water can escape only through the previous boundary, giving $H_{dr} = H_o$. A double-drained layer is bounded by two pervious strata. Pore water can escape to either boundary, and therefore $H_{dr} = H_o/2$.
The Figure also shows the solution to the equation in terms of the above dimensionless parameters. For a double-drained layer, pore pressure dissipation is modeled using the entire figure. However, for a single drained layer, only the upper or lower half is used. As expected, U_z is zero for all Z at the beginning of the consolidation process ($T = 0$). As time elapses and pore pressures dissipate, U_z gradually increases to 1.0 for all points in the layer and $\sigma'v$ increases accordingly.

From the Figure, it is possible to find the consolidation ratio (and therefore u and σ'_v) at any time t and any position z within the consolidating layer after the start of loading. The time factor T can be calculated from the above equation given the C_v for a particular deposit, the total thickness of the layer, and the boundary drainage conditions.

The Figure also provides some insight as to the progress of consolidation with time. The isochrones (curves of constant T) represent the percent consolidation for a given time throughout the compressible layer. For example, the percent consolidation at the mid-height of a doubly drained layer for T = 0.2 is approximately 23% (see point *A* in the Figure). However, at Z = 0.5, U_z = 44% for the same time factor.
Similarly, near the drainage surfaces at Z = 0.1, the clay is already 86% consolidated. This also means, at that same depth and time, 86% of the original excess pore pressure has dissipated and the effective stress has increased by a corresponding amount.

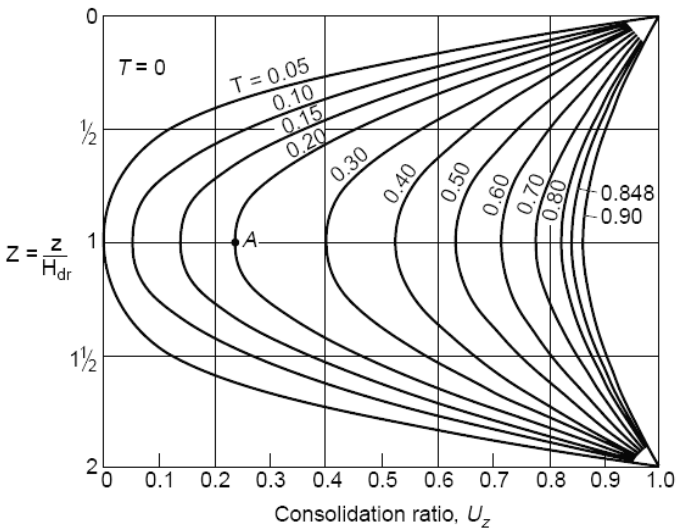
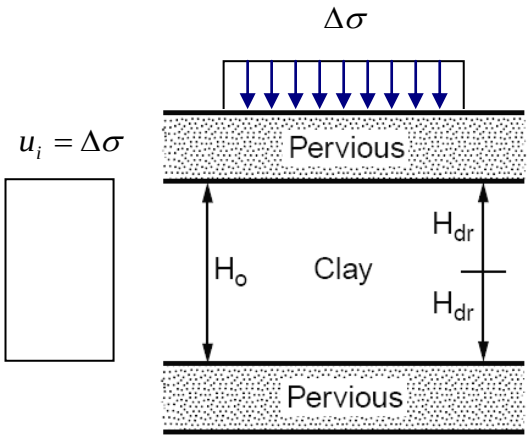


Fig. ( ) Consolidation ratio as a function of Z and T. (*Source:* Taylor, D. W. 1948. *Fundamentals of Soil Mechanics*. John Wiley & Sons, New York.)

1.5.4 Special solution



$$Consolidation\ Ratio\ U_z = \frac{\Delta\sigma'_v}{u_i} = 1 - \frac{u}{u_i}$$

$$U(z,t) = (C\cos\lambda z + D\sin\lambda z)\, e^{-\lambda^2 . C_v . t}$$

Boundary conditions:

$$U(0,t) = 0$$
$$U(2H_{dr},t) = 0$$

Initial conditions:

$$U(z,0) = u_i$$

(1) $U(0,t) = C\, e^{-\lambda^2 . C_v . t} = 0 \quad \Rightarrow \quad C = 0$

(2) $U(2H_{dr},t) = D\sin 2\lambda H_{dr}\, e^{-\lambda^2 . C_v . t} = 0$

$$\sin 2\lambda H_{dr} = 0 \quad \Rightarrow \quad 2\lambda H_{dr} = n\pi$$

$$\lambda = \frac{n\pi}{2H_{dr}} \qquad n = 1,2,3.....$$

$$U_n = D_n \sin\frac{n\pi z}{2H_{dr}} e^{\frac{-n^2\pi^2 . C_v . t}{4H^2_{dr}}}$$

$$U = \sum_{n=1}^{\infty} U_n$$

$$U = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi z}{2H_{dr}} e^{\frac{-n^2\pi^2 . C_v . t}{4H^2_{dr}}}$$

(3) $U(z,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi z}{2H_{dr}} = u_i$

$$u_i = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi z}{2H_{dr}} \quad \Rightarrow$$

$$D_n = b_n = \frac{2}{2H_{dr}} \int_o^{2H_{dr}} u_i \sin\frac{n\pi z}{2H_{dr}} dz$$

$$D_n = \frac{u_i}{H_{dr}} \left[-\frac{2H_{dr}}{n\pi} \cos\frac{n\pi z}{2H_{dr}} \right]_0^{2H_{dr}}$$

$$D_n = \frac{-2u_i}{n\pi}\left[\cos n\pi - 1\right] = \begin{cases} 0 & n = even \\ \dfrac{4u_i}{n\pi} & n = odd \end{cases}$$

$$U = \sum_{n=1,3,5}^{\infty} \frac{4u_i}{n\pi} \sin\frac{n\pi z}{2H_{dr}} e^{\frac{-n^2\pi^2.C_v.t}{4H^2{}_{dr}}}$$

$$for\ \ facilitating\ \ Let\ \ n = 2m+1 \qquad m = 0,1,2,3.....$$

$$U = \sum_{m=0}^{\infty} \frac{4u_i}{(2m+1)\pi} \sin\frac{(2m+1)\pi z}{2H_{dr}} e^{\frac{-(2m+1)^2\pi^2.C_v.t}{4H^2{}_{dr}}}$$

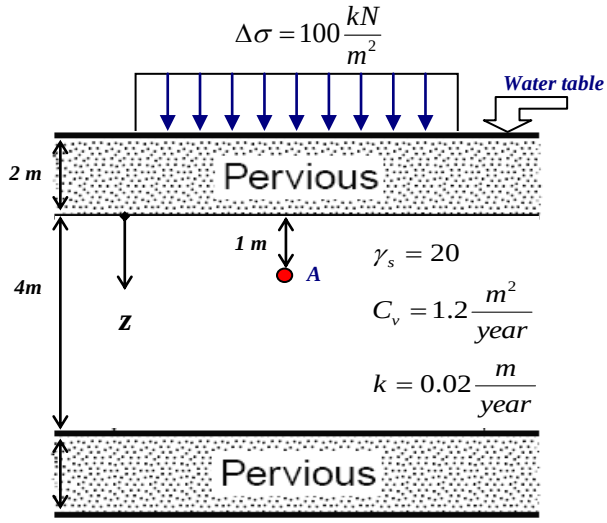
$$Let \quad M = \frac{\pi(2m+1)}{2} \quad \Rightarrow$$

$$U = \sum_{m=0}^{\infty} \frac{2u_i}{M} \sin\frac{M z}{H_{dr}} e^{\frac{-M^2.C_v.t}{H^2{}_{dr}}}$$

$$\boxed{\begin{array}{l} U = \displaystyle\sum_{m=0}^{\infty} \frac{2u_i}{M} \sin\frac{M z}{H_{dr}} e^{-M^2 T} \quad where \quad T = \frac{C_v.t}{H^2{}_{dr}} \\[2ex] U_z = \dfrac{\Delta\sigma'_v}{u_i} = 1 - \dfrac{u}{u_i} \\[2ex] U_z = 1 - \displaystyle\sum_{m=0}^{\infty} \frac{2}{M} \sin\frac{M z}{H_{dr}} e^{-M^2 T} \end{array}}$$

Example: At point ***A*** find after 8 months, (1) the consolidation ratio, (2) excess pore pressure, (3) Actual pore pressure, and (4) actual stress?

Solution:



$$H_{dr} = \frac{4}{2} = 2m$$

$$z = 1.0m$$

$$Z = \frac{z}{H_{dr}} = \frac{1}{2} = 0.5$$

$$T = \frac{C_v.t}{H^2{}_{dr}} = \frac{1.2 \times \frac{8}{12}}{2^2} = 0.2$$

Solution using soil mechanic approach:

(1) From the Chart, applying $Z = 0.5$ *and* $T = 0.2$

$$U_z = \frac{\Delta\sigma' v}{u_i} = 0.44$$

(2) $$\frac{\Delta\sigma' v}{u_i} = 0.44 \quad \Rightarrow \quad \Delta\sigma' v = 0.44 \times 100 = 44\,\frac{kN}{m^2}$$

$$u = 100 - 44 = 56\,\frac{kN}{m^2}$$

$$Or \quad U_z = \frac{\Delta\sigma' v}{u_i} = 1 - \frac{u}{u_i} \quad \Rightarrow \quad \frac{u}{u_i} = 1 - U_z = 1 - 0.44 = 0.56$$

$$u = 0.56 \times 100 = 56\,\frac{kN}{m^2}$$

(3) $$Actual\ U = u + Water\ Head = 56 + (3 \times 10) = 86\,\frac{kN}{m^2}$$

(4) $$\sigma' v = \Delta\sigma' v + Soil\ Weight = 45 + [(\gamma_s - \gamma_w) \times H] = 44 + [(20 - 10) \times 3] = 74\,\frac{kN}{m^2}$$

Solution using Engineering Analysis Approach:

$$U = \sum_{m=0}^{\infty} \frac{2u_i}{M} \sin \frac{M\,z}{H_{dr}} e^{-M^2 T} \quad where \quad T = \frac{C_v .t}{H^2{}_{dr}}$$

$$\frac{u}{u_i} = \sum_{m=0}^{\infty} \frac{2}{M} \sin \frac{M\,z}{H_{dr}} e^{-M^2 T} = \sum_{n=0}^{\infty} \frac{2}{M} \sin(0.5M)\, e^{-M^2 T}$$

m	$M = \frac{(2m+1)}{2}\pi$	$\frac{2}{M}$	$\sin 0.5M$	$e^{-0.2M^2}$	$\frac{u}{u_i}$
0	$\frac{\pi}{2}$	$\frac{4}{\pi}$	0.707	0.6105	0.55
1	$\frac{3\pi}{2}$	$\frac{4}{3\pi}$	0.707	0.0118	0.0054
2	$\frac{5\pi}{2}$	$\frac{4}{5\pi}$	- 0.707	0.0000044	-0.000000792
$\sum$					0.554

$$\frac{u}{u_i} = 0.554 \quad \Rightarrow \quad u = 0.554 \times u_i = 55.4\,\frac{kN}{m^2}$$

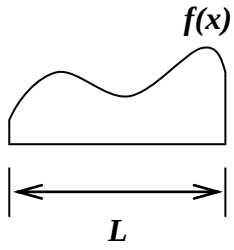
$$\Delta\sigma' v = 100 - 55.4 = 44.6\,\frac{kN}{m^2}$$

$$Actual\ U = u + Water\ Head = 55.4 + (3 \times 10) = 85.4\,\frac{kN}{m^2}$$

$$\sigma' v = \Delta\sigma' v + Soil\ Weight = 44.6 + [(\gamma_s - \gamma_w) \times H] = 44.6 + [(20-10) \times 3] = 74.6\,\frac{kN}{m^2}$$

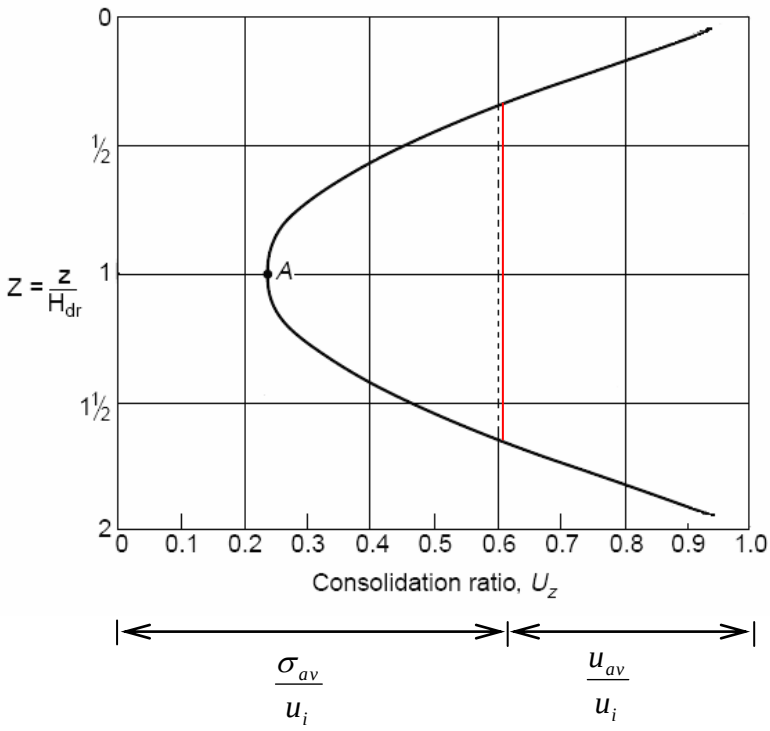
1.5.5 Average Consolidation:

For variable f(x):



$$f(x)_{av} = \frac{\int_0^L f(x)\,dx}{L}$$

$$U = \sum_{m=0}^{\infty} \frac{2u_i}{M} \sin \frac{M\,z}{H_{dr}} e^{-M^2 T}$$



$$U_{av} = \frac{\int U\,dz}{2H_{dr}} \quad \Leftrightarrow \quad \sigma_{av} = u_i - u_{av}$$

$$\sigma_{av} = u_i - \frac{\int U\,dz}{2H_{dr}}$$

$$U_H = \frac{\sigma_{av}}{u_i} = 1 - \frac{\frac{\int U\,dz}{2H_{dr}}}{u_i}$$

$$U_H = 1 - \frac{U_{av}}{u_i} = 1 - \frac{\frac{\int U\,dz}{2H_{dr}}}{u_i}$$

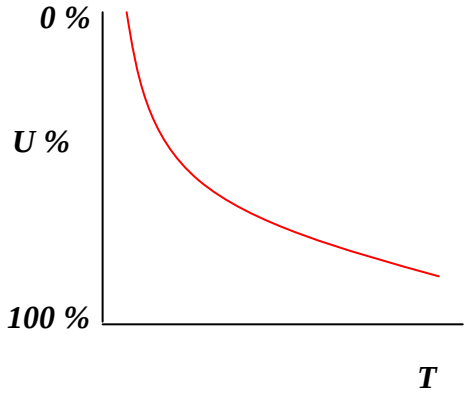
$$\int_0^{2H} U\,dz = \sum_{m=0}^{\infty} \frac{2u_i}{M} e^{-M^2 T} \int_0^{2H_{dr}} \sin \frac{M\,z}{H_{dr}}\,dz$$

$$= -\frac{2u_i H_{dr}}{M^2} . e^{-M^2 T} . \left[\cos \frac{M\,z}{H_{dr}}\right]_0^{2H_{dr}}$$

$$\left[\cos \frac{M\,z}{H_{dr}}\right]_0^{2H_{dr}} = \cos 2M - 1 = \cos(2m+1)\pi - 1 = -2$$

$$\int_0^{2H} U\,dz = \sum_{m=0}^{\infty} \frac{4u_i\,H_{dr}}{M^2}.e^{-M^2 T}$$

$$\frac{\int U\,dz}{2H_{dr}} = \sum_{m=0}^{\infty} \frac{2u_i}{M^2}.e^{-M^2 T}$$



$$U_H = 1 - \frac{U_{av}}{u_i} = 1 - \frac{\frac{\int U\,dz}{2H_{dr}}}{u_i}$$

$$U_H = 1 - \frac{\sum_{m=0}^{\infty} \frac{2u_i}{M^2}.e^{-M^2 T}}{u_i}$$

$$U_H = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2}.e^{-M^2 T}$$

1.6 Classification of partial differential equation

The simplicity and elegance of d'Alembert solution of the wave equation raises the question of whether other partial differential equations can be solved by this method. Let us consider first the possibility of finding solutions of the form $u = f(x+\lambda y)$ for the equation

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x.\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0 \qquad A, B, C \quad \text{Constants,}$$

Substituting our tentative solution, we have

$$A f''(x+\lambda y) + 2B\lambda f''(x+\lambda y) + C\lambda^2 f''(x+\lambda y) = 0$$

Which will be an identify if and only if

$$C\lambda^2 + 2B\lambda + A = 0$$

Thus there are two, one, or no (real) values of λ for which solutions of the form $f(x+\lambda y)$ exists, according as the discriminant $B^2 - AC$ is greater than, equal to, or less than zero. By analogy with the criterion for a conic to be a hyperbola, a parabola, or an ellipse, the equation is said to be a ***hyperbolic***, ***parabolic***, or ***elliptic*** equation according as $B^2 - AC \succ 0$, $B^2 - AC = 0$, *or* $B^2 - AC \prec 0$.

The simplest, and in elementary applications the most important, examples of ***hyperbolic***, ***parabolic***, and ***elliptic*** partial differential equations are respectively,

a. ***the wave equation*** $a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$ ***(hyperbolic)***

b. ***the heat equation*** $\frac{\partial^2 u}{\partial x^2} - a^2\frac{\partial u}{\partial t} = 0$ ***(parabolic)***

c. ***Laplace's equation*** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ***(elliptic)***