

Chapter 5

Laplace Transforms

1.1 Introduction:

The previous chapter introduced the concept of Fourier series. If the function is nonzero only when $t > 0$, a similar transform, the **Laplace transform**, exists. It is particularly useful in solving initial-value problems involving linear, constant coefficient, ordinary and partial differential equations. The present chapter develops the general properties and techniques of Laplace transforms.



Pierre-Simon Laplace

lived from 1749 to 1827

1.2 Definition of Laplace transforms:

Consider the function $f(t)$ such that $f(t) = 0$ for $t < 0$. Then the Laplace integral

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

defines the Laplace transform of $f(t)$. Which we shall write $\mathcal{L}[f(t)]$ or $F(s)$. The **Laplace transform** converts a function of t into a function of the transform variable s .

Example: find $\mathcal{L}(1)$, $\mathcal{L}(t)$ and $\mathcal{L}(e^{at})$?

Solution:

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\mathcal{L}(1) = \int_0^{\infty} e^{-st} (1) dt = \frac{-s}{-s} \int_0^{\infty} e^{-st} dt = \frac{-1}{s} [e^{-st}]_0^{\infty}$$

$$\mathcal{L}(1) = \frac{-1}{s} [0 - 1] = \frac{1}{s} \quad \text{where } s > 0$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}(t) = \int_0^\infty e^{-st} t dt \quad \text{take } u = t \Rightarrow du = dt \quad \text{and} \quad dv = e^{-st} dt \Rightarrow v = \frac{1}{-s} e^{-st}$$

$$\mathcal{L}(t) = \left[t \frac{1}{-s} e^{-st} \right]_0^\infty - \int_0^\infty \frac{1}{-s} e^{-st} dt = 0 - \left[\frac{1}{s^2} e^{-st} \right]_0^\infty = 0 - \left[0 - \frac{1}{s^2} \right] = \frac{1}{s^2} \quad s > 0$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} (e^{at}) dt = \frac{-(s-a)}{-(s-a)} \int_0^\infty e^{-(s-a)t} dt$$

$$\mathcal{L}[e^{at}] = \frac{-1}{(s-a)} [e^{-(s-a)t}]_0^\infty = \frac{-1}{(s-a)} [0 - 1] = \frac{1}{(s-a)} \quad \text{where } s > a$$

Example: Find $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$?

Solution: $e^{i\omega t} = \cos \omega t + i \sin \omega t$ Euler's formula

$$\mathcal{L}(e^{i\omega t}) = \int_0^\infty e^{-st} (e^{i\omega t}) dt$$

$$\mathcal{L}(e^{i\omega t}) = \int_0^\infty e^{-st} (\cos \omega t + i \sin \omega t) dt = \int_0^\infty e^{-st} \cos \omega t dt + i \int_0^\infty e^{-st} \sin \omega t dt \Rightarrow$$

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos \omega t) + i \mathcal{L}(\sin \omega t) \dots \quad (1)$$

and from the previous example we know that

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} \quad \text{taking } i\omega = a \quad \text{and multiplying by the conjugate} \Rightarrow$$

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} \times \frac{s + i\omega}{s + i\omega} = \frac{s + i\omega}{s^2 + \omega^2} \Rightarrow$$

$$\mathcal{L}(e^{i\omega t}) = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2} \dots \quad (2)$$

comparing equations (1) and (2) we find

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

1.3 Laplace transforms for derivatives:

1.3.1 First derivative:

$$\begin{aligned}\mathcal{L}[y'(t)] &= \int_0^{\infty} e^{-st} y'(t) dt \\ &= \left[e^{-st} y(t) \right]_0^{\infty} - \int_0^{\infty} y(t) (-se^{-st}) dt \\ &= [0 - y(0)] + s \int_0^{\infty} e^{-st} y(t) dt \\ \mathcal{L}[y'(t)] &= -y(0) + s \mathcal{L}[y(t)]\end{aligned}$$

$$\boxed{\mathcal{L}[y'(t)] = s \mathcal{L}[y(t)] - y(0)}$$

1.3.2 Second derivative:

$$\begin{aligned}\mathcal{L}[y''(t)] &= \mathcal{L}[G'(t)] \quad \text{where } G(t) = y'(t) \\ &\quad G'(t) = y''(t)\end{aligned}$$

From 1st derivative law

$$\begin{aligned}\mathcal{L}[G'(t)] &= s \mathcal{L}[G(t)] - G(0) \\ &= s \mathcal{L}[y'(t)] - y'(0) \\ &= s \{s \mathcal{L}[y(t)] - y(0)\} - y'(0) \\ \mathcal{L}[y''(t)] &= s^2 \mathcal{L}[y(t)] - s y(0) - y'(0)\end{aligned}$$

From the sequence of 1st and 2nd derivatives we can find that:

$$\mathcal{L}[y'''(t)] = s^3 \mathcal{L}[y(t)] - s^2 y(0) - s y'(0) - y''(0)$$

and

$$\boxed{\mathcal{L}[y^n(t)] = s^n \mathcal{L}[y(t)] - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{n-1}(0)}$$

Example: find $\mathcal{L}(t^n)$

Solution:

$$y(t) = t^n \Rightarrow y(0) = 0 \Rightarrow y'(t) = n t^{n-1}$$

$$\mathcal{L}[y'(t)] = s \mathcal{L}[y(t)] - y(0)$$

$$\mathcal{L}[n t^{n-1}] = s \mathcal{L}[t^n]$$

$$\mathcal{L}[t^n] = \frac{n}{s} \mathcal{L}[t^{n-1}]$$

if $n = 1$ then

$$\mathcal{L}[t] = \frac{1}{s} \mathcal{L}[1] = \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^2}$$

if $n = 2$ then

$$\mathcal{L}[t^2] = \frac{2}{s} \mathcal{L}[t] = \frac{2}{s} \times \frac{1}{s^2} = \frac{2}{s^3}$$

if $n = 3$ then

$$\mathcal{L}[t^3] = \frac{3}{s} \mathcal{L}[t^2] = \frac{3}{s} \times \frac{2}{s^3} \times \frac{1}{s^2} = \frac{3!}{s^4}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

1.4 Table of L.T

No.	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
8	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
9	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
10	$e^{at} f(t)$	$f(s-a)$
11	$t^n f(t)$	$(-1)^n f^{(n)}(s)$
12	$y^{(n)}(t)$	$s^n \mathcal{L}[y(t)] - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{n-1}(0)$