Applying boundary and initial conditions to the general formula of the wave equation:

(1) $y(0,t)=(A\cos\lambda t+B\sin\lambda t).\left(C\cos\frac{\lambda}{a}(0)+D\sin\frac{\lambda}{a}(0)\right)=0$

$y(0,t)=(A\cos\lambda t+B\sin\lambda t).(C\,(1)+D(0))=0 \quad\Rightarrow\quad C=0$

$y(x,t)=(A\cos\lambda t+B\sin\lambda t).D\sin\frac{\lambda}{a}x$

(2) $y(L,t)=(A\cos\lambda t+B\sin\lambda t).D\sin\frac{\lambda}{a}L=0$

$y(L,t)=\left(\overline{A}\cos\lambda t+\overline{B}\sin\lambda t\right).\sin\frac{\lambda}{a}L=0$

$\therefore \sin\frac{\lambda}{a}L=0 \quad\Rightarrow\quad \frac{\lambda}{a}L=n\pi \quad\Rightarrow\quad \lambda=\frac{n\pi a}{L} \quad\Leftrightarrow\quad n=1,2,3.....$

$\therefore y_n(x,t)=\left(\overline{A}_n\cos\frac{n\pi a}{L}t+\overline{B}_n\sin\frac{n\pi a}{L}t\right).\sin\frac{n\pi}{L}x$

$y(x,t)=\sum_{n=1}^{\infty}y_n(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left[\overline{A}_n\cos\frac{n\pi a}{L}t+\overline{B}_n\sin\frac{n\pi a}{L}t\right]$

(3) $y(x,0)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left[\overline{A}_n\cos(0)+\overline{B}_n\sin(0)\right]=0$

$y(x,0)=\sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{L}x\right)\overline{A}_n=0$

If $\sin\frac{n\pi}{L}x=0 \quad\Rightarrow\quad$ *trivial solution*

$\therefore \overline{A}_n=0$

$y(x,t)=\sum_{n=1}^{\infty}\overline{B}_n\left(\sin\frac{n\pi a}{L}t\right)\left(\sin\frac{n\pi}{L}x\right)$

(4) $y^o(x,o)=\left.\frac{\partial y}{\partial t}\right|_{t=0}=\begin{vmatrix}x\\L-x\end{vmatrix}$

$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\overline{B}_n\left(\sin\frac{n\pi}{L}x\right)\left(\frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\right)$

$y^o(x,o)=\sum_{n=1}^{\infty}\overline{B}_n\left(\sin\frac{n\pi}{L}x\right)\frac{n\pi a}{L}$

$y^o(x,o)=\sum_{n=1}^{\infty}\frac{n\pi a}{L}\overline{B}_n\left(\sin\frac{n\pi}{L}x\right)=\begin{vmatrix}x & 0\prec x\prec\frac{L}{2}\\ L-x & \frac{L}{2}\prec x\prec L\end{vmatrix}$

And by taking Fourier half-range Sin expansion:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$Let \quad \sum_{n=1}^{\infty}\frac{n\pi a}{L}\overline{B}_n\left(\sin\frac{n\pi}{L}x\right)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \Rightarrow$$

$$b_n=\frac{n\pi a}{L}\overline{B}_n \quad \Rightarrow \quad \overline{B}_n=\frac{L}{n\pi a}.b_n$$

$$but \quad b_n=\frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L}\,dx$$

$$b_n=\frac{2}{L}\left[\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}\,dx+\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$b_n=\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \Rightarrow$$

$$\overline{B}_n=\frac{L}{n\pi a}.b_n=\frac{L}{n\pi a}.\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}=\frac{4L^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}=\begin{cases}0 & n=even\\ \dfrac{4L^2}{n^3\pi^3 a} & n=1,5,9,....\\ -\dfrac{4L^2}{n^3\pi^3 a} & n=3,7,11,....\end{cases}$$

$$y(x,t)=\sum_{n=1}^{\infty}\overline{B}_n\ \left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{n\pi a}{L}t\right)$$

$$\boxed{y(x,t)=\sum_{n=1}^{\infty}\left(\frac{4L^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\right)\left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{n\pi a}{L}t\right)}$$

1.3.3.2 The d'Alembert Solution of the Wave Equation

If ***f*** is a function possessing a second derivative, then, by the chain rule,

$$\frac{\partial f(x-at)}{\partial t} = -af'(x-at) \qquad \frac{\partial f(x-at)}{\partial x} = f'(x-at)$$

$$\frac{\partial^2 f(x-at)}{\partial t^2} = a^2 f''(x-at) \qquad \frac{\partial^2 f(x-at)}{\partial x^2} = f''(x-at)$$

And from these results it is evident that $y = f(x-at)$ satisfies the Wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{.....} \quad (1)$$

It is an equally simple matter to prove that if ***g*** is an arbitrary twice-differentiable function, then $g(x+at)$ is likewise a solution of (1). Hence, since it is a linear equation, it follows that the sum

$$y = f(x-at) + g(x+at) \quad \text{.....(2)}$$

is also a solution of (1). In fact, it can be shown that if ***f*** and ***g*** are arbitrary twice-differentiable functions, then Eq. (2) is a complete solution of the wave equation.
This form of the solution of the wave equation is especially useful in revealing the significance of the parameter ***a*** and its dimensions of velocity. Suppose, specifically, that we consider the vibrations of a uniform string stretching from $-\infty \, to \, \infty$. If its transverse displacement is given by (2), we have in fact two waves traveling in opposite directions along the string, each with velocity ***a***. For consider the function $f(x-at)$. At $t=0$, it defines the curve $y = f(x)$, and at any later time $t = t_1$, it defines the curve $y = f(x-at_1)$. But these curves are identical except that the latter is translated to the right a distance equal to at_1. Thus the entire configuration moves along the string without distortion a distance of at_1 in t_1 units of time. The velocity with which the wave is propagated is therefore

$$v = \frac{at_1}{t_1} = a$$

Similarly, the function $g = f(x+at)$ defines a configuration which moves to the left along the string with constant velocity ***a***. The total displacement of the string is, of course, the algebraic sum of these two traveling waves, see the Fig. below.

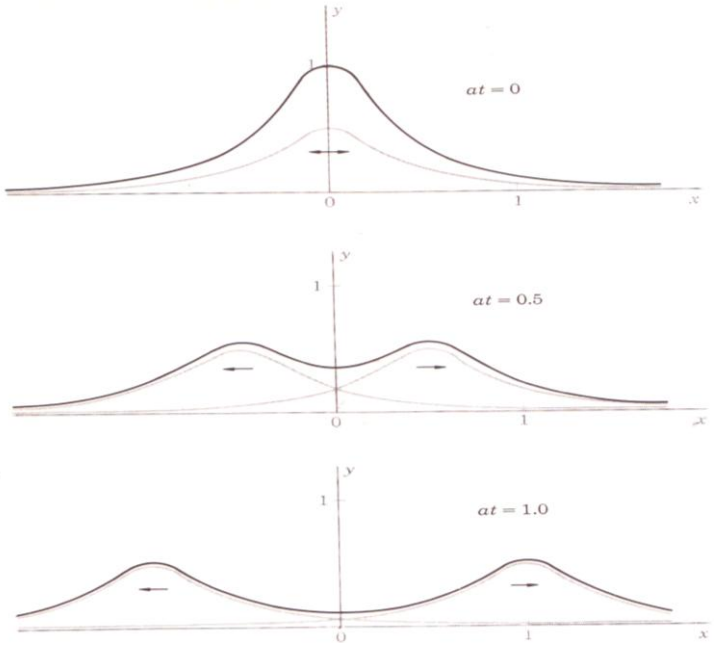


Fig. the propagation of a disturbance along a two-way infinite string

To carry the solution through in detail, let us suppose that the initial displacement of the string at any point ***x*** is given by $\phi(x)$ and that the initial velocity of the string at any point is $\theta(x)$. Then, as conditions to determine the form of ***f*** and ***g***, we have, from (2) and its derivatives with respect to ***t***,

$$y(x,t) = f(x-at) + g(x+at)$$
$$y(x,0) = f(x) + g(x) = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$y_t(x,t) = -a\, f'(x-at) + a\, g'(x+at)$$
$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = -a\, f'(x) + a\, g'(x) = \theta(x)$$

dividing by ***a***, and integrating the last equation with respect to ***t*** we get

$$y(x,0) = -f(x) + g(x) = \frac{1}{a}\int_{x_o}^{x} \theta(x)\, dx \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_{x_o}^{x} \theta(x)\, dx \quad \textit{by adding Eqs. (1) and (2)}$$

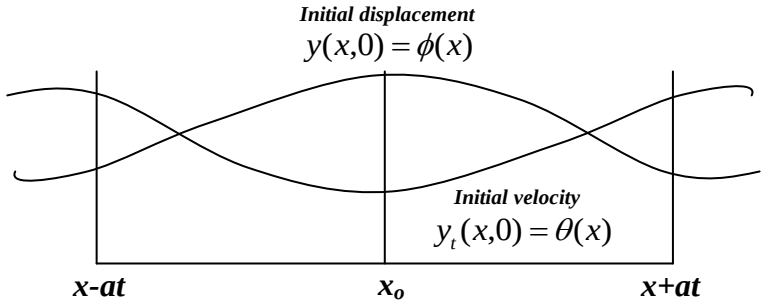
$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_{x_o}^{x} \theta(x)\, dx \quad \textit{by deduction Eqs. (1) and (2)}$$

$$g(x+at)=\frac{1}{2}\phi(x+at)+\frac{1}{2a}\int_{x_o}^{x+at}\theta(x)\,dx$$

$$f(x-at)=\frac{1}{2}\phi(x-at)-\frac{1}{2a}\int_{x_o}^{x-at}\theta(x)\,dx$$

$$y(x,t)=f(x-at)+g(x+at)=\frac{1}{2}\phi(x-at)-\frac{1}{2a}\int_{x_o}^{x-at}\theta(x)\,dx+\frac{1}{2}\phi(x+at)+\frac{1}{2a}\int_{x_o}^{x+at}\theta(x)\,dx$$

$$\boxed{y(x,t)=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\theta(x)\,dx}$$



Example: Use the d'Alembert approach to find the solution of a wave function if the initial value of the displacement is given by $u(x,0)=x^2$ and the initial velocity is $u_t(x,0)=\sin x$?

Solution:

$$u(x,t)=\frac{1}{2}\left[\phi(x+at)+\phi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\theta(x)\,dx$$

$$u(x,t)=\frac{1}{2}\left[(x+at)^2+(x-at)^2\right]+\frac{1}{2a}\int_{x-at}^{x+at}\sin x\,dx$$

$$u(x,t)=\frac{1}{2}\left[x^2+2a\,xt+a^2t^2+x^2-2a\,xt+a^2t^2\right]+\frac{(-1)}{2a}\left[\cos(x+at)-\cos(x-at)\right]$$

$$u(x,t)=\frac{1}{2}\left[2x^2+2a^2t^2\right]+\frac{1}{2a}\left[\cos(x-at)-\cos(x+at)\right]$$

$$\underline{Note}:\sin mx\sin nx=\frac{1}{2}\left[\cos(m-n)x-\cos(m+n)x\right]$$

$$u(x,t)=x^2+a^2t^2+\frac{1}{a}\sin x\sin at$$

1.4 Flow of heat in conducting bodies

1- Heat flows in the direction of decreasing temperature.
2- The rate of heat flow through an area is proportion to the area and to the temperature gradient.

$$\frac{Q}{t} \propto A\frac{\partial u}{\partial x} \Rightarrow$$
$$Q \propto A.t.\frac{\partial u}{\partial x}$$
$$Q = k.A.t.\frac{\partial u}{\partial x}$$

Where ***k*** is the characteristic of the material and it is called the ***heat conductivity***.

3- Quantity of heat lost or gained by a body when its temperature changes is proportional to the mass and temperature change of the body.

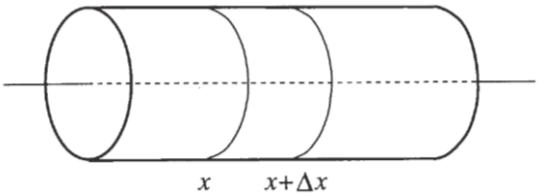
$$\Delta H \propto M\ \Delta u$$

$$\Delta H = C.M.\Delta u$$

Where ***C*** is the heat energy which must be supplied to unit mass of the substance in order to raise it through unit temperature range, which is a constant called the ***specific heat***.

1.4.1 Derivation of the Mathematical model of Partial Differential Equation

Consider a rod composed of a uniform heat-conducting material, with length ***L*** and with uniform cross-sectional area ***A***. It will be assumed that the lateral surface of the rod is insulated and that at any time the temperature in the rod is the same throughout any given cross section but may vary from one cross section to another. Because of these variations in temperature, heat energy will be transported lengthwise along the rod from the hotter parts to the colder parts. Such a unidirectional transfer of heat energy is called a ***one-dimensional heat flow***.



Inflow Heat $Q = -k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x}$

Outflow Heat $Q = -k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}$

The minus sign express the familiar fact that heat flows in the direction of decreasing temperature.

Then the heat loss or gain equals:

$$\Delta H = Q = -k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x} - \left[-k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}\right]$$

$$but \quad \Delta H = C.M.\Delta u = C.\rho.A.\Delta x.\Delta u \quad \Rightarrow$$

$$-k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x} - \left[-k.A.\Delta t.\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}\right] = C.\rho.A.\Delta x.\Delta u$$

$$\left.\begin{matrix}\Delta x \xrightarrow{\lim} 0 \\ \Delta t \xrightarrow{\lim} 0\end{matrix}\right| \frac{k\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right]}{\Delta x} = \left.\begin{matrix}\Delta x \xrightarrow{\lim} 0 \\ \Delta t \xrightarrow{\lim} 0\end{matrix}\right| C.\rho.\frac{\Delta u}{\Delta t}$$

$$f'(x) = \Delta x \xrightarrow{\lim} 0 \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$k\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = C.\rho.\frac{\partial u}{\partial t}$$

$$k\frac{\partial^2 u}{\partial x^2} = C.\rho.\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = \frac{k}{C.\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}}$$

where ***K*** is a positive constant called ***Diffusivity*** of the conducting material.

And from the one-dimensional heat flow equation we can estimate without derivation the two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

And in case of steady state heat flow $\frac{\partial u}{\partial t} = 0$,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ This is called the Laplace equation.

1.4.2 General solution

$$\frac{\partial u}{\partial t} = K\,\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \alpha^2\,\frac{\partial^2 u}{\partial x^2} \quad where\ \alpha^2 = K$$

$$U(x,t) = X_{(x)}.T_{(t)}$$

$$\frac{\partial u}{\partial t} = X.T' \quad and \quad \frac{\partial^2 u}{\partial x^2} = X''.T$$

$$X.T' = \alpha^2.X''.T$$

$$\frac{T'}{T.\alpha^2} = \frac{X''}{X} = \mu$$

Case1: $\mu \succ 0$ is not a solution.

Case2: $\mu = 0$ is not a solution also.

Case3: $\mu \prec 0$

$let\ \mu = -\lambda^2$

$$\frac{T'}{T.\alpha^2} = -\lambda^2 \quad \Rightarrow \quad T' + \lambda^2.\alpha^2.T = 0$$

$$\frac{dT}{dt} = -\lambda^2.\alpha^2.T$$

$$\frac{dT}{T} = -\lambda^2.\alpha^2.dt$$

$$\ln T = -\lambda^2.\alpha^2.t + c$$

$$T = e^{-\lambda^2.\alpha^2.t+c} = e^c.e^{-\lambda^2.\alpha^2.t}$$

$$T = C.e^{-\lambda^2.\alpha^2.t}$$

$$\frac{X''}{X} = -\lambda^2 \quad \Rightarrow \quad X'' = -\lambda^2.X$$

$$X'' + \lambda^2.X = 0$$

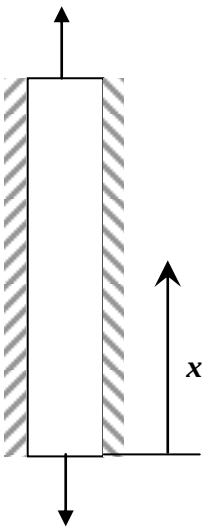
$$m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m_{1,2} = \mp\lambda i$$

$$X = A\cos\lambda x + B\sin\lambda x$$

$$\boxed{U(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right)C.e^{-\lambda^2.\alpha^2.t}}$$

Example: A rod L (cm) long with insulated lateral surface is initially at temperature (100 c^o), if both ends are kept at zero temperature; find the temperature at any point ?

Solution:



$$U(x,t)=\left(A\cos\lambda x+B\sin\lambda x\right)C.e^{-\lambda^2.\alpha^2.t}$$
$$U(x,t)=\left(\overline{A}\cos\lambda x+\overline{B}\sin\lambda x\right)e^{-\lambda^2.\alpha^2.t}$$

Boundary Conditions:

$U(0,t)=0 \quad and \quad U(L,t)=0$

(1) $U(0,t)=(\overline{A}+0)\,e^{-\lambda^2.\alpha^2.t}=0 \quad \Rightarrow \quad \overline{A}=0$

$\therefore \quad U(x,t)=\overline{B}\sin\lambda x\,.\,e^{-\lambda^2.\alpha^2.t}$

(2) $U(L,t)=\overline{B}\sin\lambda L\,.\,e^{-\lambda^2.\alpha^2.t}=0 \quad \Rightarrow \quad \sin\lambda L=0$

$\therefore \quad \lambda L=n\pi \quad \Rightarrow \quad \lambda=\dfrac{n\pi}{L} \quad \Leftrightarrow \quad n=1,2,3,....$

$\therefore \quad U_n=\overline{B}_n\sin\dfrac{n\pi}{L}x.e^{-\alpha^2.\frac{n^2\pi^2}{L^2}.t}$

$\therefore \quad U(x,t)=\sum_{n=1}^{\infty}\overline{B}_n\sin\dfrac{n\pi x}{L}.e^{-\alpha^2.\frac{n^2\pi^2}{L^2}.t}$

Initial Conditions:

$U(x,0)=(U_o)$

(3) $U(x,0)=\sum_{n=1}^{\infty}\overline{B}_n\sin\dfrac{n\pi x}{L}=U_o$

$U_o=f(x)=\sum_{n=1}^{\infty}b_n\sin\dfrac{n\pi x}{L} \quad \Rightarrow \quad \overline{B}_n=b_n$

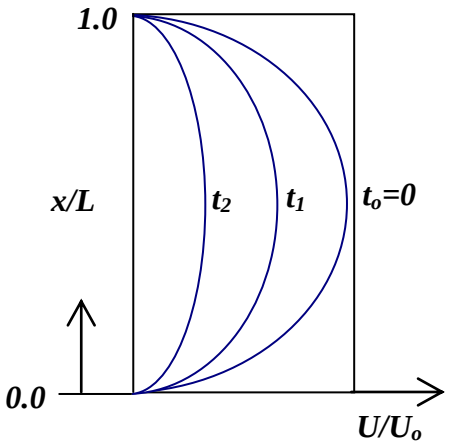
$b_n=\dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx=\dfrac{2U_o}{L}\int_0^L\sin\dfrac{n\pi x}{L}\,dx$

$$b_n=\frac{2U_o}{L}\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L=\frac{2U_o}{n\pi}\left[1-\cos n\pi\right]=\begin{vmatrix}0 & n=even\\ \dfrac{4U_o}{n\pi} & n=Odd\end{vmatrix}$$

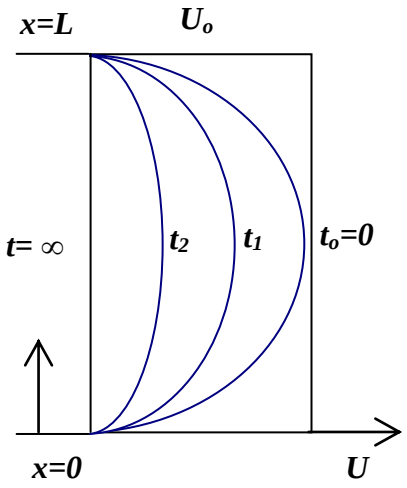
$$U(x,t)=\sum_{n=1}^{\infty}\frac{2U_o}{n\pi}\left(1-\cos n\pi\right)\sin\frac{n\pi x}{L}\,e^{-\alpha^2.\frac{n^2\pi^2}{L^2}.t}$$

Or

$$U(x,t)=\frac{4U_o}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\,e^{-\alpha^2.\frac{n^2\pi^2}{L^2}.t}$$



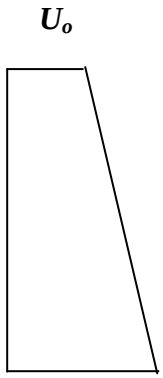
Normalized Chart



Special Case Chart

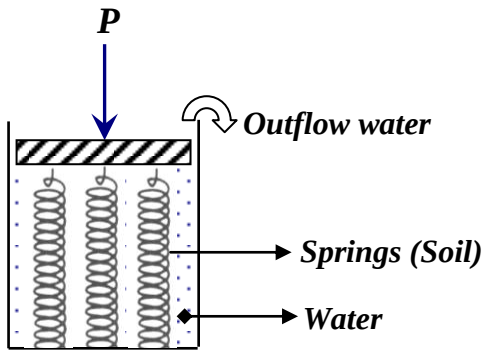
Hint

In case U_o is variable (with respect to x); we should find the related equation that represent the change in U_o with respect to x. ***(i.e):***



$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

1.5 One-Dimensional Consolidation



Section in the Soil

1.5.1 Assumptions:

At any time during the process of consolidation, the amount of settlement is directly related to the proportion of excess pore pressure that has been dissipated. The theory of consolidation is used to predict the progress of excess pore pressure dissipation as a function of time. Therefore, the same theory is also used to predict the rate of consolidation settlement. The one-dimensional theory of Terzaghi is most commonly used for prediction of consolidation settlement rate. The assumptions of the classical Terzaghi theory are as follows:

1. Drainage and compression are one-dimensional.
2. The compressible soil layer is homogenous and completely saturated.
3. The mineral grains and pore water are incompressible.
4. Darcy's law governs the outflow of water from the soil.
5. The applied load increment produces only small strains. Therefore, the thickness of the layer remains unchanged during the consolidation process.
6. The hydraulic conductivity and compressibility of the soil are constant.
7. The relationship between void ratio and vertical effective stress is linear and unique. This assumption also implies that there is no secondary compression settlement.
8. Total stress remains constant throughout the consolidation process.

Theory relates three quantities:

a. excess pore pressure.
b. depth ***z***.
c. time ***t***.