

Chapter 4

Fourier series

1.1 Introduction:

One of the crowning glories of nineteenth century mathematics was the discovery of the infinite series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$



Jean Baptiste Joseph Fourier

1768 - 1830

Fourier series arose as a formal solution to the classic wave equation. Later on, it was used to describe physical processes in which events recur in a regular pattern. It is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument of nearly every recondite question in modern physics.

1.2 Periodic functions:

A function $f(x)$ is called periodic if it is defined for all real x and if there is some positive number p such that

$$f(x+p) = f(x) \quad \dots \quad (1) \quad \text{for all } x$$

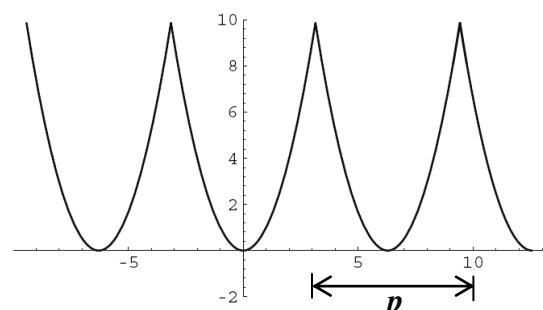
This number p is called a period of $f(x)$.

From **Eq. (1)** we have

$$f(x+2p) = f[(x+p)+p] = f(x+p) = f(x)$$

and for any Integer n

$$f(x+np) = f(x) \quad \text{for all } x$$



Periodic Function

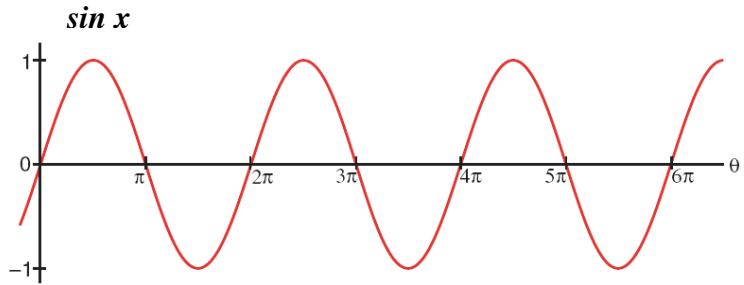
Hence $2p, 3p, 4p, \dots$ are also periods of $f(x)$.

Furthermore if $f(x)$ and $g(x)$ have period p , then the function

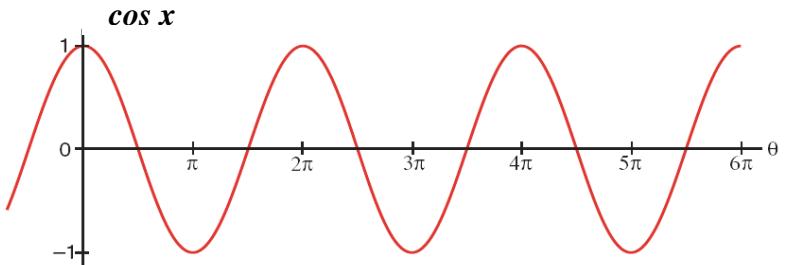
$$h(x) = af(x) + bg(x) \quad a, b \text{ constants.}$$

1.3 Trigonometric series:

The **sin** and **cos** functions have the period of 2π



Which will result in the following expansion



$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

where

$a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are real constants

Such a series is called a **trigonometric series**

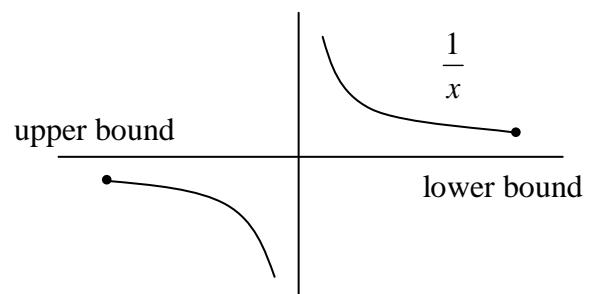
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where a_n, b_n are called the coefficients of the series.

1.4 Bounds of a Function:

If for all values of x in a given interval, $f(x)$ is never greater than some fixed number m , the number m is called the upper bound for $f(x)$. If $f(x)$ never less than some number m , then m is called the lower bound of the function.

If the function has upper and lower bound then the function is said to be bounded.



1.5 Continuity of a Function

If $f(c)$ exists then,

$$\lim_{x \rightarrow c} f(x) = f(c)$$

1.6 Fourier Series

Theorem 1

If $f(x)$ is a periodic function with a period ($2L$) and if $f(x)$ can be represented by a trigonometric series, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} dx$$

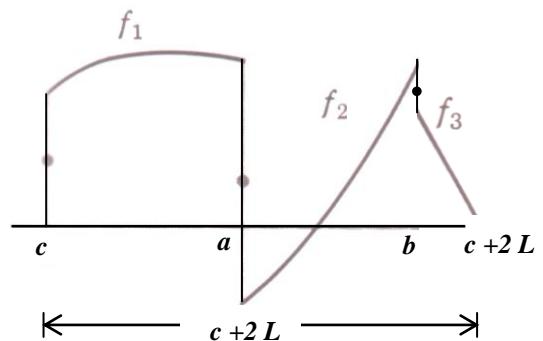
$$a_0 = \frac{1}{L} \int_c^{c+2L} f(x) dx$$

Where a_n, b_n, a_0 are **Euler's coefficients**. And $n = 1, 2, 3, \dots$

Theorem 2

If $f(x)$ is a bounded periodic function of period ($2L$) which in any one period has at most a finite number of max and min and finite number of points of discontinuity then the Fourier series of $f(x)$ converges to $f(x)$ at all points of where $f(x)$ is continuous also the series converges to the average value of the right and left hand limits of $f(x)$ at each points where $f(x)$ is discontinuous.

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx \Rightarrow$$



$$a_n = \frac{1}{L} \left[\int_c^a f_1(x) \cos \frac{n\pi x}{L} dx + \int_a^b f_2(x) \cos \frac{n\pi x}{L} dx + \int_b^{c+2L} f_3(x) \cos \frac{n\pi x}{L} dx \right]$$

Note: In Fourier series we will need certain integrals such us:

$$\int \sin mx \sin nx dx = \frac{1}{2} \int [\cos(m-n)x - \cos(m+n)x] dx$$

$$\int \cos mx \cos nx dx = \frac{1}{2} \int [\cos(m-n)x + \cos(m+n)x] dx$$

$$\int \sin mx \cos nx dx = \frac{1}{2} \int [\sin(m-n)x + \sin(m+n)x] dx$$

$$\int \sin^2 nx dx = \frac{1}{2} \int (1 - \cos 2nx) dx$$

$$\int \sin nx \cos nx dx = \frac{1}{2} \int \sin 2nx dx$$

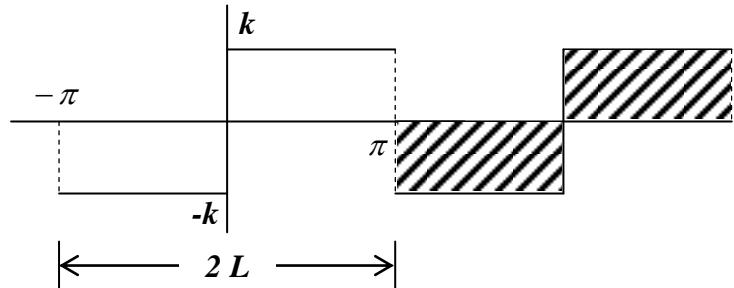
$$\int \cos^2 nx dx = \frac{1}{2} \int (1 + \cos 2nx) dx$$

Example: find Fourier series of the following periodic function $f(x)$?

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$$

$$\text{and } f(x+2\pi) = f(x)$$

Solution:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{L} \int_c^{c+2L} f(x) dx$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) dx + \int_0^\pi (k) dx \right] = \frac{1}{\pi} \left\{ [(-k)x]_{-\pi}^0 + [(k)x]_0^\pi \right\} = \frac{1}{\pi} [-k\pi + k\pi] = 0$$

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \cos \frac{n\pi x}{\pi} dx + \int_0^\pi (k) \cos \frac{n\pi x}{\pi} dx \right] = \frac{1}{\pi} \left\{ \left[-\frac{k}{n} \sin nx \right]_{-\pi}^0 + \left[\frac{k}{n} \sin nx \right]_0^\pi \right\} = 0$$

$$\begin{aligned}
 b_n &= \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} dx \\
 b_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \sin \frac{n\pi x}{\pi} dx + \int_0^\pi (k) \sin \frac{n\pi x}{\pi} dx \right] \\
 b_n &= \frac{1}{\pi} \left\{ \left[\frac{k}{n} \cos nx \right]_{-\pi}^0 + \left[\frac{-k}{n} \cos nx \right]_0^\pi \right\} = \frac{1}{\pi} \left\{ \left[\frac{k}{n} \cos 0 - \frac{k}{n} \cos(-n\pi) \right] + \left[\frac{-k}{n} \cos(n\pi) - \frac{-k}{n} \cos 0 \right] \right\}
 \end{aligned}$$

$$\because \cos(-\alpha) = \cos \alpha \Rightarrow \cos(-n\pi) = \cos n\pi$$

$$\therefore b_n = \frac{k}{n\pi} [(1 - \cos n\pi) + (-\cos n\pi + 1)] = \frac{k}{n\pi} (2 - 2\cos n\pi) = \frac{2k}{n\pi} (1 - \cos n\pi)$$

$$\cos n\pi = \begin{cases} -1 & n = \text{odd} \\ +1 & n = \text{even} \\ -1^{(n)} & \text{for all} \end{cases}$$

$$bn = \frac{2k}{n\pi} [1 - (-1^{(n)})]$$

$$n=1 \Rightarrow b_1 = \frac{4k}{\pi}$$

$$n=2 \Rightarrow b_2 = 0 \quad \text{same for } b_4, b_6, b_8, \dots = 0$$

$$n=3 \Rightarrow b_3 = \frac{4k}{3\pi}$$

$$n=5 \Rightarrow b_5 = \frac{4k}{5\pi}$$

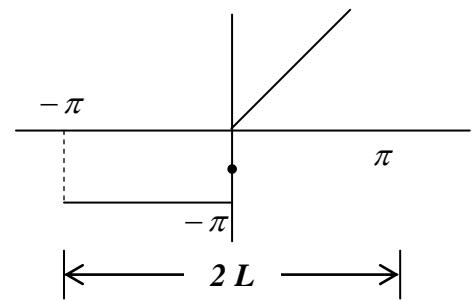
$$f(x) = \frac{4k}{\pi} \sin x + \frac{4k}{3\pi} \sin 3x + \frac{4k}{5\pi} \sin 5x + \dots$$

$$f(x) = \frac{2k}{n\pi} \sum_{n=1}^{\infty} [1 - (-1)^n] \sin nx$$

Example: Obtain the Fourier series of the periodic function defined by:

$$f(x) = \begin{cases} -\pi & \text{for } -\pi < x < 0 \\ x & \text{for } 0 < x < \pi \end{cases}$$

And prove that: $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$



Solution: $f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_o = \frac{1}{L} \int_c^{c+2L} f(x) dx$$

$$a_o = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi dx + \int_0^\pi x dx \right] = \frac{1}{\pi} \left\{ \left[-\pi x \right]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^\pi \right\} = \frac{1}{\pi} \left[(-\pi(0 + \pi)) + \left(\frac{\pi^2}{2} - 0 \right) \right] = -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) \cos \frac{n\pi x}{\pi} dx + \int_0^\pi (x) \cos \frac{n\pi x}{\pi} dx \right] = \frac{1}{\pi} \left\{ \left[\frac{-\pi}{n} \sin nx \right]_{-\pi}^0 + \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^\pi \right\}$$

$$a_n = \frac{1}{\pi} \left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right) = \frac{1}{n^2 \pi} (\cos n\pi - 1) = \begin{cases} 0 & n = \text{even} \\ -\frac{2}{n^2 \pi} & n = \text{odd} \end{cases}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) \sin \frac{n\pi x}{\pi} dx + \int_0^\pi (x) \sin \frac{n\pi x}{\pi} dx \right] = \frac{1}{\pi} \left\{ \left[\frac{\pi}{n} \cos nx \right]_{-\pi}^0 + \left[-x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi \right\}$$

$$b_n = \frac{1}{n\pi} \left[\pi \left[1 - \underline{\cos(-n\pi)} \right] - (\pi \cos n\pi - 0) \right] = \frac{1}{n} [1 - \cos n\pi - \cos n\pi] = \frac{1}{n} (1 - 2\cos n\pi) = \begin{cases} -\frac{1}{n} & n = \text{even} \\ \frac{3}{n} & n = \text{odd} \end{cases}$$

$$\cos(-\theta) = \cos(\theta)$$

$$\therefore a_o = -\frac{\pi}{2}$$

$$a_n = \begin{cases} 0 & n = \text{even} \\ -\frac{2}{n^2 \pi} & n = \text{odd} \end{cases}$$

$$b_n = \begin{cases} -\frac{1}{n} & n = \text{even} \\ \frac{3}{n} & n = \text{odd} \end{cases}$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots + \frac{\cos nx}{n^2} \right)$$

$$+ 3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \dots + \frac{(1 - 2 \cos n\pi)}{n} \sin nx$$

or $f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{\cos n\pi - 1}{n^2 \pi} \cos nx + \frac{(1 - 2 \cos n\pi)}{n} \sin nx \right)$

from the fig. above

$$\text{at } x=0 \quad f(x) \text{ converges to } \frac{(0 + (-\pi))}{2} = \frac{-\pi}{2} \Rightarrow$$

$$\frac{-\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) + 0$$

$$\frac{-\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{-\pi}{4} = -\frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \quad O.K$$

Example: What is the Fourier expansion of the periodic function whose definition in one period is?

$$f(t) = \begin{cases} 0 & -\pi < t \leq 0 \\ \sin t & 0 \leq t \leq \pi \end{cases}$$

Solution: $f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$

$$a_o = \frac{1}{L} \int_c^{c+2L} f(t) dt$$

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \left[\int_{-\pi}^0 (0) dt + \int_0^{\pi} \sin t dt \right]$$

$$a_o = \frac{1}{\pi} [-\cos t]_0^{\pi} = -\frac{1}{\pi} (\cos \pi - \cos 0) = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

$$a_n = \frac{1}{L} \int_c^{c+2L} f(t) \cos nt dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (0) \cos nt dt + \int_0^{\pi} \sin t \cos nt dt \right]$$

and $\because \sin t \cos nt = \frac{1}{2} [\sin(1-n)t + \sin(1+n)t]$ then

$$a_n = \frac{1}{2\pi} \left[\int_0^{\pi} (\sin(1-n)t + \sin(1+n)t) dt \right] = \frac{1}{2\pi} \left\{ \left[-\frac{\cos(1-n)t}{(1-n)} \right]_0^{\pi} + \left[-\frac{\cos(1+n)t}{(1+n)} \right]_0^{\pi} \right\}$$

$$a_n = \frac{-1}{2\pi} \left\{ \left[\frac{\cos(1-n)\pi}{(1-n)} \right]_0^{\pi} + \left[\frac{\cos(1+n)\pi}{(1+n)} \right]_0^{\pi} \right\} = \frac{-1}{2\pi} \left[\frac{\cos(1-n)\pi}{(1-n)} + \frac{\cos(1+n)\pi}{(1+n)} - \frac{1}{1-n} - \frac{1}{1+n} \right]$$

$$a_n = \frac{-1}{2\pi} \left[\frac{\cos(\pi - n\pi)}{(1-n)} + \frac{\cos(\pi + n\pi)}{(1+n)} - \frac{1}{1-n} - \frac{1}{1+n} \right] = \frac{-1}{2\pi} \left[\frac{-\cos n\pi}{(1-n)} + \frac{-\cos n\pi}{(1+n)} - \frac{1}{1-n} - \frac{1}{1+n} \right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{\cos n\pi}{(1-n)} + \frac{\cos n\pi}{(1+n)} + \frac{1}{1-n} + \frac{1}{1+n} \right] = \frac{1}{2\pi} \left[\frac{(1+n)\cos n\pi + (1-n)\cos n\pi + (1+n) + (1-n)}{(1-n)(1+n)} \right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{\cos n\pi + n\cos n\pi + \cos n\pi - n\cos n\pi + 1 + n + 1 - n}{1 - n^2} \right] = \frac{2\cos n\pi + 2}{2\pi(1 - n^2)} = \frac{1 + \cos n\pi}{\pi(1 - n^2)}$$

$$a_n = \begin{cases} \frac{2}{\pi(1 - n^2)} & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$$

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \sin t \cos x dx = \frac{1}{2\pi} \int_0^{\pi} [\sin(1-1)x + \sin(1+1)x] dx = \frac{1}{2\pi} \int_0^{\pi} \sin 2x dx$$

$$a_1 = \frac{-1}{2\pi} \left[\frac{\cos 2x}{2} \right]_0^{\pi} = \frac{-1}{4\pi} [\cos 2\pi - \cos 0] = \frac{-1}{4\pi} [1 - 1] = 0$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(t) \sin nt dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (0) \sin nt dt + \int_0^{\pi} \sin t \sin nt dt \right] = \frac{1}{\pi} \left[\frac{1}{2} \int_0^{\pi} (\cos(1-n)t - \cos(1+n)t) dt \right]$$

$$b_n = \frac{1}{2\pi} \left\{ \left[\frac{\sin(1-n)t}{(1-n)} \right]_0^{\pi} - \left[\frac{\sin(1+n)t}{(1+n)} \right]_0^{\pi} \right\} = \frac{1}{2\pi} \left[\frac{\sin(1-n)\pi}{(1-n)} - \frac{\sin(1+n)\pi}{(1+n)} - 0 - 0 \right]$$

$$b_n = \frac{1}{2\pi} \left[\frac{\sin n\pi}{(1-n)} - \frac{-\sin n\pi}{(1+n)} \right] = \frac{1}{2\pi} \left[\frac{\sin n\pi}{(1-n)} + \frac{\sin n\pi}{(1+n)} \right] = \frac{2\sin n\pi}{2\pi(1 - n^2)}$$

$$b_n = \frac{2\sin n\pi}{2\pi(1 - n^2)} = 0 \quad \text{for all } n \text{ except } n = 1$$

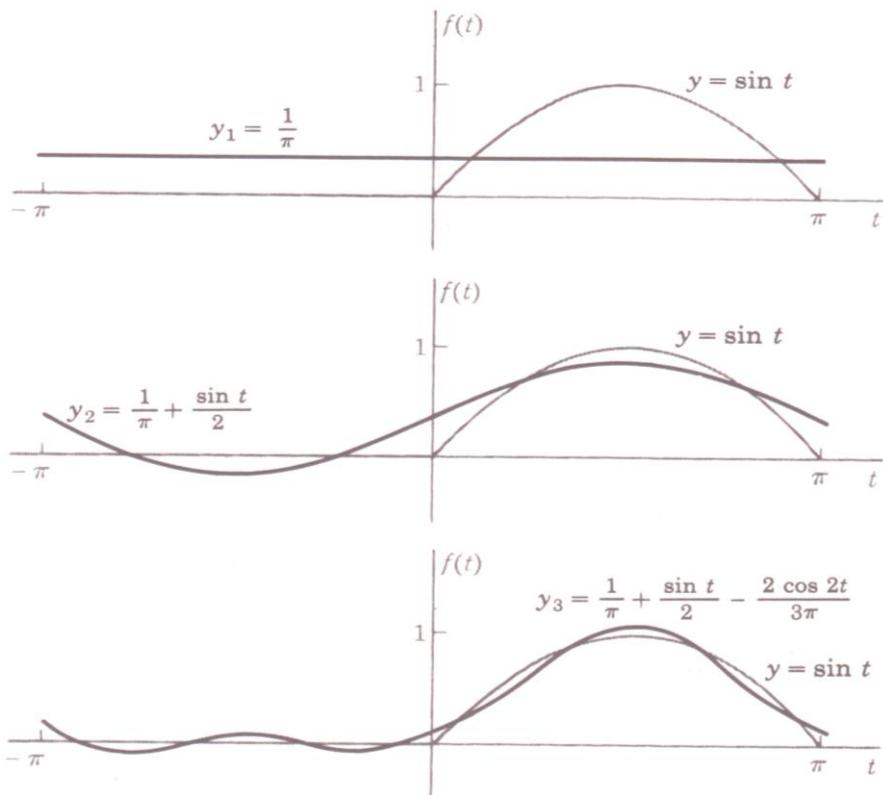
$$b_1 = \frac{1}{\pi} \int_0^\pi \sin t \sin t \, dt = \frac{1}{\pi} \int_0^\pi \sin^2 t \, dt = \frac{1}{2\pi} \int_0^\pi (1 - \cos 2t) \, dt$$

$$b_1 = \frac{1}{2\pi} \left[\int_0^\pi dt - \int_0^\pi \cos 2t \, dt \right] = \frac{1}{2\pi} \left\{ [t]_0^\pi - \left[\frac{1}{2} \sin 2t \right]_0^\pi \right\}$$

$$b_1 = \frac{1}{2\pi} \left\{ [\pi - 0] - \frac{1}{2} [\sin 2\pi - \sin 0] \right\} = \frac{1}{2}$$

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \dots \right)$$



1.6.1 Even and Odd Functions

$f(x)$ is said to be **Even** if $f(-x) = f(x)$ where the function is symmetric w.r.t the **Y-axis**.

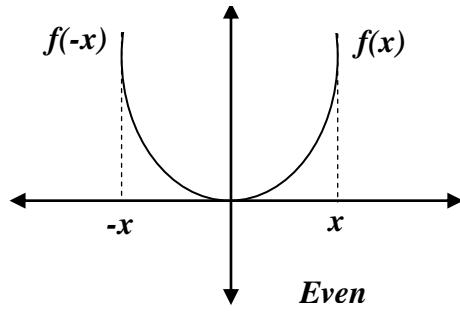
Theorem 3

For Even functions:

$$a_o = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = 0$$



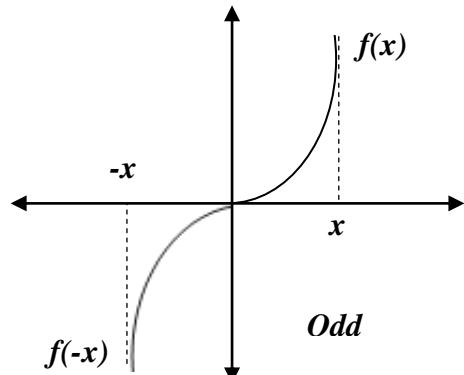
$f(x)$ is said to be **Odd** if $f(-x) = -f(x)$ where the function is symmetric w.r.t the **point of origin**.

Theorem 4

For Odd functions:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$a_o = a_n = 0$$



Example: Obtain the Fourier series of the function? $f(x) = x$ $-\pi < x < \pi$

Solution: Odd function symmetric about the point of origin.

$$a_o = a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{\pi} \int_0^\pi x \sin \frac{n\pi x}{\pi} dx$$

$$\because \int_0^\pi x \sin nx dx = -\frac{\pi}{n} \cos n\pi \Rightarrow$$

$$b_n = \frac{2}{\pi} \times \frac{-\pi}{n} \cos n\pi = -\frac{2}{n} \cos n\pi$$

$$b_n = \begin{cases} -\frac{2}{n}(-1) = \frac{2}{n} & n = \text{odd} \\ -\frac{2}{n}(1) = -\frac{2}{n} & n = \text{even} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

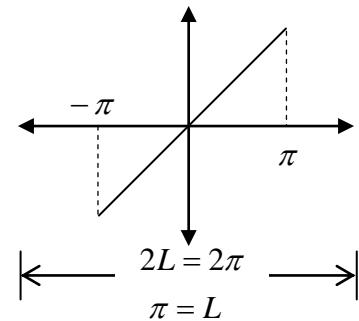
$$f(x) = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \dots \right)$$

$$\text{Substituting } x = \frac{\pi}{2} \Rightarrow f(x) = \frac{\pi}{2}$$

$$\frac{\pi}{2} = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right)$$

$$= \sum_{i=1}^n 2 (-1)^{i+1} \frac{1}{2i-1}$$

$$\approx \frac{3.14}{2}$$



Example: find the Fourier expansion for the function?

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Solution:

$$b_n = 0$$

$$a_o = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \int_0^\pi x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos \frac{n\pi x}{\pi} dx = \frac{2}{\pi} \int_0^\pi x \cos nx dx$$

$$a_n = \frac{2}{\pi} \cos \frac{n\pi - 1}{n^2} = \begin{cases} 0 & n = even \\ -\frac{4}{\pi n^2} & n = odd \end{cases}$$

$$\begin{aligned} f(x) &= \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \end{aligned}$$

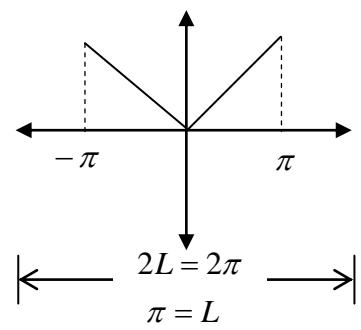
$$at \ x = 0 \quad f(x) = 0$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi}{2} = \frac{4}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$= \sum_{i=1}^n \frac{1}{(2i-1)^2}$$



1.6.2 Half Range Expansion

In some problems we are concerned with interval $(0, L)$ instead of the usual interval of length $(2L)$, furthermore, the conditions of the problems may require us to expand the given function in a series of \sin or \cos only:

This is achieved by:

$$\left. \begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \end{aligned} \right\} \dots \dots \dots \text{for sin series only}$$

$$\left. \begin{aligned} f(x) &= \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ a_o &= \frac{2}{L} \int_0^L f(x) dx \quad : \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \end{aligned} \right\} \dots \dots \dots \text{for cos series only}$$

Example: Expand the function $f(x) = x$: $0 < x < \pi$ in Fourier sin only and cos only?

Solution:

(1) Expansion by sin:

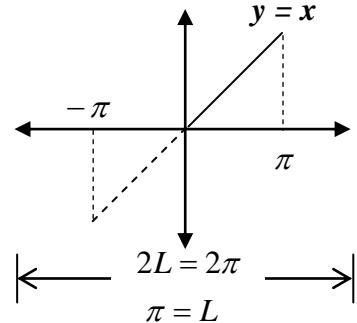
$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx dx$$

$$b_n = \frac{2}{\pi} \left(\frac{-\pi \cos n\pi}{n} \right) = -2 \frac{(-1)^n}{n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = -2 \left(-\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \dots \right)$$

$$f(x) = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots + (-1)^{n+1} \frac{\sin nx}{n} \right)$$



(2) Expansion by cos:

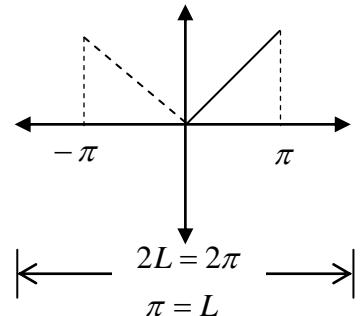
$$b_n = 0$$

$$a_o = \frac{2}{\pi} \int_0^\pi x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = \frac{2}{\pi} \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos \frac{n\pi x}{\pi} dx$$

$$a_n = \frac{2(\cos n\pi - 1)}{n^2 \pi} = \begin{cases} 0 & n = \text{even} \\ -\frac{4}{n^2 \pi} & n = \text{odd} \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$$

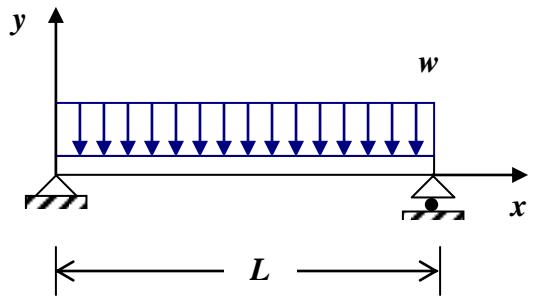


1.6.3 Applications:

Example: Find the Max moment and Max deflection of the simply supported beam shown below?

Solution:

Usually we use the sin series to reduce the solution procedures and minimize the coefficients.



$$w = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}; \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$EI \frac{d^4 y}{dx^4} = w = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

and by integration:

$$EI \frac{d^3 y}{dx^3} = -\frac{L}{n\pi} \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L} + A$$

$$EI \frac{d^2 y}{dx^2} = -\left(\frac{L}{n\pi}\right)^2 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} + Ax + B = \text{Moment}$$

$$\text{at } x=0 \text{ and } x=L : M=0 \Rightarrow A=B=0$$

$$M = EI \frac{d^2 y}{dx^2} = -\left(\frac{L}{n\pi}\right)^2 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$EI \frac{dy}{dx} = \left(\frac{L}{n\pi}\right)^3 \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{L} + C$$

$$EI y = \left(\frac{L}{n\pi}\right)^4 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} + Cx + D$$

$$\text{at } x=0 \text{ and } x=L : y=0 \Rightarrow C=D=0$$

$$EI y = \left(\frac{L}{n\pi}\right)^4 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{L} \int_0^L w \sin \frac{n\pi x}{L} dx = \frac{2w}{L} \int_0^L \sin \frac{n\pi x}{L} dx$$

$$b_n = -\frac{2w}{L} \left[\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right]_0^L = -\frac{2w}{n\pi} (\cos n\pi - 1)$$

$$b_n = \begin{cases} 0 & n = \text{even} \\ \frac{4w}{n\pi} & n = \text{odd} \end{cases}$$

$$-M = EI \frac{d^2y}{dx^2} = -\left(\frac{L}{n\pi}\right)^2 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$-M = -\left(\frac{L}{n\pi}\right)^2 \sum_{n=1}^{\infty} \frac{4w}{n\pi} \sin \frac{n\pi x}{L} = \frac{-4wL^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}$$

$$\text{from structure at } x = \frac{L}{2} : M = \frac{wL^2}{8}$$

by Fourier series :

$$-M = \frac{-4wL^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi L/2}{L} = \frac{-4wL^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2}$$

$$-M = \frac{-4wL^2}{\pi^3} \left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \right) = -0.12507 wL^2 \approx -\frac{wL^2}{8}$$

$$M = 0.12507 wL^2 \approx \frac{wL^2}{8}$$

$$EI y = \left(\frac{L}{n\pi} \right)^4 \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$EI y = \frac{4wL^4}{\pi^5} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{L}$$

$$\text{from structure at } x = \frac{L}{2} : y = \frac{5}{384} \frac{wL^4}{EI}$$

by Fourier series :

$$EI y = \frac{4wL^4}{\pi^5} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{L}}{n^5}$$

$$EI y = \frac{4wL^4}{\pi^5} \left(\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} \right)$$

$$EI y = \frac{4wL^4}{\pi^5} \Rightarrow EI y = \frac{wL^4}{76.31EI} = \frac{5.01wL^4}{384EI} \approx \frac{5wL^4}{384EI}$$