

1.7 Further applications

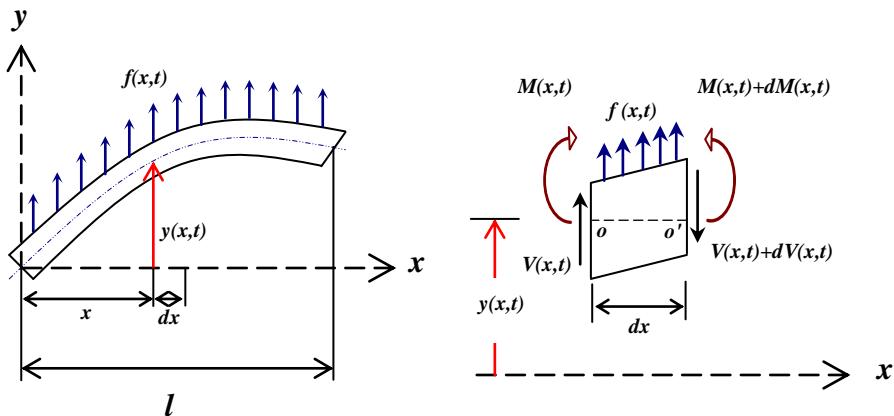
1.7.1 Vibration of Beam

Consider the free body diagram of an element of a beam shown in Fig. below where $M(x,t)$ is the bending moment, $V(x,t)$ is the shearing force, and $f(x,t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x) dx \frac{\partial^2 y}{\partial t^2}(x,t)$$

The force equation in the y direction gives

$$-(V + dV) + f(x,t) dx + V = \rho A(x) dx \frac{\partial^2 y}{\partial t^2}(x,t) \quad \dots \quad (1)$$



Where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the z -axis passing through the point o in the Fig. leads to

$$(M + dM) - (V + dV) dx + f(x,t) dx \frac{dx}{2} - M = 0 \quad \dots \quad (2)$$

By writing

$$dV = \frac{\partial V}{\partial x} dx \quad \text{and} \quad dM = \frac{\partial M}{\partial x} dx$$

And disregarding terms involving second powers in dx , Eqs. (1) and (2) can be written as

$$-\frac{\partial V}{\partial x}(x,t) + f(x,t) = \rho A(x) \frac{\partial^2 y}{\partial t^2}(x,t) \quad \dots \quad (3)$$

$$\frac{\partial M}{\partial x}(x,t) - V(x,t) = 0 \quad \dots \quad (4)$$

By using the relation $V = \frac{\partial M}{\partial x}$, from Eqs. (3) and (4) becomes

$$-\frac{\partial^2 M}{\partial x^2}(x,t) + f(x,t) = \rho A(x) \frac{\partial^2 y}{\partial t^2}(x,t) \quad \dots \quad (5)$$

From the elementary theory of bending of beam (also known as the **Euler-Bernoulli or thin beam theory**), the relationship between bending moment and deflection can be expressed as,

$$M(x,t) = EI(x) \frac{\partial^2 y}{\partial t^2}(x,t) \quad \dots \quad (6)$$

Where E is the Young's modulus and I(x) us the moment of inertia of the beam cross section about the z-axis. Inserting Eq. (6) into Eq. (5) we obtain the equation of motion for the forced vibration of a non-uniform beam:

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 y}{\partial x^2}(x,t) \right] + \rho A(x) \frac{\partial^2 y}{\partial t^2}(x,t) = f(x,t) \quad \dots \quad (7)$$

For a uniform beam Eq. (7) reduces to

$$EI \frac{\partial^4 y}{\partial x^4}(x,t) + \rho A \frac{\partial^2 y}{\partial t^2}(x,t) = f(x,t) \quad \dots \quad (8)$$

For free vibration $f(x,t) = 0$, and so the equation of motion becomes

$$a^2 \frac{\partial^4 y}{\partial x^4}(x,t) + \frac{\partial^2 y}{\partial t^2}(x,t) = 0$$

$$\text{where } a = \sqrt{\frac{EI}{\rho A}}$$

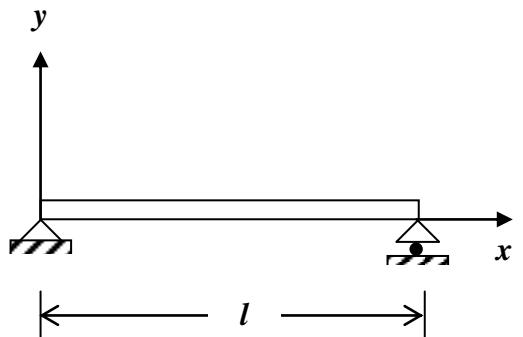
Example: Find the equation of vibration of the beam to satisfy the following initial and boundary conditions?

Solution:

Boundary conditions:

$$y(0,t) = 0 \quad \frac{\partial^2 y}{\partial x^2} \Big|_{0,t} = 0$$

$$y(l,t) = 0 \quad \frac{\partial^2 y}{\partial x^2} \Big|_{l,t} = 0$$



Initial conditions:

$$y(x,0) = f(x)$$

$$\frac{\partial y}{\partial t} \Big|_{x,0} = g(x)$$

Separation of variables:

$$a^2 \frac{\partial^4 y}{\partial x^4} = - \frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = X \cdot T$$

$$\frac{\partial^4 y}{\partial x^4} = X^{(IV)} \cdot T \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = X \cdot T''$$

$$a^2 \frac{X^{(IV)}}{X} = - \frac{T''}{T} = \mu$$

(1) If $\mu = 0$

$$a^2 \frac{X^{(IV)}}{X} = 0$$

$$-\frac{T''}{T} = 0 \Rightarrow T'' = 0 \Rightarrow T = At + B \quad \text{Trivial solution}$$

(2) If $\mu > 0 \Leftrightarrow \mu = \lambda^2$

$$-T'' - \mu T = 0$$

$$T'' + \lambda^2 T = 0$$

$$m^2 + \lambda^2 = 0 \Rightarrow m_{1,2} = \mp \lambda i$$

$$T = E \cos \lambda t + F \sin \lambda t$$

$$a^2 X^{(IV)} - \mu X = 0$$

$$m^4 + \frac{\lambda^2}{a^2} = 0$$

$$(m^2 - \frac{\lambda}{a})(m^2 + \frac{\lambda}{a}) = 0 \Rightarrow m_{1,2} = \mp \sqrt{\frac{\lambda}{a}} \quad m_{3,4} = \mp \sqrt{\frac{\lambda}{a}} i$$

$$X = C_5 e^{\sqrt{\frac{\lambda}{a}}x} + C_6 e^{-\sqrt{\frac{\lambda}{a}}x} + C_3 \cos \sqrt{\frac{\lambda}{a}}x + C_4 \sin \sqrt{\frac{\lambda}{a}}x$$

$$X = C_5 \left(\sinh \sqrt{\frac{\lambda}{a}}x + \cosh \sqrt{\frac{\lambda}{a}}x \right) + C_6 \left(\cosh \sqrt{\frac{\lambda}{a}}x - \sinh \sqrt{\frac{\lambda}{a}}x \right) + C_3 \cos \sqrt{\frac{\lambda}{a}}x + C_4 \sin \sqrt{\frac{\lambda}{a}}x$$

$$X = (C_5 - C_6) \sinh \sqrt{\frac{\lambda}{a}}x + (C_5 + C_6) \cosh \sqrt{\frac{\lambda}{a}}x + C_3 \cos \sqrt{\frac{\lambda}{a}}x + C_4 \sin \sqrt{\frac{\lambda}{a}}x$$

$$X = C_1 \sinh \sqrt{\frac{\lambda}{a}}x + C_2 \cosh \sqrt{\frac{\lambda}{a}}x + C_3 \cos \sqrt{\frac{\lambda}{a}}x + C_4 \sin \sqrt{\frac{\lambda}{a}}x$$

$$y(x,t) = X_{(x)} \cdot T_{(t)}$$

$$y(x,t) = \left(C_1 \sinh \sqrt{\frac{\lambda}{a}}x + C_2 \cosh \sqrt{\frac{\lambda}{a}}x + C_3 \cos \sqrt{\frac{\lambda}{a}}x + C_4 \sin \sqrt{\frac{\lambda}{a}}x \right) \cdot E \cos \lambda t + F \sin \lambda t$$

OR

$$y(x,t) = \left(A \cos \sqrt{\frac{\lambda}{a}}x + B \sin \sqrt{\frac{\lambda}{a}}x + C \sinh \sqrt{\frac{\lambda}{a}}x + D \cosh \sqrt{\frac{\lambda}{a}}x \right) \cdot T_{(t)}$$

Boundary conditions:

$$y(x,t) = \left(A \cos \sqrt{\frac{\lambda}{a}} x + B \sin \sqrt{\frac{\lambda}{a}} x + C \sinh \sqrt{\frac{\lambda}{a}} x + D \cosh \sqrt{\frac{\lambda}{a}} x \right) \cdot T_{(t)}$$

$$\frac{\partial y}{\partial x} = \sqrt{\frac{\lambda}{a}} \left(-A \sin \sqrt{\frac{\lambda}{a}} x + B \cos \sqrt{\frac{\lambda}{a}} x + C \cosh \sqrt{\frac{\lambda}{a}} x + D \sinh \sqrt{\frac{\lambda}{a}} x \right) \cdot T_{(t)}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\lambda}{a} \left(-A \cos \sqrt{\frac{\lambda}{a}} x - B \sin \sqrt{\frac{\lambda}{a}} x + C \sinh \sqrt{\frac{\lambda}{a}} x + D \cosh \sqrt{\frac{\lambda}{a}} x \right) \cdot T_{(t)}$$

$$y(0,t) = (A + D) \cdot T_{(t)} = 0$$

$$A + D = 0 \quad \dots \quad (1)$$

$$\frac{\partial^2 y}{\partial x^2} \Big|_{0,t} = \frac{\lambda}{a} (-A + D) \cdot T_{(t)} = 0$$

$$-A + D = 0 \quad \dots \quad (2)$$

from (1) and (2) $\Rightarrow A = 0$ and $D = 0$

$$y(l,t) = \left(B \sin \sqrt{\frac{\lambda}{a}} l + C \sinh \sqrt{\frac{\lambda}{a}} l \right) \cdot T_{(t)} = 0$$

$$B \sin \sqrt{\frac{\lambda}{a}} l + C \sinh \sqrt{\frac{\lambda}{a}} l = 0 \quad \dots \quad (3)$$

$$\frac{\partial^2 y}{\partial x^2} \Big|_{l,t} = \left(-B \sin \sqrt{\frac{\lambda}{a}} l + C \sinh \sqrt{\frac{\lambda}{a}} l \right) \cdot T_{(t)} = 0$$

$$-B \sin \sqrt{\frac{\lambda}{a}} l + C \sinh \sqrt{\frac{\lambda}{a}} l = 0 \quad \dots \quad (4)$$

adding (3) to (4) $\Rightarrow 2C \sinh \sqrt{\frac{\lambda}{a}} l = 0 \Rightarrow C = 0$

substituting in either (3) or (4) $\Rightarrow B \sin \sqrt{\frac{\lambda}{a}} l = 0$ either $B = 0 \Rightarrow$ trivial solution

$$\sin \sqrt{\frac{\lambda}{a}} l \Rightarrow \sqrt{\frac{\lambda}{a}} l = n\pi \Rightarrow \sqrt{\frac{\lambda}{a}} = \frac{n\pi}{l} \Rightarrow \sqrt{\lambda} = \frac{\sqrt{a} n\pi}{l} \Rightarrow \lambda = \frac{a \cdot n^2 \pi^2}{l^2}$$

$$y_n(x,t) = B_n \sin \frac{n\pi}{l} x \cdot T_{(t)}$$

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \cdot \left(E \cos \frac{a \cdot n^2 \pi^2}{l^2} t + F \sin \frac{a \cdot n^2 \pi^2}{l^2} t \right)$$

$$y(x,t) = \sum_{n=1}^{\infty} \left(\bar{E}_n \cos \frac{a \cdot n^2 \pi^2}{l^2} t + \bar{F}_n \sin \frac{a \cdot n^2 \pi^2}{l^2} t \right) \cdot \sin \frac{n\pi}{l} x$$

Initial conditions:

$$y(x,t) = \sum_{n=1}^{\infty} \left(\bar{E}_n \cos \frac{a \cdot n^2 \pi^2}{l^2} t + \bar{F}_n \sin \frac{a \cdot n^2 \pi^2}{l^2} t \right) \cdot \sin \frac{n\pi}{l} x$$

$$y(x,0) = \sum_{n=1}^{\infty} \bar{E}_n \cdot \sin \frac{n\pi}{l} x = f(x)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-\bar{E}_n \frac{a \cdot n^2 \pi^2}{l^2} \sin \frac{a \cdot n^2 \pi^2}{l^2} t + \bar{F}_n \frac{a \cdot n^2 \pi^2}{l^2} \cos \frac{a \cdot n^2 \pi^2}{l^2} t \right) \cdot \sin \frac{n\pi}{l} x$$

$$\left. \frac{\partial y}{\partial t} \right|_{x,o} = \sum_{n=1}^{\infty} \frac{a \cdot n^2 \pi^2}{l^2} \cdot \bar{F}_n \sin \frac{n\pi}{l} x = g(x)$$

Applying Fourier sin series :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \Rightarrow \bar{E}_n = b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{l} x dx$$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \Rightarrow \frac{a \cdot n^2 \pi^2}{l^2} \bar{F}_n = b_n \Rightarrow \bar{F}_n = \frac{l^2}{a \cdot n^2 \pi^2} b_n = \frac{l^2}{a \cdot n^2 \pi^2} \times \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{l} x dx$$