

1.5 Properties of Laplace Transforms

A) $\mathcal{L}[F(t) + G(t)] = \mathcal{L}[F(t)] + \mathcal{L}[G(t)]$

B) If $C = \text{constant}$ then,

$$\mathcal{L}[C \cdot F(t)] = C \cdot \mathcal{L}[F(t)]$$

Example: Find $\mathcal{L}[3 - 5e^{2t} + 4\sin t - 7\cos 3t]$?

Solution:

$$\begin{aligned} &= 3\mathcal{L}[1] - 5\mathcal{L}[e^{2t}] + 4\mathcal{L}[\sin t] - 7\mathcal{L}[\cos 3t] \\ &= \frac{3}{s} - 5 \frac{1}{s-2} + 4 \frac{1}{s^2+1} - 7 \frac{s}{s^2+9} \\ &= \frac{3}{s} - \frac{5}{s-2} + \frac{4}{s^2+1} - \frac{7s}{s^2+9} \end{aligned}$$

C) If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[e^{at} f(t)] = f(s-a)$$

Example: Find $\mathcal{L}[e^{3t} \cos 4t]$?

Solution:

$$\begin{aligned} \mathcal{L}[\cos 4t] &= \frac{s}{s^2+16} = F(s) \\ \mathcal{L}[e^{3t} f(t)] &= f(s-3) = \frac{(s-3)}{(s-3)^2+16} \\ &= \frac{s-3}{s^2-6s+25} \end{aligned}$$

D) If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

or $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

Example: Find $\mathcal{L}[t \sin t]$?

Solution:

$$\mathcal{L}[\sin t] = \frac{1}{s^2 + 1} = F(s)$$

$$\mathcal{L}[t \sin t] = (-1)^1 \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right)$$

$$= \frac{2s}{(s^2 + 1)^2}$$

Example: Find $\mathcal{L}[t^2 \sin t]$?

Solution:

$$\begin{aligned}\mathcal{L}[t^2 \sin t] &= (-1)^2 \frac{d^2}{ds^2} \left(\frac{1}{s^2 + 1} \right) \\ &= \frac{6s^2 - 2}{(s^2 + 1)^3}\end{aligned}$$

1.6 Inverse of Laplace transforms

If $\sin(x) = y$ then $x = \sin^{-1}(y)$

$$\mathcal{L}[f(t)] = F(s) \text{ then } f(t) = \mathcal{L}^{-1}[F(s)]$$

Example:

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

1.6.1 Properties of Inverse of Laplace transform

A) $\mathcal{L}^{-1}[F(s) + G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)]$

B) If $C = \text{constant}$ then,

$$\mathcal{L}^{-1}[C \cdot F(s)] = C \cdot \mathcal{L}^{-1}[F(s)] = C \cdot f(t)$$

C) If $\mathcal{L}^{-1}[F(s)] = f(t)$ then

$$\mathcal{L}^{-1}[f(s-a)] = e^{at} f(t)$$

D) If $\mathcal{L}^{-1}[F(s)] = f(t)$ then

$$\mathcal{L}^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$$

E) If $\mathcal{L}^{-1}[F(s)] = f(t)$ then

$$\mathcal{L}^{-1}[s F(s) - f(0)] = f'(t)$$

Example:

$$\begin{aligned} & \mathcal{L}^{-1}\left[\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right] \\ &= 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - 3\mathcal{L}^{-1}\left[\frac{s}{s^2+16}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] \\ &= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t \end{aligned}$$

Example:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{3s-1}{s^2+2s+5}\right] &= \mathcal{L}^{-1}\left[\frac{3(s+1)-1-3}{s^2+2s+1+4}\right] = \mathcal{L}^{-1}\left[\frac{3(s+1)-4}{s^2+2s+1+4}\right] = \mathcal{L}^{-1}\left[\frac{3(s+1)-4}{(s+1)^2+4}\right] \\
&= \mathcal{L}^{-1}\left[\frac{3(s+1)}{(s+1)^2+4} - \frac{4}{(s+1)^2+4}\right] = 3\mathcal{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+4}\right] - 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] \\
\mathcal{L}[e^{at} \cos \omega t] &= \frac{(s-a)}{(s-a)+\omega^2} \quad \text{and} \quad \mathcal{L}[e^{at} \sin \omega t] = \frac{\omega}{(s-a)^2+\omega^2} \\
\therefore 3\mathcal{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+4}\right] - 2\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] &= 3e^{-t} \cos 2t - 2e^{-t} \sin 2t
\end{aligned}$$

Example:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{5s^2-7S+17}{(s-1)(s^2+4)}\right] &\Rightarrow \\
\frac{5s^2-7S+17}{(s-1)(s^2+4)} &= \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+4)} \\
\frac{5s^2-7S+17}{(s-1)(s^2+4)} &= \frac{A(s^2+4)+(Bs+C)(s-1)}{(s-1)(s^2+4)} \Rightarrow \\
5s^2-7S+17 &= A(s^2+4)+(Bs+C)(s-1) \Rightarrow \\
A &= 3 \\
B &= 2 \\
C &= -5 \\
\frac{5s^2-7S+17}{(s-1)(s^2+4)} &= \frac{3}{(s-1)} + \frac{2s-5}{(s^2+4)} \\
\mathcal{L}^{-1}\left[\frac{5s^2-7S+17}{(s-1)(s^2+4)}\right] &= 3\mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right] + 2\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)}\right] \\
\mathcal{L}^{-1}\left[\frac{5s^2-7S+17}{(s-1)(s^2+4)}\right] &= 3e^t + 2\cos 2t - \frac{5}{2}\sin 2t
\end{aligned}$$

1.7 Solution of Ordinary D.E's by Laplace transforms:

1.7.1 D.E's with constant coefficients:

Example: $y'' + 4y = 16t \Leftrightarrow y(0) = 3 \quad y'(0) = -6$

Solution: Let $Y = \mathcal{L}[y]$

$$\mathcal{L}[y'' + 4y] = \mathcal{L}[16t]$$

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = 16\mathcal{L}[t]$$

$$s^2(Y) - s y(0) - y'(0) + 4Y = \frac{16}{s^2}$$

$$y(0) = 3 \quad \text{and} \quad y'(0) = -6$$

$$s^2Y - 3s + 6 + 4Y = \frac{16}{s^2}$$

$$(s^2 + 4)Y = 3s - 6 + \frac{16}{s^2}$$

$$Y = \frac{3s - 6}{s^2 + 4} + \frac{16}{s^2(s^2 + 4)} \quad \text{from partial fraction} \quad \frac{16}{s^2(s^2 + 4)} = \frac{A}{s^2} + \frac{B}{(s^2 + 4)} \quad \text{where } A = 4 \quad B = -4$$

$$Y = \frac{3s}{s^2 + 4} - \frac{6}{s^2 + 4} + \frac{4}{1} \left[\frac{1}{s^2} - \frac{1}{(s^2 + 4)} \right]$$

$$Y = \frac{3s}{s^2 + 4} - \frac{10}{s^2 + 4} + \frac{4}{s^2}$$

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1} \left[\frac{3s}{s^2 + 4} - \frac{10}{s^2 + 4} + \frac{4}{s^2} \right]$$

$$\therefore y(t) = 3\cos 2t - 5\sin 2t + 4t$$

1.7.2 D.E's with variable coefficients: (*First degree polynomial only*)

From the properties of Laplace transforms we recall that:

$$\mathcal{L}[t^n y(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

Hence for $n=1$

$$\mathcal{L}[t y(t)] = -F'(s) \quad \text{or in another expression} \quad = -\frac{d}{ds} \mathcal{L}[y(t)]$$

similarly

$$\begin{aligned} \mathcal{L}[t y'(t)] &= -\frac{d}{ds} \mathcal{L}[y'(t)] = -\frac{d}{ds} [s F(s) - y(0)] \\ &= -s F'(s) - F(s) \end{aligned}$$

similarly for $y''(t)$

$$\begin{aligned} \mathcal{L}[t y''(t)] &= -\frac{d}{ds} \mathcal{L}[y''(t)] = -\frac{d}{ds} [s^2 F(s) - s y(0) - y'(0)] \\ &= -s^2 F'(s) - 2s F(s) + y(0) \end{aligned}$$

Example: Solve the following D.E ?

$$t y'' - t y' + y = 0 \quad \text{where } y'(0) = 1 \quad \text{and} \quad y(0) = 0$$

$$\mathcal{L}[t y''] - \mathcal{L}[t y'] + \mathcal{L}[y] = 0 \quad \text{and} \quad F(s) = \mathcal{L}[y] = Y$$

$$[-s^2 Y' - 2s Y + 0] - [-s Y' - Y] + Y = 0$$

$$-s^2 Y' - 2s Y + s Y' + Y = 0$$

$$(-s^2 + s) Y' + (-2s + 2) Y = 0$$

$$-s(s-1) Y' - 2(s-1) Y = 0 \quad \div \quad -(s-1)$$

$$s \frac{dY}{ds} + 2Y = 0$$

$$\int \frac{dY}{2Y} = -\int \frac{ds}{s} \Rightarrow \frac{1}{2} \ln Y = -\ln s + c \Rightarrow \ln Y = -2 \ln s + 2c$$

$$\ln Y = \ln(s^{-2}) + 2c \Rightarrow Y = e^{\ln(s^{-2}) + 2c} \Rightarrow Y = e^{\ln(s^{-2})} \times e^{2c}$$

$$Y = s^{-2} \times k \Rightarrow Y = \frac{k}{s^2}$$

$$\because Y = \frac{k}{s^2} \Rightarrow \mathcal{L}[y] = \frac{k}{s^2}$$

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{k}{s^2}\right] = k \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$y = k t$$

$$\text{applying the initial condition} \quad y'(0) = 1$$

$$y' = k = 1$$

$$y = t$$

1.7.3 Simultaneous Linear D.E's

Example: Solve the D.E's ?

$$\left. \begin{array}{l} \frac{dy}{dt} + 6y = \frac{dx}{dt} \\ 3x - \frac{dx}{dt} = 2\frac{dy}{dt} \end{array} \right| \quad \begin{array}{l} y(0) = 3 \quad \text{and} \quad x(0) = 2 \end{array}$$

Solution: assume $\mathcal{L}[y] = Y$ and $\mathcal{L}[x] = X$

$$\begin{aligned} \mathcal{L}[y'] + 6\mathcal{L}[y] &= \mathcal{L}[x'] \\ sY - y(0) + 6Y &= sX - x(0) \\ sY - 3 + 6Y &= sX - 2 \quad \dots \quad (1) \end{aligned}$$

$$\begin{aligned} 3\mathcal{L}[x] - \mathcal{L}[x'] &= 2\mathcal{L}[y] \\ 3X - [sX - x(0)] &= 2[sY - y(0)] \\ 3X - [sX - 2] &= 2[sY - 3] \quad \dots \quad (2) \end{aligned}$$

$$\begin{aligned} -sX + (s+6)Y &= 1 \quad \dots \quad (1) \\ (3-s)X - 2sY &= -8 \quad \dots \quad (2) \end{aligned}$$

$$\begin{aligned} X &= \frac{2s+16}{s^2+s-6} = \frac{2s+16}{(s-2)(s+3)} = \frac{4}{s-2} - \frac{2}{s+3} \\ Y &= \frac{3s-1}{s^2+s-6} = \frac{3s-1}{(s-2)(s+3)} = \frac{1}{s-2} + \frac{2}{s+3} \end{aligned}$$

$$\begin{aligned} x &= \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}\left[\frac{4}{s-2} - \frac{2}{s+3}\right] \\ x &= 4e^{2t} - 2e^{-3t} \end{aligned}$$

$$\begin{aligned} y &= \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{2}{s+3}\right] \\ y &= e^{2t} + 2e^{-3t} \end{aligned}$$