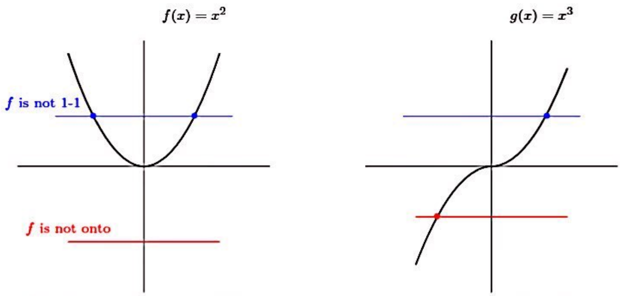


f is neither onto nor 1-1 g is onto and 1-1

Definition 4.3.3

Let $f : A \to B$ be a function. If the **inverse relation** f^{-1} of f is a function, then we say that f^{-1} is the **inverse function** of f. In particular, if f^{-1} is a function, then $f^{-1} : B \to A$ is defined by

$$f^{-1} = \{(y, x) : (x, y) \in f\}.$$

Example 4.3.7

Let $f = \{(1, 2), (4, 2)\}$ be a function. Decide whether f^{-1} is a function.

Solution:

No. Since $f^{-1} = \{(2, 1), (2, 4)\}$ where 2 is mapped to two distinct elements.

Theorem 4.3.3

Let $f : A \to B$ and $g : B \to A$. Then, $g = f^{-1}$ iff $f \circ g = I_B$ and $g \circ f = I_A$, where $I_A : A \to A$ is the **identity function** defined by $I_A(x) = x$ for all $x \in A$.

Example 4.3.8

Let $f(x) = 2x + 1$ and let $g(x) = \dfrac{x-1}{2}$. Show that $g = f^{-1}$.

Solution:

For all $x \in \mathbb{R}$, $(f \circ g)(x) = f(g(x)) = f(\frac{x-1}{2}) = 2\frac{x-1}{2} + 1 = x - 1 + 1 = x = I_{\mathbb{R}}$. Therefore, $g = f^{-1}$.

Theorem 4.3.4

Let $f : A \to B$ be a function. Then,

1. f^{-1} is a function from $\text{Rng}(f)$ to A iff f is one-to-one.
2. If f^{-1} is a function, then f^{-1} is one-to-one.

Proof:

1. " $\Rightarrow$ ": Assume that f^{-1} is a function. Let $f(x) = f(y) = z$, then $(x, z), (y, z) \in f$. Thus, $(z, x), (z, y) \in f^{-1}$. Since f^{-1} is a function, $x = y$. Therefore, f is 1-1.
   " $\Leftarrow$ ": Assume that f is 1-1. Let $(x, y), (x, z) \in f^{-1}$ (we need to show that $y = z$). Then, $(y, x), (z, x) \in f$. Since f is 1-1, $y = z$. Thus, f^{-1} is a function. By Definition 3.1.6, $\text{Dom}(f^{-1}) = \text{Rng}(f)$ and $\text{Rng}(f^{-1}) = \text{Dom}(f)$.
2. Assume that f^{-1} is a function. Let $f^{-1}(x) = f^{-1}(y) = z$, then $(x, z), (y, z) \in f^{-1}$. Thus, $(z, x), (z, y) \in f$ and since f is a function, $x = y$. Therefore, f^{-1} is 1-1.

Definition 4.3.4

A function $f : A \to B$ is called a **1-1 corresponding** or a **bijection** if it is both 1-1 and onto B. In that case, we write $f : A \xrightarrow[onto]{1-1} B$.

Theorem 4.3.5

Let $f : A \xrightarrow[onto]{1-1} B$ and $g : B \xrightarrow[onto]{1-1} C$. Then,

1. $g \circ f : A \xrightarrow[onto]{1-1} C$ is a bijection.
2. $f^{-1} : B \xrightarrow[onto]{1-1} A$ is a bijection.

Proof:

1. By Theorem 4.3.1 and Theorem 4.3.2, if f and g are one-to-one and onto, the composite function $g \circ f$ is also one-to-one and onto.
2. By Theorem 4.3.4, if f is one-to-one, then f^{-1} is a function and hence it is a one-to-one

function. To show that f^{-1} is onto A, let $a \in A$. Then, $f(a) = b \in B$. Thus, $(a, b) \in f$ and hence $(b, a) \in f^{-1}$ and therefore $f^{-1}(b) = a$.

Section 4.4: Images of Sets

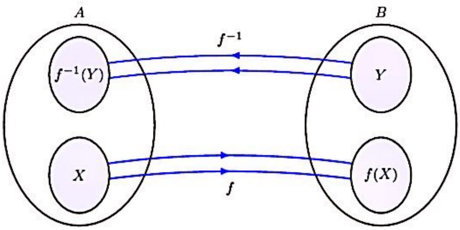
Definition 4.4.1

Let $f : A \to B$. If $X \subseteq A$, the **image of** X or image set of X is

$$f(X) = \{y \in B : y = f(x) \text{ for some } x \in X\}.$$

If $Y \subseteq B$, then the **inverse image of** Y is

$$f^{-1}(Y) = \{x \in A : f(x) = y \text{ for some } y \in Y\}.$$

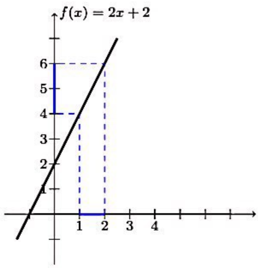


Example 4.4.1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x+2$. Find $f(\{1,4\})$, $f([1,2])$, $f(\mathbb{N})$, $f^{-1}(\{2,3\})$, and $f^{-1}([2,4])$.

Solution:

- $f(\{1,4\}) = \{4,10\}$.
- $f([1,2]) = [4,6]$.
- $f(\mathbb{N}) = \{4,6,8,10,12,\cdots\}$.
- $f^{-1}(\{2,3\}) = \{0,\frac{1}{2}\}$.
- $f^{-1}([2,4]) = [0,1]$.



$f([1,2])$