,(x_2 - 2x_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\sum fx_2 = -k\,(x_2 - x_1)$$

$$m\frac{d^2x_2}{dt^2} = -k\,(x_2 - x_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$m\,x_1'' = k\,(x_2 - 2x_1)$$
$$m\,x_2'' = -k\,(x_2 - x_1)$$

$$Let \quad \frac{k}{m} = \omega^2 \quad \Rightarrow$$

$$\left.\begin{aligned} (D^2 + 2\omega^2)\,x_1 - \omega^2\,x_2 &= 0 \\ -\omega^2\,x_1 + (D^2 + \omega^2)\,x_2 &= 0 \end{aligned}\right] \quad Ommiting\ x_2$$

$$(D^2 + \omega^2)(D^2 + 2\omega^2)\,x_1 - \omega^2\,(D^2 + \omega^2)\,x_2 = 0$$
$$-\omega^4\,x_1 + \omega^2\,(D^2 + \omega^2)\,x_2 = 0$$

$$(D^4 + 3\omega^2 D^2 + 2\omega^4)\,x_1 - \omega^4\,x_1 = 0$$
$$(D^4 + 3\omega^2 D^2 + \omega^4)\,x_1 = 0$$

$m^4 + 3\omega^2 m^2 + \omega^4 = 0$

$m^2 = (-3 \mp \sqrt{5})\dfrac{\omega^2}{2}$

$m^2 = -0.38\,\omega^2 \quad , \quad -2.62\,\omega^2$

$m_{1,2} = \mp 0.62\,\omega i \quad , \quad m_{3,4} = \mp 1.62\,\omega i$

$x_1 = A_1 \cos 0.62\,\omega t + B_1 \sin 0.62\,\omega t + A_2 \cos 1.62\,\omega t + B_2 \sin 1.62\,\omega t$

$$at\ t = 0 \quad \Leftrightarrow \quad x_1 = -x_o \quad and \quad x_2 = 0 \quad \Rightarrow \frac{dx_1}{dt} = 0 \quad \frac{dx_2}{dt} = 0$$

$A_1 = -0.28\,x_o \quad A_2 = -0.72\,x_o \quad B_1 = B_2 = 0$

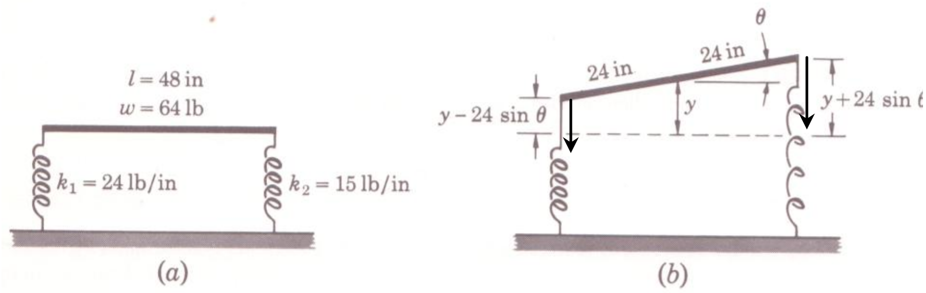
$x_1 = -x_o(0.28 \cos 0.62\,\omega t + 0.72 \cos 1.62\,\omega t)$

$$T = \frac{2\,\pi}{0.62\,\omega} \qquad f_1 = \frac{0.62\,\omega}{2\,\pi}$$

$$f_2 = \frac{1.62\,\omega}{2\,\pi}$$

Example: A uniform bar 4 ft long and weighing 16 lb/ft is supported as shown in fig. on springs of modulus 24 and 15 lb/in, respectively. If the springs are guided so that only vertical displacement of the center of the bar is possible, and if friction is neglected, find the (natural) frequencies at which the system would begin to vibrate if disturbed slightly from its equilibrium position?

Solution:



Assuming $\oplus\uparrow$ *and counterclockwise positive moment*

From the Fig. it is clear that the instantaneous deflections of the left- and the right-hand springs are, respectively:

$y-24\sin\theta \quad : \quad y+24\sin\theta$

or, if we make the usual small-angle approximation $\sin\theta=\theta$,

$y-24\theta \quad : \quad y+24\theta$

Hence the unbalanced forces which the springs apply to the ends of the bar are,

$-24\,(y-24\theta) \quad : \quad -15\,(y+24\theta)$

Newton's law applied to the translation of the center of gravity of the bar therefore gives the equation,

$$m\frac{d^2y}{dt^2}=\sum fy$$

$$\frac{64}{384}\frac{d^2y}{dt^2}=-24(y-24\theta)-15(y+24\theta)$$

or

$$\frac{d^2y}{dt^2}+234\,y-1296\theta=0 \quad \text{...} \quad (1)$$

Applying Newton's second law in tortional form to the rotation of the bar about its center of gravity, we have (using the fact that the moment of inertia of a uniform bar of length ***l*** about its midpoint is $\frac{1}{12}ml^2$,

$$I_{mid}\,\theta^{\bullet\bullet}=\sum M_{mid}$$

$$\frac{64}{384}\frac{(48)^2}{12}\theta^{\bullet\bullet}=24\left[24(y-24\theta)\right]-24\left[15(y+24\theta)\right]$$

or

$$\frac{d^2\theta}{dt^2}-\frac{27}{4}y+702\theta=0 \quad \text{...} \quad (2)$$

Since we are only asked to find the natural frequencies of the mechanical configuration, there is no need to actually solve the simultaneous D.E's (1) and (2). For the required frequencies are completely determined by the corresponding characteristics equation:

$$\begin{vmatrix} m^2+234 & -1296 \\ -\frac{27}{4} & m^2+702 \end{vmatrix}=m^4+936m^2+155520=(m^2+216)(m^2+720)=0$$

The roots of the equation are $\pm 6\sqrt{6}\, i$ *and* $\pm 12\sqrt{5}\, i$.

The components $\boldsymbol{y}$ and $\boldsymbol{\theta}$ corresponding to the characteristic numbers $\pm 6\sqrt{6}\, i$ are both periodic functions which oscillate with frequency $w_1 = 6\sqrt{6}\ rad/\sec$. Likewise, The components corresponding to the characteristic numbers are oscillatory function with frequency $w_2 = 12\sqrt{5}\, rad/\sec$. thus, $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ are the natural frequencies of the mechanical system. Under the assumption of small-amplitude vibration. Converted from ***radians per second***, they are:

$$f_1 = \frac{6\sqrt{6}}{2\pi} = 2.34\ Hz \quad and \quad f_1 = \frac{12\sqrt{5}}{2\pi} = 4.27\ Hz$$

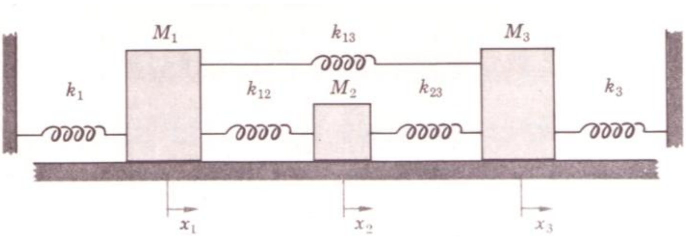
Example: Let as consider the mass-spring system shown in the Fig. below, determine the forces which act on each of the masses as a result of the arbitrary displacements $\boldsymbol{x_1}$**,** $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ of the respective masses?

Solution:

Taking into considration that,

$f = \oplus$ If the spring is stretched.

$f = -$ If the spring is compressed.



And taking the positive direction of $\boldsymbol{x}$ as the base for the system movement and applying it to the forces we have:

Spring modulus	Change in the spring length	Force
$\boldsymbol{k_1}$	$\boldsymbol{X_1}$	$\boldsymbol{k_1\, x_1}$
$\boldsymbol{k_{12}}$	$\boldsymbol{x_2 - x_1}$	$\boldsymbol{k_{12}\,(x_2 - x_1)}$
$\boldsymbol{k_{13}}$	$\boldsymbol{x_3 - x_1}$	$\boldsymbol{k_{13}\,(x_3 - x_1)}$
$\boldsymbol{k_{23}}$	$\boldsymbol{x_3 - x_2}$	$\boldsymbol{k_{23}\,(x_3 - x_2)}$
$\boldsymbol{k_3}$	$\boldsymbol{- x_3}$	$\boldsymbol{k_3\,(- x_3)}$

$$f_1 = -k_1\, x_1 + k_{12}(x_2 - x_1) + k_{13}(x_3 - x_1)$$
$$f_2 = -k_{12}(x_2 - x_1) + k_{23}(x_3 - x_2)$$
$$f_3 = k_{13}(x_3 - x_1) - k_{23}(x_3 - x_2) - k_3\, x_3$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad k = \begin{bmatrix} -(k_1 + k_{12} + k_{13}) & k_{12} & k_{13} \\ k_{12} & -(k_{12} + k_{23}) & k_{23} \\ k_{13} & k_{23} & -(k_{13} + k_{23} + k_3) \end{bmatrix}$$

$\boldsymbol{k}$ is usually called the stiffness matrix of the system.