Chapter 6

Partial Differential Equations

1.1 Introduction:

In our previous work, most notably in Chap. 1, we have seen how the analysis of mechanical or electrical systems containing lumped parameters often leads to ordinary differential equations in which the time ***t*** is the (only) independent variable. However, the assumption that all masses exists as conceptualized mass points; that all springs are weightless; or that elements of an electric circuits are concentrated in ideal resisters, capacitors, and inductors, rather than continuously distributed, is frequently not sufficiently accurate. In such cases, a more realistic approach must take into account the fact that the dependant variable depend not only on ***t*** but also on one or more space variables. Because there is more than one independent variable, the formulation of such problems leads to partial, rather than ordinary, differential equations. In this chapter we shall discuss such equations as they commonly arise in applied mathematics. We shall begin by examining in some detail the derivation form physical principles of a number of important partial differential equations. Then, knowing the forms of most common occurrence, we shall investigate methods of solution and their application to specific problems.

Figure Although largely self-educated in mathematics, Jean Le Rond d'Alembert (1717–1783) gained equal fame as a mathematician and *philosophe* of the continental Enlightenment. By the middle of the eighteenth century, he stood with such leading European mathematicians and mathematical physicists as Clairaut, D. Bernoulli, and Euler. Today we best remember him for his work in fluid dynamics and applying partial differential equations to problems in physics. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

1.2 Some important linear partial differential equations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$ ***One-dimensional wave equation.***

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ ***Two-dimensional wave equation.***

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$ ***One-dimensional heat equation.***

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ ***Two-dimensional Laplace equation.***

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ ***Three-dimensional Laplace equation.***

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ ***Two-dimensional Poisson equation.***

Example: Verify that the following functions are solutions of Laplace equation $(U_{xx} + U_{yy}) = 0$?

1) $U = 2xy$
2) $U = \sin x \sinh y$

Solution:

$U = 2xy$

$U_x = 2y \qquad U_y = 2x$

$U_{xx} = 0 \qquad U_{yy} = 0$

$U_{xx} + U_{yy} = 0 \quad \Leftrightarrow$ *Laplace equation*

$U = \sin x \sinh y$

$U_x = \cos x \sinh y \quad \Rightarrow \quad U_{xx} = -\sin x \sinh y$

$U_y = \sin x \cosh y \quad \Rightarrow \quad U_{yy} = \sin x \sinh y$

$U_{xx} + U_{yy} = -\sin x \sinh y + \sin x \sinh y = 0 \quad \Rightarrow$ *Laplace equation*

Example: Verify that the following function is a solutions of Heat equation $U_t = C^2 U_{xx}$

$U = e^{-t}\sin 3x$?

Solution:

$U = e^{-t}\sin 3x$

$U_t = -e^{-t}\sin 3x$

$U_x = 3e^{-t}\cos 3x \quad \Rightarrow \quad U_{xx} = -9e^{-t}\sin 3x$

$$C^2 = \frac{U_t}{U_{xx}} = \frac{-e^{-t}\sin 3x}{-9e^{-t}\sin 3x} = \frac{1}{9} \quad \Rightarrow \quad C = \mp\frac{1}{3}$$

It is a solution of heat equation.

Example: Verify that the following function is a solution of Wave equation $U_{tt} = a^2 U_{xx}$

$U = x^3 + 3xt^2$?

Solution:

$U = x^3 + 3xt^2$

$U_t = 6xt \quad \Rightarrow \quad U_{tt} = 6x$

$U_x = 3x^2 + 3t^2 \quad \Rightarrow \quad U_{xx} = 6x$

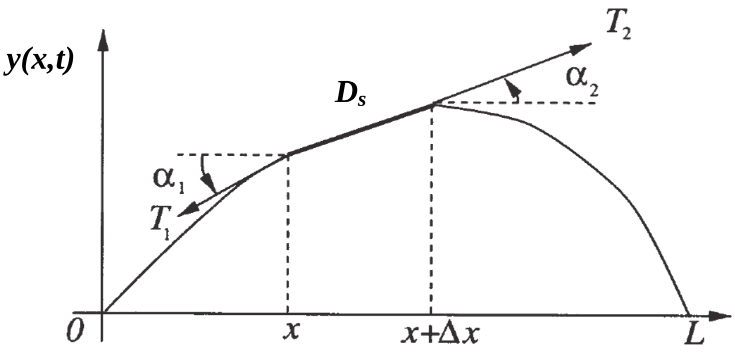
$$a^2 = \frac{U_{tt}}{U_{xx}} = \frac{6x}{6x} = 1$$

It is a solution of the wave equation.

1.3 Vibration in a stretched string

Assumptions:

1- Moment of Inertia ***(I)*** is so small.
2- No bending.
3- Only pure tension.



$$T_1 = T_{(x)} \quad \& \quad T_2 = T_{(x+\Delta x)}$$

1.3.1 Finding the differential equation of the mathematical model

$$\sum f_y = m\frac{d^2 y}{dt^2}$$

$$T_{(x+\Delta x)}\sin\alpha_2 - T_{(x)}\sin\alpha_1 = \rho\Delta s\frac{\partial^2 y}{\partial t^2}$$

α *depends on* x *where* $\alpha_1 = \alpha_{(x)}$ *and* $\alpha_2 = \alpha_{(x+\Delta x)}$

$$\frac{T_{(x+\Delta x)}\sin\alpha_{(x+\Delta x)} - T_{(x)}\sin\alpha_{(x)}}{\Delta x} = \rho\frac{\Delta s}{\Delta x}\frac{\partial^2 y}{\partial t^2}$$

$\because$ α *and* Δs *are very small then* $\Delta s = \Delta x$ *and*
$\sin\alpha_{(x+\Delta x)} = \tan\alpha_{(x+\Delta x)}$

$$\lim_{\Delta x\to 0}\frac{T_{(x+\Delta x)}\tan\alpha_{(x+\Delta x)} - T_{(x)}\tan\alpha_{(x)}}{\Delta x} = \lim_{\Delta x\to 0}\rho\frac{\partial^2 y}{\partial t^2}$$

knowing that $\lim_{\Delta x\to 0}\frac{f_{(x+\Delta x)} - f_{(x)}}{\Delta x} = f'_{(x)}$ *then*

$$\frac{\partial}{\partial x}\left(T_{(x)}\tan\alpha_{(x)}\right) = \rho\frac{\partial^2 y}{\partial t^2} \quad \Leftrightarrow \quad \tan\alpha_{(x)} = \frac{\partial y_{(x,t)}}{\partial x}$$

$$\frac{\partial}{\partial x}\left(T_{(x)}\frac{\partial y_{(x,t)}}{\partial x}\right) = \rho\frac{\partial^2 y}{\partial t^2} \quad \Leftrightarrow \quad \because T \text{ is constant}$$

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial t^2} \quad \Leftrightarrow \quad c^2 = \frac{\rho}{T}}$$

1.3.2 Another method for deriving the wave equation:

$$\frac{dy}{dx} = f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$f'(x).\Delta x = f(x+\Delta x) - f(x)$$

$$\boxed{f(x+\Delta x) \approx f(x) + f'(x).\Delta x}$$

$$\sum f_y = m\frac{d^2 y}{dt^2} = \rho \Delta s \frac{\partial^2 y}{\partial t^2}$$
$$\sum f_y = T_{(x+\Delta x)} \sin \alpha_{(x+\Delta x)} - T_{(x)} \sin \alpha_{(x)}$$
$$\sum f_y = T \sin \alpha\big|_{(x+\Delta x)} - T \sin \alpha\big|_{(x)}$$
$$\sum f_y = \left[T \sin \alpha\big|_{(x)} + \frac{\partial}{\partial x}(T \sin \alpha).\Delta x \right] - T \sin \alpha\big|_{(x)}$$
$$\sum f_y = \frac{\partial}{\partial x}(T \sin \alpha).\Delta x \quad \Rightarrow \quad if \;\; \alpha \; is \; small \quad \Rightarrow \quad \sin \alpha \approx \tan \alpha \approx \frac{\partial y}{\partial x}$$
$$\sum f_y = \frac{\partial}{\partial x}(T \frac{\partial y}{\partial x}).\Delta x$$
$$\sum f_y = T \frac{\partial^2 y}{\partial x^2} \Delta x \quad \Leftrightarrow \quad \sum f_y = m\frac{\partial^2 y}{\partial t^2} = \rho \Delta s \frac{\partial^2 y}{\partial t^2} \quad \Rightarrow$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} \quad where \quad c^2 = \frac{\rho}{T}$$

Or

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad where \quad a^2 = \frac{T}{\rho}}$$

1.3.3 General solution:

1.3.3.1 Separation of variables

We begin by presenting the most classical method of solving the wave equation: separation of variables. Despite its current widespread use, its initial application to the vibrating string problem was immersed in controversy involving the application of a half-range Fourier sine series to represent the initial conditions. On one side, Daniel Bernoulli claimed (1775) that he could represent any general initial condition with this technique. To d'Alembert and Euler, however, the half-range Fourier sine series, with its period of 2L, could not possibly represent any arbitrary function. However, by 1807 Bernoulli was proven correct by the use of separation of variables in the heat conduction problem and it rapidly grew in acceptance. In the following examples we show how to apply this method to solve the wave equation.

Separation of variables consists of four distinct steps. The basic idea is to convert a second-order partial differential equation into two ordinary differential equations. First, we assume that the solution equals the product $X_{(x)}.T_{(t)}$. *Direct substitution into the partial differential equation and boundary conditions yields tow ordinary differential equations and the corresponding boundary conditions. Step tow involves solving a boundary-value problem of the Sturm-Liouville type. In step three we find the corresponding time dependence. Finally we construct the complete solution as a sum of all product solutions. Upon applying the Initial coefficients, we have an eigenfunction expansion and must compute the Fourier coefficients. The substitution of these coefficients into the summation yields the final solution.*

$$\boxed{y(x,t) = X_{(x)}.T_{(t)}}$$

$$y = X\ T$$

$$\frac{\partial^2 y}{\partial t^2} = X\ \frac{d^2T}{dt^2} = X\ T''$$

$$\frac{\partial^2 y}{\partial x^2} = T\ X''$$

And substituting in the original Wave differential equation

$$X.T'' = a^2\, T.X''$$

$$\boxed{\frac{T''}{T} = a^2\ \frac{X''}{X}}$$

Now the left-hand member of the equation is clearly independent of ***x***. hence (in spite of its appearance) the right-hand side of the equation must also be independent of ***x***, since it is identically equal to the expression on the left. Similarly, each member of the equation must be independent of ***t***. therefore, being independent of both ***x*** and ***t***, each side of the equation must be a constant, say ***μ***, and we can write

$$\frac{T''}{T} = a^2 \frac{X''}{X} = \mu$$

Thus the determination of solutions of the original partial differential equation has been reduced to the determination of solutions of the two ordinary differential equations

$$T'' = \mu T \quad and \quad X'' = \frac{\mu}{a^2} X$$

Case 1: $\mu \succ 0$

$$a^2 X'' - \mu X = 0 \quad , let\ \mu = \lambda^2$$
$$a^2 m^2 - \lambda^2 = 0 \quad characteristic\ equation$$
$$a^2 m^2 = \lambda^2 \quad \Rightarrow \quad m_{1,2} = \mp \frac{\lambda}{a}$$
$$X = A e^{\frac{\lambda}{a}x} + B e^{-\frac{\lambda}{a}x}$$

$$T'' - \mu T = 0$$
$$let\ \mu = \lambda^2$$
$$T'' - \lambda^2 T = 0$$
$$m^2 - \lambda^2 = 0 \quad characteristic\ equation$$
$$m^2 = \lambda^2 \quad \Rightarrow \quad m_{1,2} = \mp \lambda$$
$$T = C e^{\lambda t} + D e^{-\lambda t}$$
$$y(x,t) = \left(A e^{\frac{\lambda}{a}x} + B e^{-\frac{\lambda}{a}x} \right).\left(C e^{\lambda t} + D e^{-\lambda t} \right)$$

And this solution is rejected physically because it is not periodic.

Case 2: $\mu = 0$

$$X'' = 0 \quad \Rightarrow \quad X' = C \quad \Rightarrow \quad X = Ax + B$$
$$T'' = 0 \quad \Rightarrow \quad T' = C \quad \Rightarrow \quad T = Ct + D$$
$$y = (Ax + B).(Ct + D)$$

And this solution is rejected physically also because it is not periodic.