Lecture Notes

on

Electric Circuit I- (First Semester)

(EE2310)

By

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No.	Subject	Week (4 hrs/w)
1	**Revision for DC Circuits** 1.1- Basic concepts. 1.2- Basic Laws. 1.3- Methods of Analysis. 1.4- Circuit theorems. 1.5- Examples. 1.6- Capacitors and inductors.	1
2	**Operational Amplifiers (Op. Amp.)** 2.1- Introduction to Op. Amp. 2.2- Ideal Op. Amp. 2.3- Inverting Op. Amp. 2.4- Non-inverting Op. Amp. 2.5- Summing Op. Amp. 2.6- Subtracting Op. Amp. 2.6- Cascaded Op. Amp. 2.7- Integrator Op. Amp. 2.8- Differentiator Op. Amp. 2.7- Examples.	2
3	**First-Order Circuits** 3.1- Introduction 3.2- Source-free RC circuit 3.3- Source-free RL circuit 3.4- Step response of RC circuit	3

	3.5- Step response of RL circuit 3.6- First-order Op. Amp. circuit 3.7- Examples	
4	**Second-Order Circuits** 4.1- Introduction 4.2- Source-free series RLC circuit. 4.3- Source-free parallel RLC circuit. 4.4- Step response of series RLC circuit. 4.5- Step response of parallel RLC circuit. 4.6- Second-order Op. Amp. circuit. 4.7- Examples.	**3**
5	**Three-Phase Circuits** 5.1- Introduction 5.2- Balanced three-phase voltages. 5.3- Balanced Wye-Wye connection. 5.4- Balanced Wye-Delta connection. 5.5- Balanced Delta-Delta connection. 5.6- Balanced Delta-Wye connection. 5.7- Power in balanced system. 5.8- Unbalanced three-phase systems. 5.9- Examples.	**3**

References:

- Charles K. Alexander, Matthew N. O. Sadiku "Fundamentals of Electric Circuits" Fifth edition.
- James W. Nilsson, Susan A. Riedel "Electric Circuits" Ninth edition

Chapter One

Revision for DC Circuits

1.1- Basic concepts.

1.2- Basic Laws.

1.3- Methods of Analysis.

1.4- Circuit theorems.

1.5- Examples.

1.6- Capacitors and inductors.

1.1- Basic concepts:

- In electrical engineering, we are often interested in communicating or transferring energy from one point to another.
- To do this, an interconnection of electrical devices is required.
- Such interconnection is referred to as an electric circuit, and each component of the circuit is known as an element.

1.1.1: An electric circuit is an interconnection of electrical elements.

A **simple** electric circuit is shown in Fig. 1.1. It consists of three basic elements: a battery, a lamp, and connecting wires.

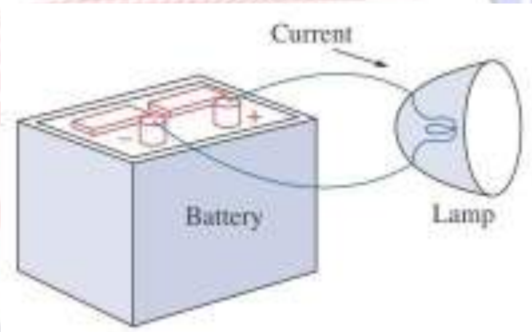


Figure 1.1: A simple electric circuit

A **complicated** real circuit is displayed in Fig. 1.2, representing the schematic diagram for a radio receiver. This circuit can be analyzed using different techniques for describing the behavior of a circuit like this.

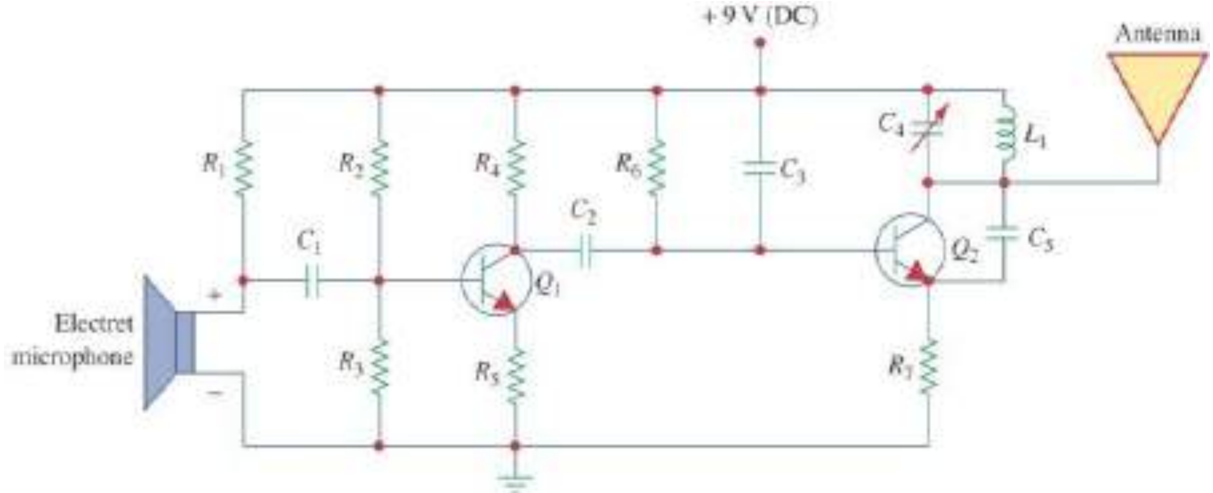


Figure 1.2: Electric circuit of a radio transmitter

As electrical engineers, we deal with measurable quantities. Our measurement, must be communicated in a standard measurement language such as the international system of units (SI) as shown in table 1.1.

TABLE 1.1

Six basic SI units and one derived unit relevant to this text.

Quantity	Basic unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Charge	coulomb	C

Table 1.2 shows the SI prefixes and their symbols.

TABLE 1.2

The SI prefixes.

Multiplier	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

1.1.2: Electric current is the time rate of change of charge measured in amperes (A). Mathematically, the relationship between current i, charge q, and time t is:

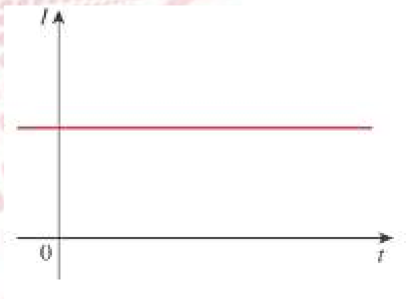
$$\boxed{i \triangleq \frac{dq}{dt}} \tag{1.1}$$

1 ampere = 1 coulomb/second

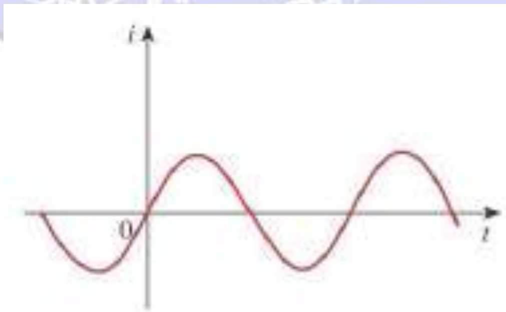
The **charge** transferred between time t_0 and t is obtained by integrating both sides of Eq. (1.1). We obtain

$$\boxed{Q \triangleq \int_{t_0}^{t} i\, dt} \tag{1.2}$$

A **direct current** (DC) is a current that remains constant with time.



An **alternating current** (AC) is a current that varies sinusoidally with time.



1.1.3: Voltage (potential difference) is the energy required to move a unit charge through an element, measured in volts (V). The voltage between two points ***a and b*** in an electric circuit is the energy (or work) needed to move a unit charge from a to b as shown in fig. 1.3; mathematically,

$$v_{ab} \triangleq \frac{dw}{dq} \qquad (1.3)$$

1 volt = 1 joule/coulomb = 1 newton-meter/coulomb

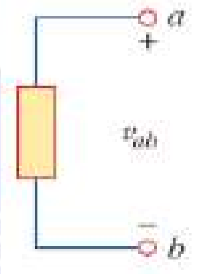


Figure 1.3: voltage across a and b

A **voltage drop** from a to b is equivalent to a voltage rise from **b to a** as shown in fig. 1.4.

$$v_{ab} = -v_{ba} \qquad (1.4)$$

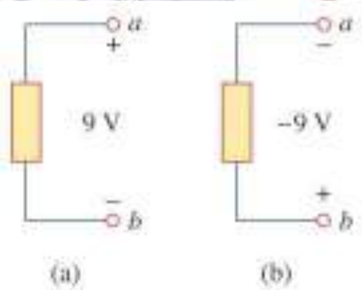


Figure 1.4: (a) Point a is 9 V above point b; (b) point b is -9 V above point a

1.1.4: Power is the time rate of expending or absorbing energy, measured in watts (W), mathematically:

$$\boxed{p \triangleq \frac{dw}{dt}} \qquad (1.5)$$

Where p is power in watts (W), w is energy in joules (J), and t is time in seconds (s). From Eqs. (1.1), (1.3), and (1.5), it follows that

$$p = \frac{dw}{dt} = \frac{dw}{dq} \cdot \frac{dq}{dt} = vi \qquad (1.6)$$

In the case, $p = +vi \ or \ vi > 0$ implies that the element is absorbing power as shown in fig. 1.5 (a). However, if $p = -vi \ or \ vi < 0$ as shown in fig. 1.5 (b), the element is releasing or suppling power.

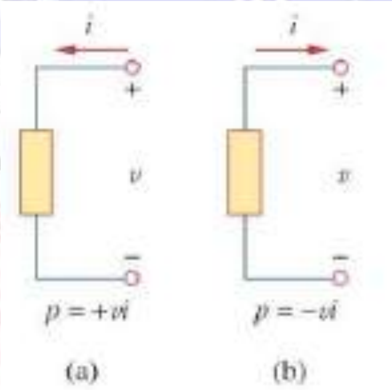


Figure 1.5: (a) absorbing power (b) supplying power

Energy is the capacity to do work, measured in joules (J). From Eq. (1.6), the energy absorbed or supplied by an element from time t_0 to time t is:

$$w = \int_{t_0}^{t} p \, dt = \int_{t_0}^{t} vi \, dt \qquad (1.7)$$

The electric power utility companies measure energy in watt-hours (Wh), where

$$1 \text{ Wh} = 3{,}600 \text{ J}$$

1.1.5 Circuit Elements:

- There are two types of elements found in electric circuits: **passive** elements and **active** elements.
- An active element is capable of generating energy while a passive element is not. Examples of **passive** element are:
  1. Resistors.
  2. Capacitors.
  3. Inductors.

Typical **active** elements include:

1. Generators.
2. Batteries.
3. Operational amplifiers.

- The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them.
- There are two kinds of sources: **independent** and **dependent** sources.

An ideal independent source is an active element that provides a specified voltage or current that is completely independent of other circuit elements as shown in figure (1.6).

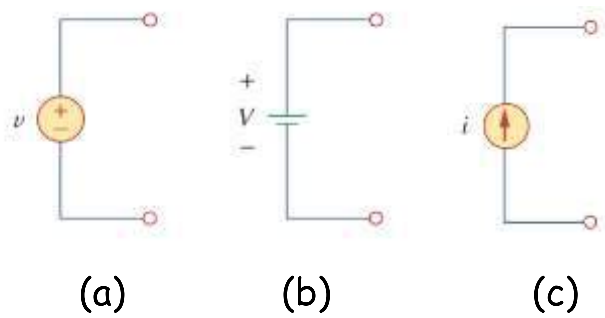


Fig. 1.6: independent sources (a) and (b) voltage, (c) current

An ideal dependent (or controlled) source is an active element in which the source quantity is controlled by another voltage or current as shown in figure 1.7.

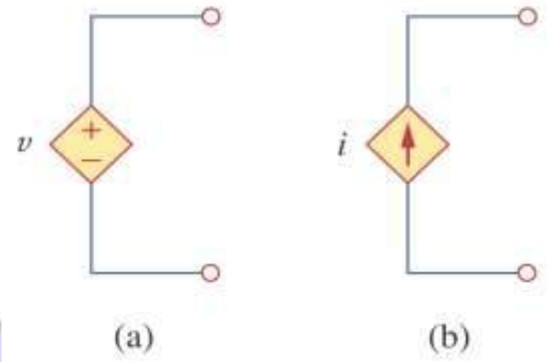


Fig. 1.7: (a) dependent voltage source and (b) dependent current source

1.2- Basic Laws

- A branch represents a single element such as a voltage source or a resistor as shown in figure 1.8.

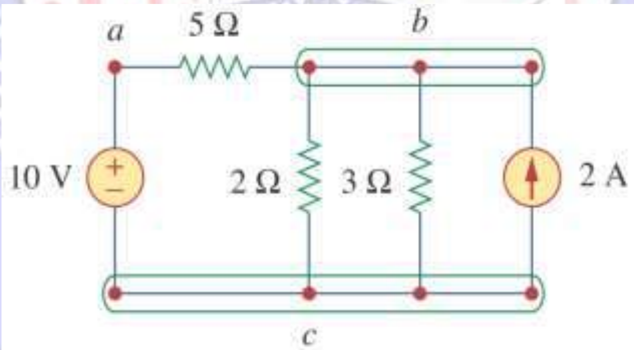


Fig. 1.8: Branches

- Figure 1.8 has 5 branches, namely, the 10-V voltage source, the 2-A current source, and the three resistors.
- A node is the point of connection between two or more branches as shown in figure 1.9.

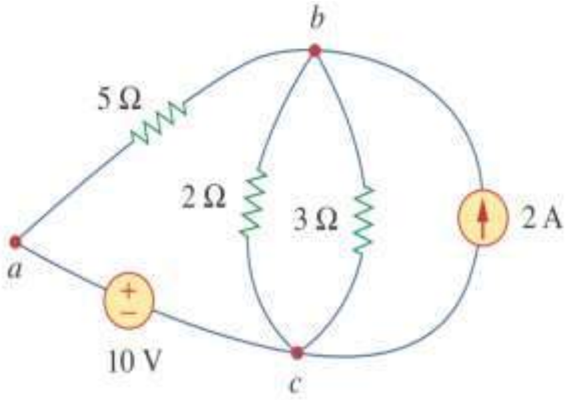


Fig. 1.9: Nodes

- Fig. 1.9 has 3 nodes *a, b, and c.*
- A loop is any closed path in a circuit.
- How many branches and nodes does the circuit in Fig. 1.10 have?

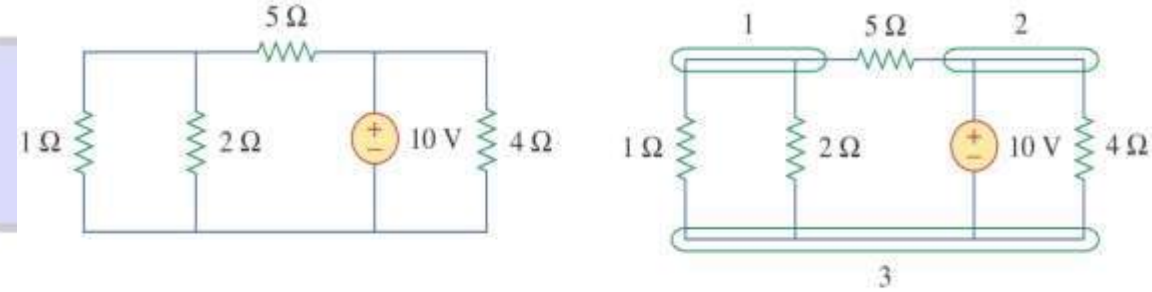


Fig. 1.10: Branches and nodes

1.2.1- Kirchhoff's current law (**KCL**) states that the algebraic sum of currents entering a node (or a closed boundary) is zero as shown in figure 1.11.

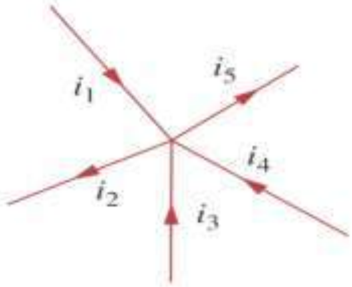


Fig. 1.11: KCL

- Mathematically, KCL in fig. 1.11 implies that;

$$\sum_{n=1}^{N} i_n = 0$$

- Applying KCL gives;

$$i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$$

1.2.2- Kirchhoff's voltage law (**KVL**) states that the algebraic sum of all voltages around a closed path (or loop) is zero as shown in figure 1.12.

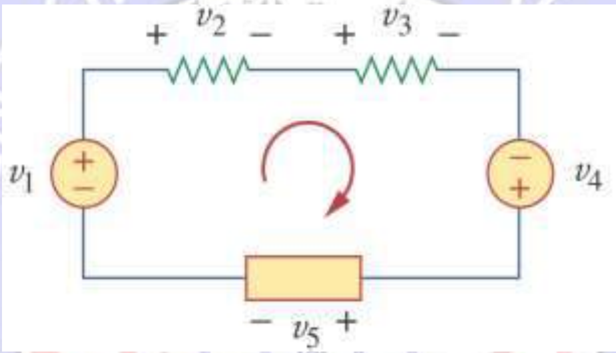


Fig. 1.12: KVL

- Mathematically, KVL states that;

$$\sum_{m=1}^{M} v_m = 0$$

- Applying KVL gives;

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

- For the circuit in Fig. 2.21(a), find voltages v_1 and v_2.

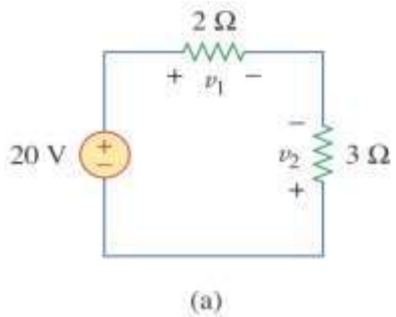


Fig. 2.21: Example 1

Solution:

To find v_1 and v_2, we apply Ohm's law and Kirchhoff's voltage law. Assume that current i flows through the loop as shown in Fig. 2.21(b). From Ohm's law,

$$v_1 = 2i, \qquad v_2 = -3i \tag{2.5.1}$$

Applying KVL around the loop gives

$$-20 + v_1 - v_2 = 0 \tag{2.5.2}$$

Substituting Eq. (2.5.1) into Eq. (2.5.2), we obtain

$$-20 + 2i + 3i = 0 \qquad \text{or} \qquad 5i = 20 \qquad \Rightarrow \qquad i = 4\text{ A}$$

Substituting i in Eq. (2.5.1) finally gives

$$v_1 = 8\text{ V}, \qquad v_2 = -12\text{ V}$$

- Determine v_o and i in the circuit shown in Fig. 2.23(a).

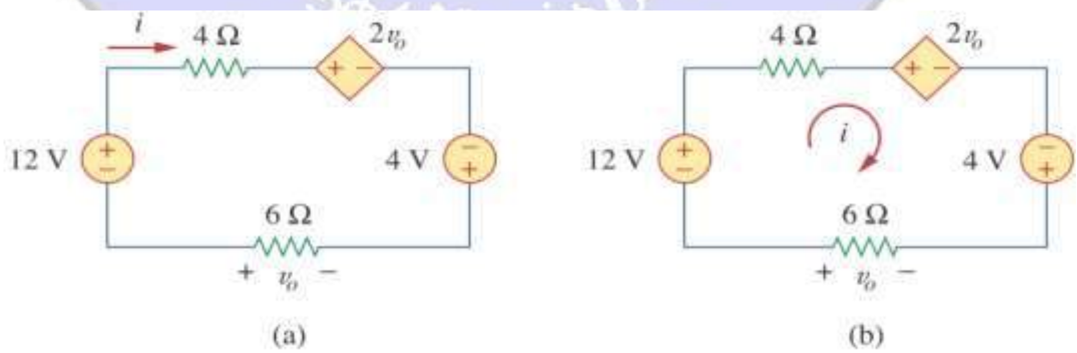


Fig. 2.23: Example 2

Solution:

We apply KVL around the loop as shown in Fig. 2.23(b). The result is

$$-12 + 4i + 2v_o - 4 + 6i = 0 \qquad \textbf{(2.6.1)}$$

Applying Ohm's law to the 6-Ω resistor gives

$$v_o = -6i \qquad \textbf{(2.6.2)}$$

Substituting Eq. (2.6.2) into Eq. (2.6.1) yields

$$-16 + 10i - 12i = 0 \qquad \Rightarrow \qquad i = -8 \text{ A}$$

and $v_o = 48$ V.

- Find currents and voltages in the circuit shown in Fig. 2.27(a).

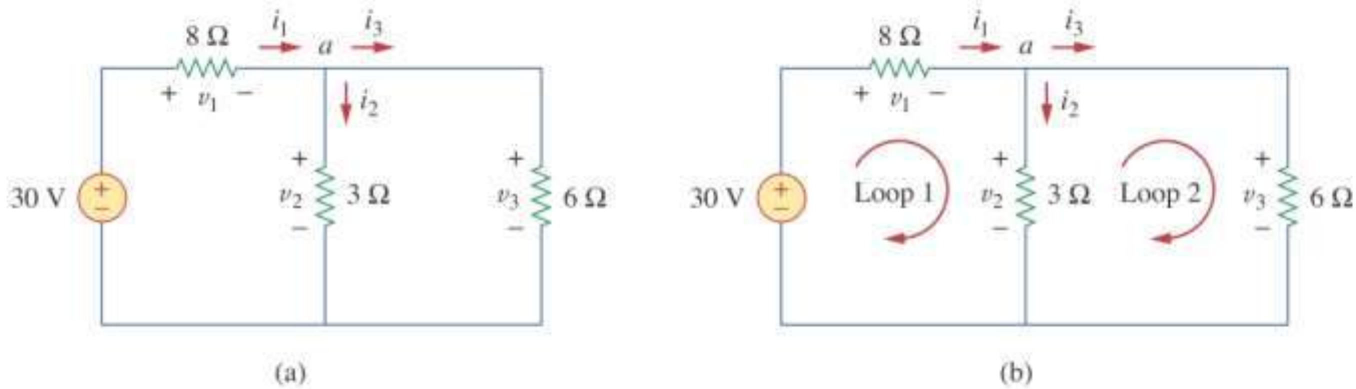


Fig. 2.27: Example 3

Solution:

We apply Ohm's law and Kirchhoff's laws. By Ohm's law,

$$v_1 = 8i_1, \qquad v_2 = 3i_2, \qquad v_3 = 6i_3 \qquad \textbf{(2.8.1)}$$

Since the voltage and current of each resistor are related by Ohm's law as shown, we are really looking for three things: (v_1, v_2, v_3) or (i_1, i_2, i_3). At node a, KCL gives

$$i_1 - i_2 - i_3 = 0 \tag{2.8.2}$$

Applying KVL to loop 1 as in Fig. 2.27(b),

$$-30 + v_1 + v_2 = 0$$

We express this in terms of i_1 and i_2 as in Eq. (2.8.1) to obtain

$$-30 + 8i_1 + 3i_2 = 0$$

or

$$i_1 = \frac{(30 - 3i_2)}{8} \tag{2.8.3}$$

Applying KVL to loop 2,

$$-v_2 + v_3 = 0 \quad \Rightarrow \quad v_3 = v_2 \tag{2.8.4}$$

as expected since the two resistors are in parallel. We express v_1 and v_2 in terms of i_1 and i_2 as in Eq. (2.8.1). Equation (2.8.4) becomes

$$6i_3 = 3i_2 \quad \Rightarrow \quad i_3 = \frac{i_2}{2} \tag{2.8.5}$$

Substituting Eqs. (2.8.3) and (2.8.5) into (2.8.2) gives

$$\frac{30 - 3i_2}{8} - i_2 - \frac{i_2}{2} = 0$$

or $i_2 = 2$ A. From the value of i_2, we now use Eqs. (2.8.1) to (2.8.5) to obtain

$$i_1 = 3 \text{ A}, \qquad i_3 = 1 \text{ A}, \qquad v_1 = 24 \text{ V}, \qquad v_2 = 6 \text{ V}, \qquad v_3 = 6 \text{ V}$$

1.2.3- Series Resistors and Voltage Division:

- Figure 2.29 shows a single-loop circuit with two resistors in series.

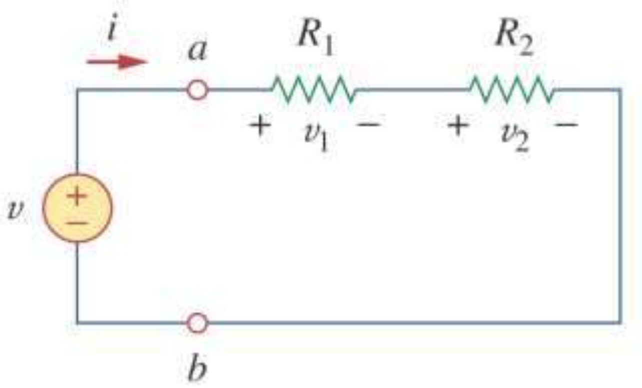


Fig. 2.29: Two resistors in series

- To determine the voltage across each resistor in Fig. 2.29, we use;

$$v_1 = \frac{R_1}{R_1 + R_2}v, \qquad v_2 = \frac{R_2}{R_1 + R_2}v$$

1.2.4- Parallel Resistors and Current Division:

- Figure 2.31 shows the two resistors which are connected in parallel.

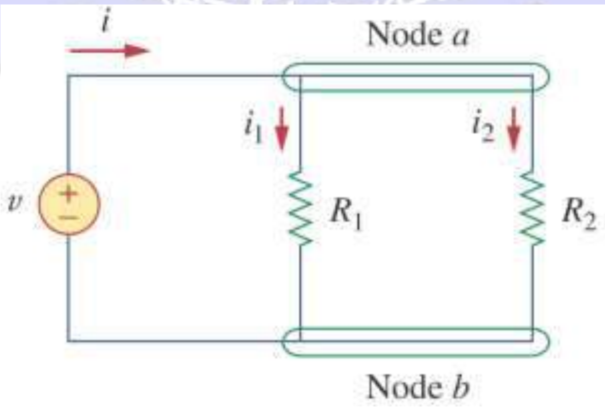


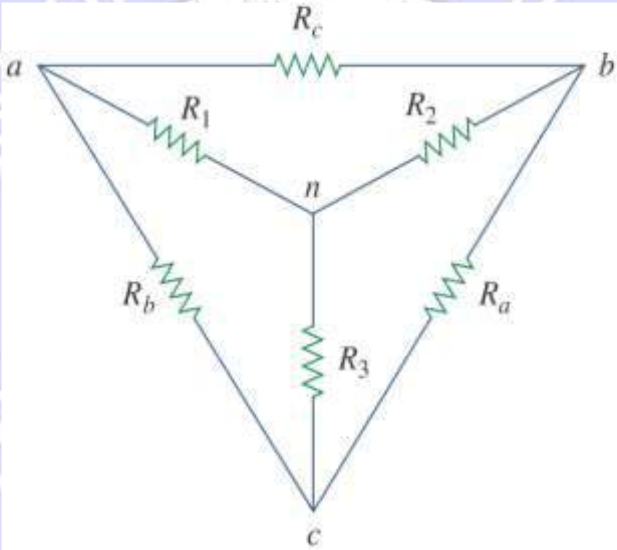
Fig. 2.31: Two resistors in parallel

- To determine the current in each resistor in Fig. 2.29, we use;

$$i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}$$

1.2.5- Delta to Wye Conversion:

- Each resistor in the wye network is the product of the resistors in the two adjacent delta branches, divided by the sum of the three delta resistors.



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

1.2.6- Wye to Delta Conversion:

- Each resistor in the delta network is the sum of all possible products of Y resistors taken two at time, divided by the opposite Y resistor.

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

- Convert the delta network in Fig. 2.50(a) to an equivalent Y network.

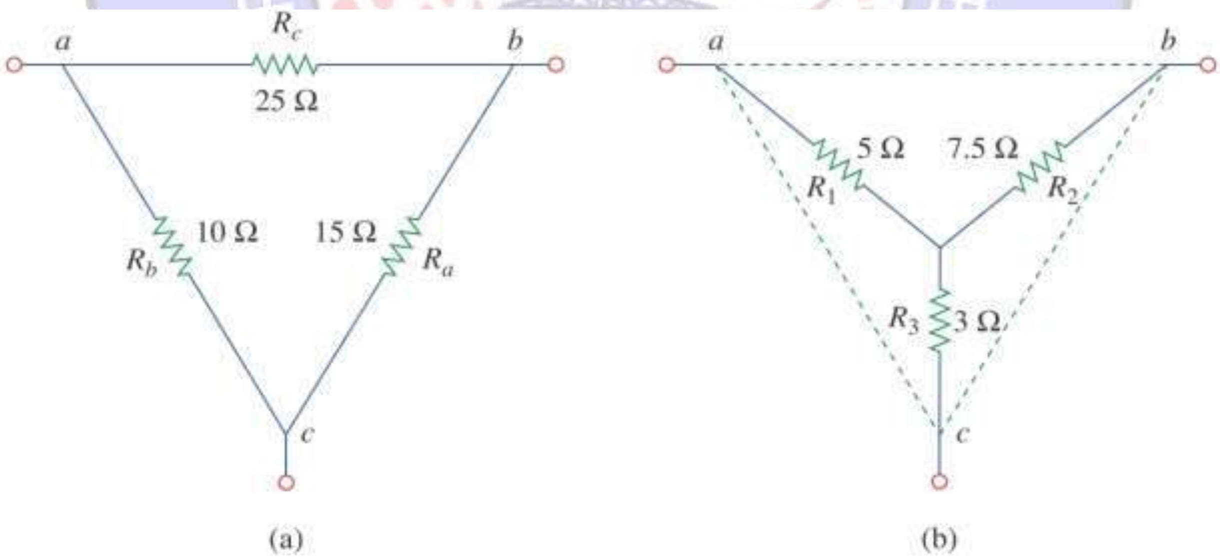


Fig. 2.50: Example 4

Solution:

Using Eqs. (2.49) to (2.51), we obtain

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} = \frac{10 \times 25}{15 + 10 + 25} = \frac{250}{50} = 5\ \Omega$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c} = \frac{25 \times 15}{50} = 7.5\ \Omega$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} = \frac{15 \times 10}{50} = 3\ \Omega$$

The equivalent Y network is shown in Fig. 2.50(b).

1.3- Methods of Analysis

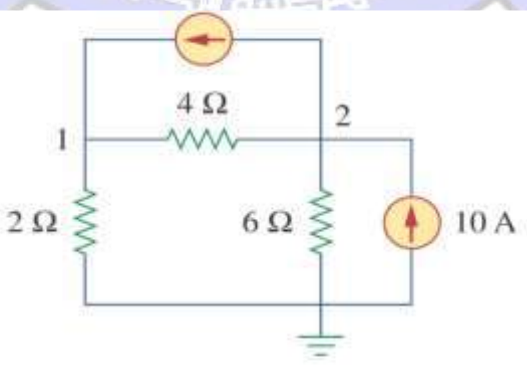
1.3.1- Nodal Analysis:

- Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}} \tag{3.3}$$

- Calculate the node voltages in the circuit shown in Fig. 3.3(a).



(a)

Fig. 3.3(a): Example 5

At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

Multiplying each term in the last equation by 4, we obtain

$$20 = v_1 - v_2 + 2v_1$$

or

$$3v_1 - v_2 = 20 \qquad \textbf{(3.1.1)}$$

At node 2, we do the same thing and get

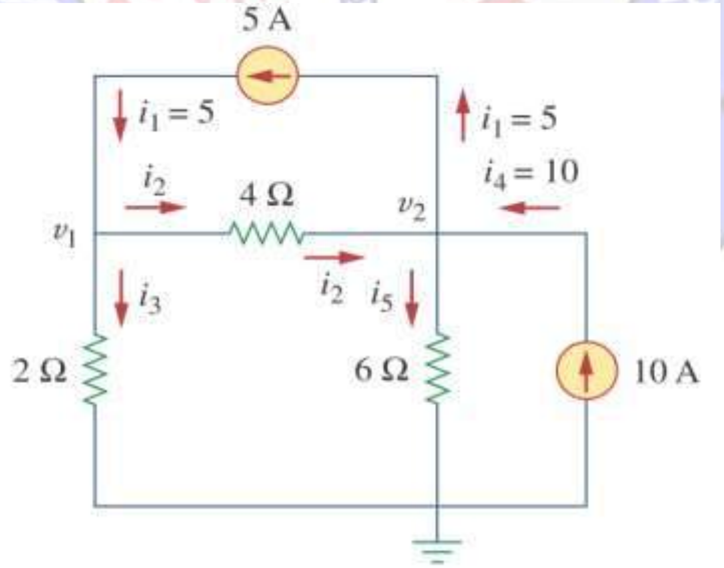
$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

or

$$-3v_1 + 5v_2 = 60 \qquad \textbf{(3.1.2)}$$



(b)

Fig. 3.3(b):

Now we have two simultaneous Eqs. (3.1.1) and (3.1.2). We can solve the equations using any method and obtain the values of v_1 and v_2.

METHOD 1 Using the elimination technique, we add Eqs. (3.1.1) and (3.1.2).

$$4v_2 = 80 \quad \Rightarrow \quad v_2 = 20 \text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \quad \Rightarrow \quad v_1 = \frac{40}{3} = 13.333 \text{ V}$$

METHOD 2 To use Cramer's rule, we need to put Eqs. (3.1.1) and (3.1.2) in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \tag{3.1.3}$$

The determinant of the matrix is

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

We now obtain v_1 and v_2 as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20 \text{ V}$$

giving us the same result as did the elimination method.

If we need the currents, we can easily calculate them from the values of the nodal voltages.

$$i_1 = 5 \text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668 \text{ A}, \qquad i_3 = \frac{v_1}{2} = 6.666 \text{ A}$$

$$i_4 = 10 \text{ A}, \qquad i_5 = \frac{v_2}{6} = 3.333 \text{ A}$$

The fact that i_2 is negative shows that the current flows in the direction opposite to the one assumed.

- **Nodal Analysis with Voltage Sources:**
  A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it as shown in figure. 3.7.

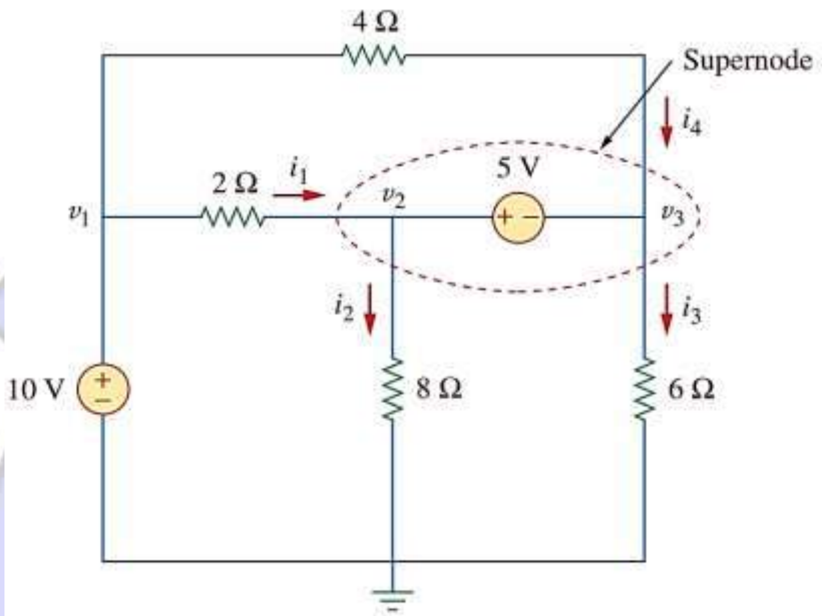


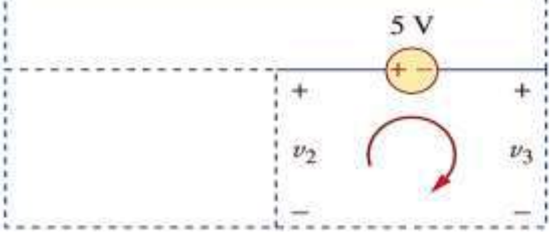
Fig. 3.7: A circuit with a supernode

$$v_1 = 10 \text{ V} \tag{3.10}$$

$$i_1 + i_4 = i_2 + i_3 \tag{3.11a}$$

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6} \tag{3.11b}$$

$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5 \tag{3.12}$$



- For the circuit shown in Fig. 3.9, find the node voltages.

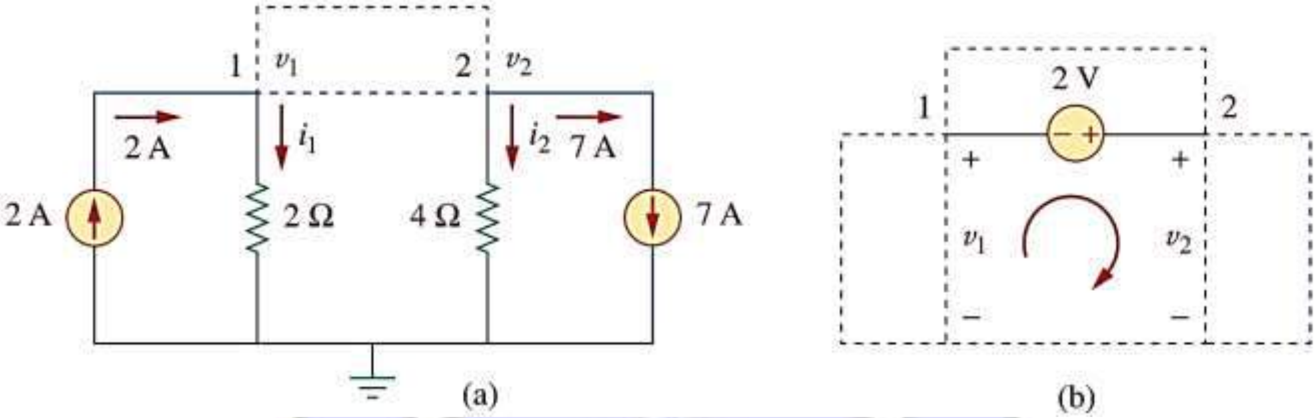


Fig. 3.9: Example 6

Solution:

The supernode contains the 2-V source, nodes 1 and 2, and the 10-Ω resistor. Applying KCL to the supernode as shown in Fig. 3.10(a) gives

$$2 = i_1 + i_2 + 7$$

Expressing i_1 and i_2 in terms of the node voltages

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \quad \Rightarrow \quad 8 = 2v_1 + v_2 + 28$$

or

$$v_2 = -20 - 2v_1 \tag{3.3.1}$$

To get the relationship between v_1 and v_2, we apply KVL to the circuit in Fig. 3.10(b). Going around the loop, we obtain

$$-v_1 - 2 + v_2 = 0 \quad \Rightarrow \quad v_2 = v_1 + 2 \tag{3.3.2}$$

From Eqs. (3.3.1) and (3.3.2), we write

$$v_2 = v_1 + 2 = -20 - 2v_1$$

or

$$3v_1 = -22 \quad \Rightarrow \quad v_1 = -7.333 \text{ V}$$

and $v_2 = v_1 + 2 = -5.333$ V. Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.

1.3.2- Mesh Analysis:

- A mesh is a loop which does not contain any other loops within it.
- For the circuit in Fig. 3.18, find the branch currents and using mesh analysis.

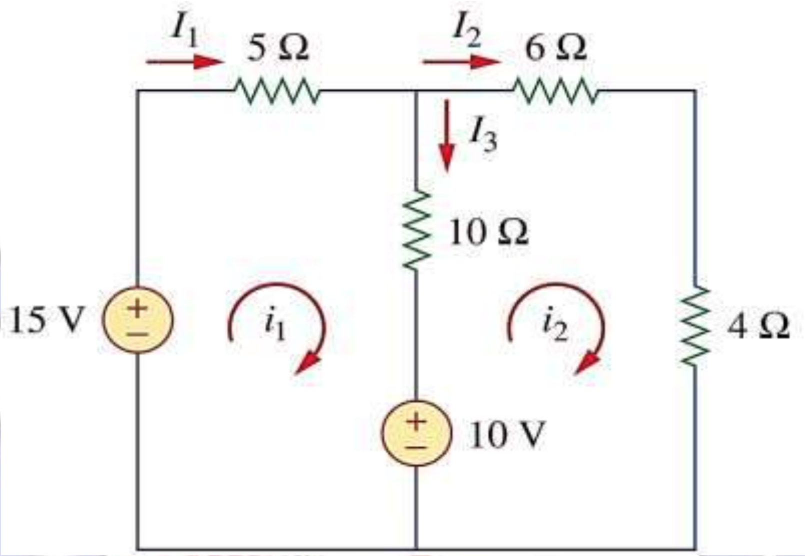


Fig. 3.18: Example 7

Solution:

We first obtain the mesh currents using KVL. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1 \tag{3.5.1}$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$i_1 = 2i_2 - 1 \tag{3.5.2}$$

METHOD 1 Using the substitution method, we substitute Eq. (3.5.2) into Eq. (3.5.1), and write

$$6i_2 - 3 - 2i_2 = 1 \quad \Rightarrow \quad i_2 = 1 \text{ A}$$

From Eq. (3.5.2), $i_1 = 2i_2 - 1 = 2 - 1 = 1$ A. Thus,

$$I_1 = i_1 = 1 \text{ A}, \qquad I_2 = i_2 = 1 \text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

METHOD 2 To use Cramer's rule, we cast Eqs. (3.5.1) and (3.5.2) in matrix form as

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We obtain the determinants

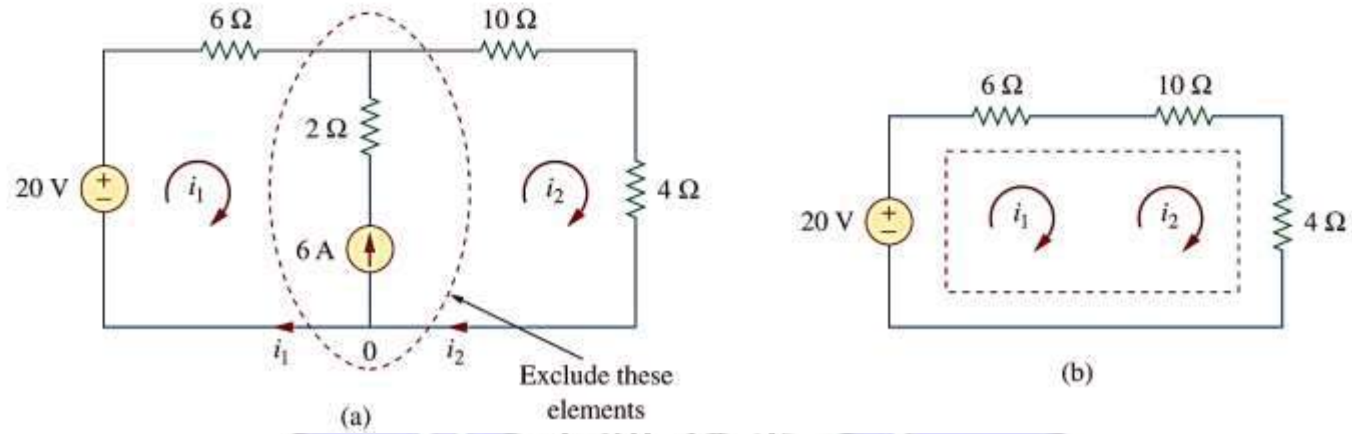
$$\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 2 = 4, \qquad \Delta_2 = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

Thus,

$$i_1 = \frac{\Delta_1}{\Delta} = 1 \text{ A}, \qquad i_2 = \frac{\Delta_2}{\Delta} = 1 \text{ A}$$

- **Mesh Analysis with Current Sources:**
- A supermesh results when two meshes have a (dependent or independent) current source in common as shown below.



$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

$$6i_1 + 14i_2 = 20 \tag{3.18}$$

$$i_2 = i_1 + 6 \tag{3.19}$$

$$i_1 = -3.2\text{ A}, \qquad i_2 = 2.8\text{ A} \tag{3.20}$$

1.4- Circuit Theorems

1.4.1- Superposition Theorem:

- The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

- Use the superposition theorem to find v in the circuit of Fig. 4.6.

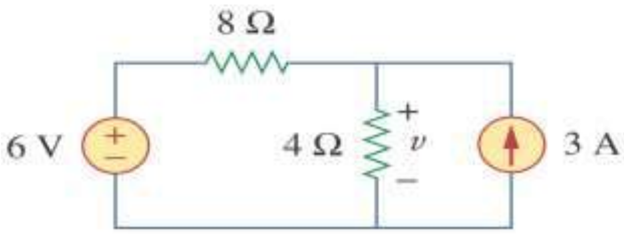


Fig. 4.6: Example 8

Solution:

Since there are two sources, let

$$v = v_1 + v_2$$

where v_1 and v_2 are the contributions due to the 6-V voltage source and the 3-A current source, respectively. To obtain v_1, we set the current source to zero, as shown in Fig. 4.7(a). Applying KVL to the loop in Fig. 4.7(a) gives

$$12i_1 - 6 = 0 \quad \Rightarrow \quad i_1 = 0.5 \text{ A}$$

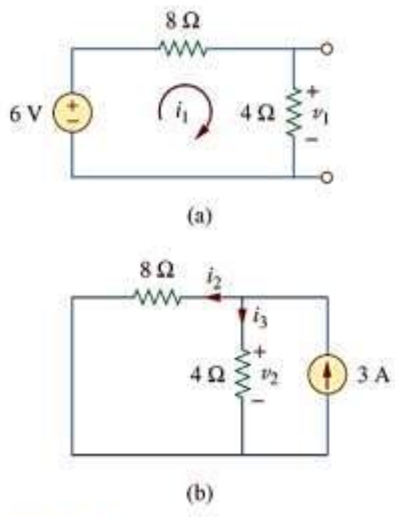


Figure 4.7
For Example 4.3: (a) calculating v_1, (b) calculating v_2.

Thus,

$$v_1 = 4i_1 = 2 \text{ V}$$

We may also use voltage division to get v_1 by writing

$$v_1 = \frac{4}{4+8}(6) = 2 \text{ V}$$

To get v_2, we set the voltage source to zero, as in Fig. 4.7(b). Using current division,

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A}$$

Hence,

$$v_2 = 4i_3 = 8 \text{ V}$$

And we find

$$v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$$

1.4.2- Thevenin's Theorem:

- Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th} , where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.
- Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals a-b. Then find the current through $R_L = 6, 16, and\ 36\ \Omega$.

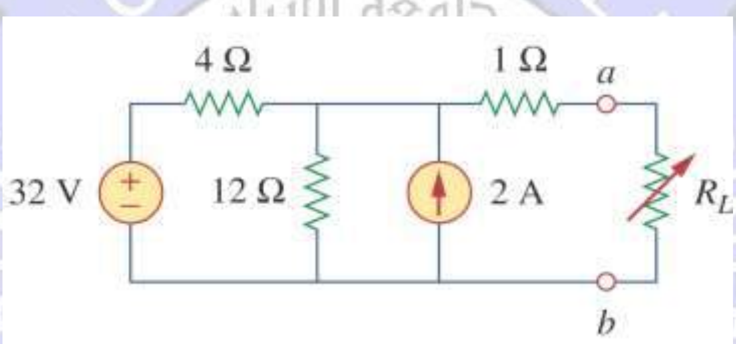


Fig. 4.27: Example 9

Solution:

We find R_{Th} by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

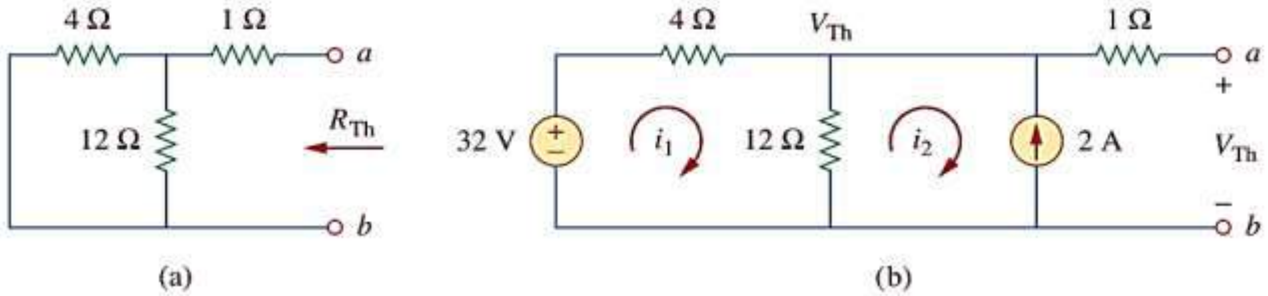


Figure 4.28
For Example 4.8: (a) finding R_{Th}, (b) finding V_{Th}.

To find V_{Th}, consider the circuit in Fig. 4.28(b). Applying mesh analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\text{ A}$$

Solving for i_1, we get $i_1 = 0.5$ A. Thus,

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\text{ V}$$

Alternatively, it is even easier to use nodal analysis. We ignore the 1-Ω resistor since no current flows through it. At the top node, KCL gives

$$\frac{32 - V_{Th}}{4} + 2 = \frac{V_{Th}}{12}$$

or

$$96 - 3V_{\text{Th}} + 24 = V_{\text{Th}} \quad \Rightarrow \quad V_{\text{Th}} = 30 \text{ V}$$

as obtained before. We could also use source transformation to find V_{Th}.

The Thevenin equivalent circuit is shown in Fig. 4.29. The current through R_L is

$$I_L = \frac{V_{\text{Th}}}{R_{\text{Th}} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3 \text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5 \text{ A}$$

When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75 \text{ A}$$

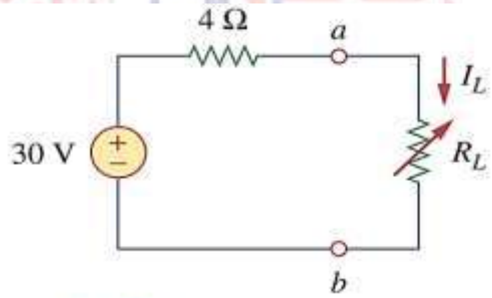


Figure 4.29
The Thevenin equivalent circuit for Example 4.8.

1.4.3- Norton's Theorem:

- Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

$$\boxed{R_N = R_{Th}} \quad (4.9)$$

$$I_N = i_{sc} \quad (4.10)$$

$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}} \quad (4.11)$$

- Find the Norton equivalent circuit of the circuit in Fig. 4.39 at terminals a-b.

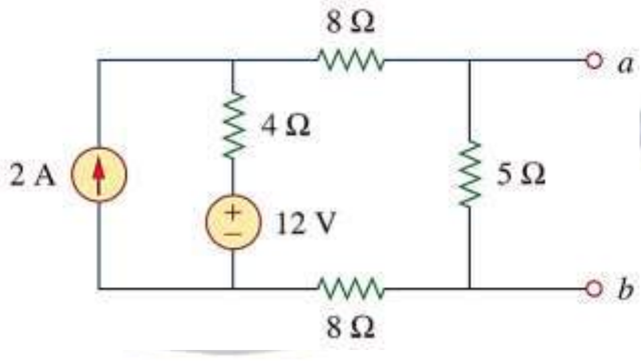


Fig. 4.39: Example 10

Solution:

We find R_N in the same way we find R_{Th} in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find R_N. Thus,

$$R_N = 5 \parallel (8 + 4 + 8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

To find I_N, we short-circuit terminals a and b, as shown in Fig. 4.40(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2\text{ A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1\text{ A} = i_{sc} = I_N$$

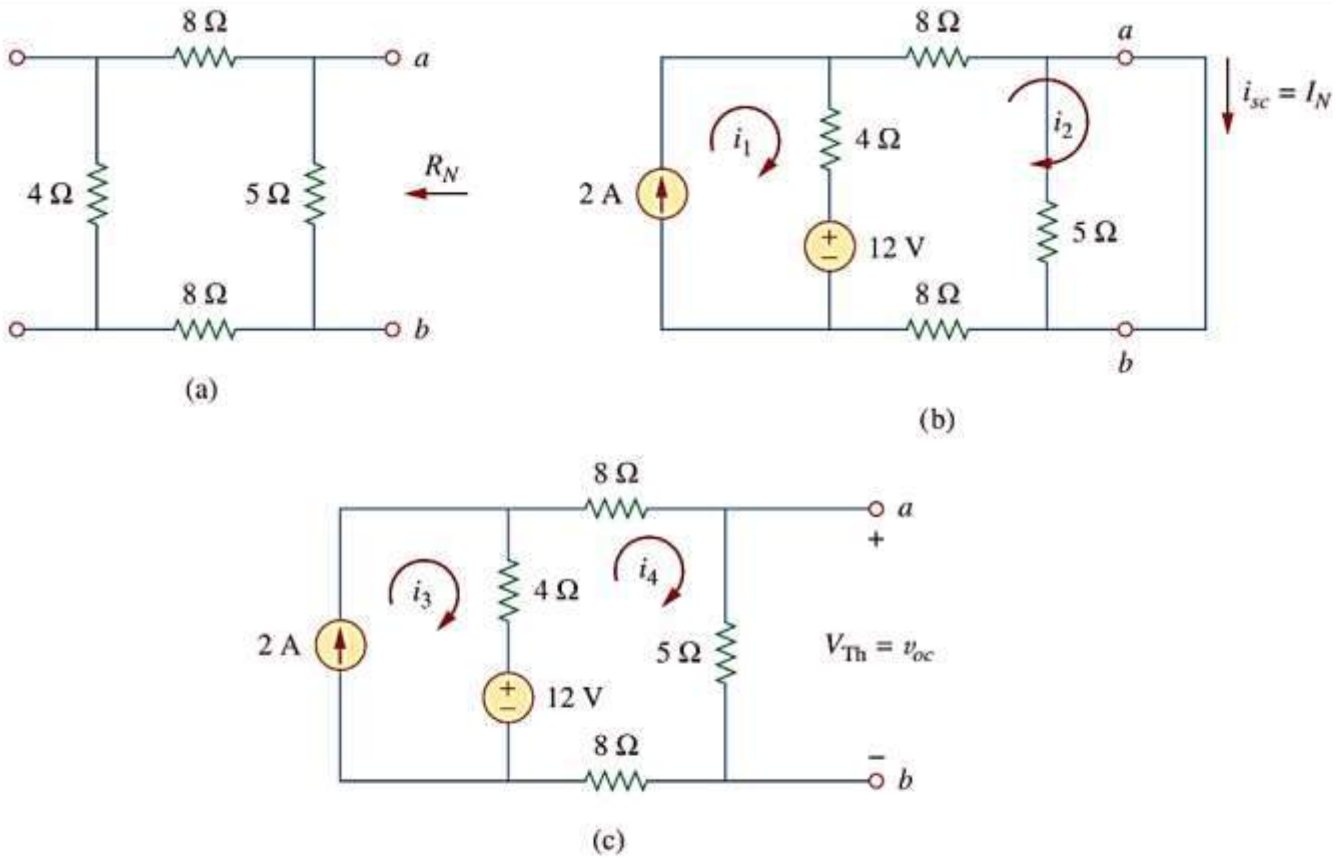


Figure 4.40
For Example 4.11; finding: (a) R_N, (b) $I_N = i_{sc}$, (c) $V_{Th} = v_{oc}$.

Alternatively, we may determine I_N from $V_{\text{Th}}/R_{\text{Th}}$. We obtain V_{Th} as the open-circuit voltage across terminals a and b in Fig. 4.40(c). Using mesh analysis, we obtain

$$i_3 = 2 \text{ A}$$

$$25i_4 - 4i_3 - 12 = 0 \quad \Rightarrow \quad i_4 = 0.8 \text{ A}$$

and

$$v_{oc} = V_{\text{Th}} = 5i_4 = 4 \text{ V}$$

Hence,

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}} = \frac{4}{4} = 1 \text{ A}$$

as obtained previously. This also serves to confirm Eq. (4.12c) that $R_{\text{Th}} = v_{oc}/i_{sc} = 4/1 = 4\ \Omega$. Thus, the Norton equivalent circuit is as shown in Fig. 4.41.

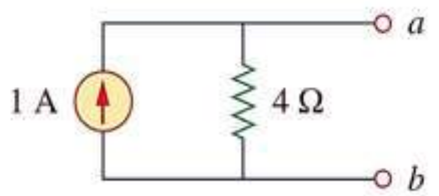
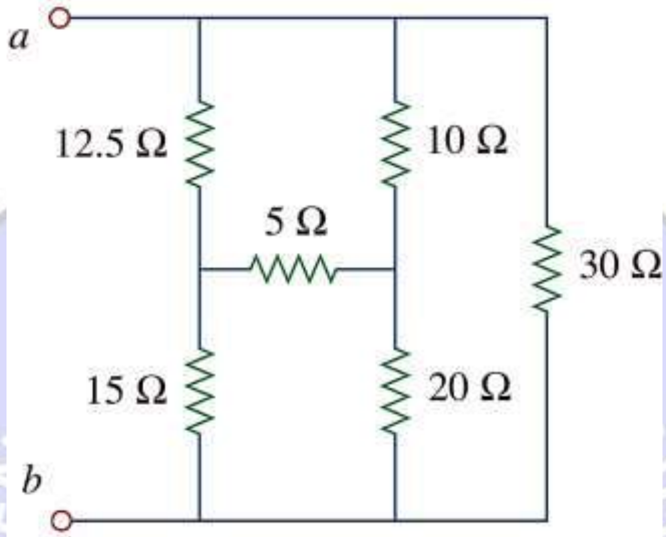


Figure 4.41
Norton equivalent of the circuit in Fig. 4.39.

1.5- Examples:

Example 2.15:

Obtain the equivalent resistance R_{ab} for the circuit in Fig. 2.52



$$R_1 = 10\,\Omega, \qquad R_2 = 20\,\Omega, \qquad R_3 = 5\,\Omega$$

Thus from Eqs. (2.53) to (2.55) we have

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = \frac{10 \times 20 + 20 \times 5 + 5 \times 10}{10}$$

$$= \frac{350}{10} = 35\,\Omega$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} = \frac{350}{20} = 17.5\,\Omega$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = \frac{350}{5} = 70\,\Omega$$

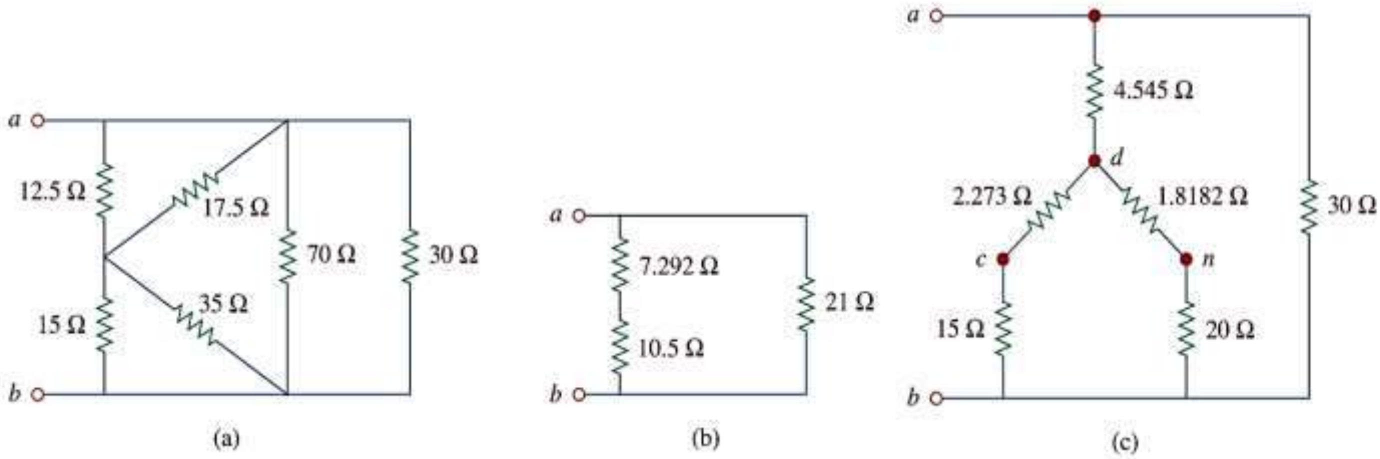


Figure 2.53
Equivalent circuits to Fig. 2.52, with the voltage source removed.

With the Y converted to Δ, the equivalent circuit (with the voltage source removed for now) is shown in Fig. 2.53(a). Combining the three pairs of resistors in parallel, we obtain

$$70 \parallel 30 = \frac{70 \times 30}{70 + 30} = 21\ \Omega$$

$$12.5 \parallel 17.5 = \frac{12.5 \times 17.5}{12.5 + 17.5} = 7.292\ \Omega$$

$$15 \parallel 35 = \frac{15 \times 35}{15 + 35} = 10.5\ \Omega$$

so that the equivalent circuit is shown in Fig. 2.53(b). Hence, we find

$$R_{ab} = (7.292 + 10.5) \parallel 21 = \frac{17.792 \times 21}{17.792 + 21} = \mathbf{9.632\ \Omega}$$

Example 3.4:

Find the node voltages in the circuit of Fig. 3.12.

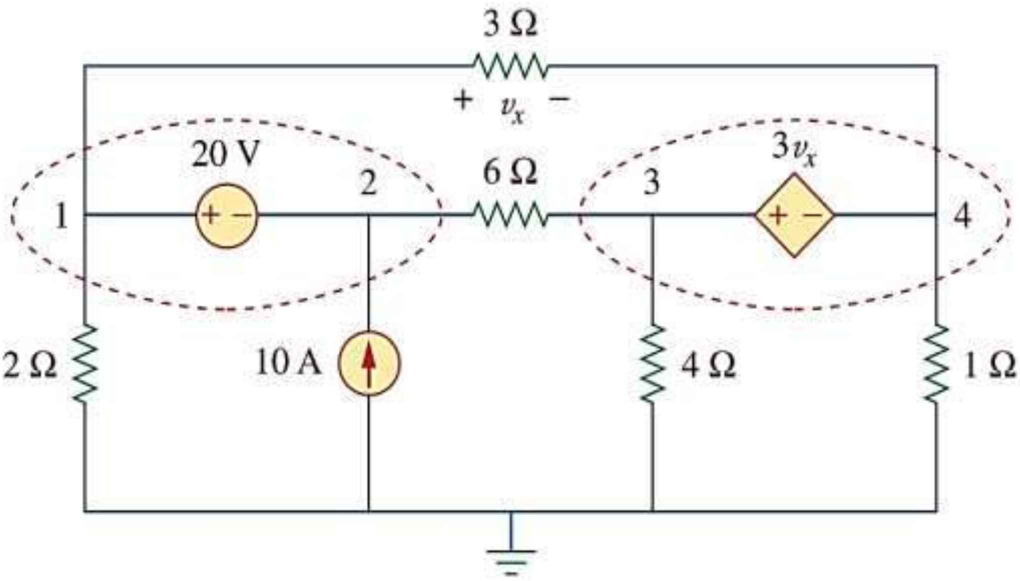


Figure 3.12
For Example 3.4.

Solution:

Nodes 1 and 2 form a supernode; so do nodes 3 and 4. We apply KCL to the two supernodes as in Fig. 3.13(a). At supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

Expressing this in terms of the node voltages,

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

or

$$5v_1 + v_2 - v_3 - 2v_4 = 60 \qquad \textbf{(3.4.1)}$$

At supernode 3-4,

$$i_1 = i_3 + i_4 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

or

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \qquad (3.4.2)$$

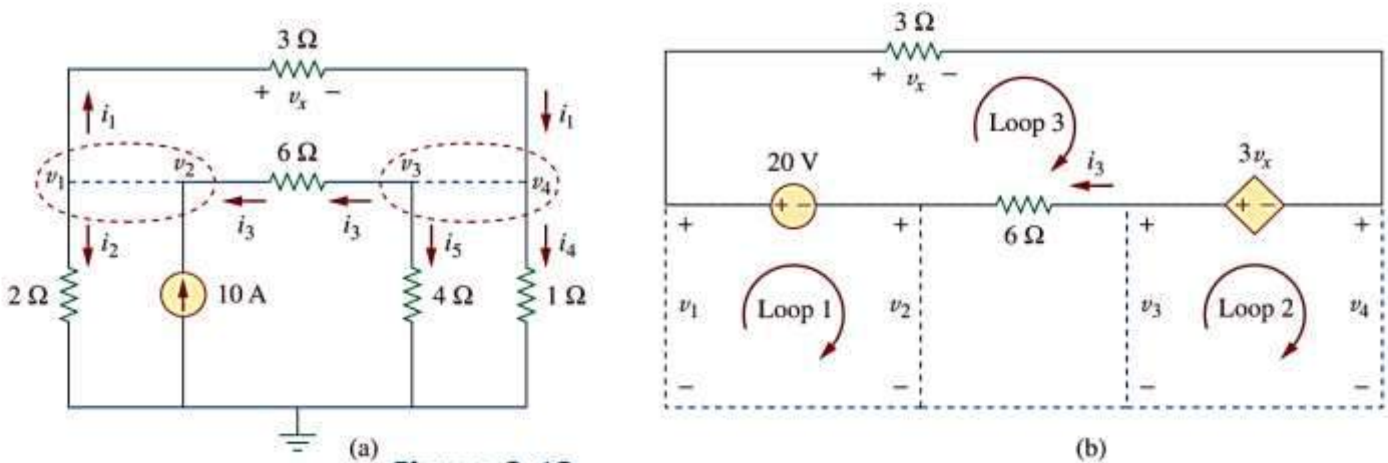


Figure 3.13
Applying: (a) KCL to the two supernodes, (b) KVL to the loops

We now apply KVL to the branches involving the voltage sources as shown in Fig. 3.13(b). For loop 1,

$$-v_1 + 20 + v_2 = 0 \quad \Rightarrow \quad v_1 - v_2 = 20 \qquad (3.4.3)$$

For loop 2,

$$-v_3 + 3v_x + v_4 = 0$$

But $v_x = v_1 - v_4$ so that

$$3v_1 - v_3 - 2v_4 = 0 \qquad (3.4.4)$$

For loop 3,

$$v_x - 3v_x + 6i_3 - 20 = 0$$

But $6i_3 = v_3 - v_2$ and $v_x = v_1 - v_4$. Hence,

$$-2v_1 - v_2 + v_3 + 2v_4 = 20 \qquad (3.4.5)$$

We need four node voltages, v_1, v_2, v_3, and v_4, and it requires only four out of the five Eqs. (3.4.1) to (3.4.5) to find them. Although the fifth equation is redundant, it can be used to check results. We can solve Eqs. (3.4.1) to (3.4.4) directly using *MATLAB*. We can eliminate one node voltage so that we solve three simultaneous equations instead of four. From Eq. (3.4.3), $v_2 = v_1 - 20$. Substituting this into Eqs. (3.4.1) and (3.4.2), respectively, gives

$$6v_1 - v_3 - 2v_4 = 80 \qquad \textbf{(3.4.6)}$$

and

$$6v_1 - 5v_3 - 16v_4 = 40 \qquad \textbf{(3.4.7)}$$

Equations (3.4.4), (3.4.6), and (3.4.7) can be cast in matrix form as

$$\begin{bmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 40 \end{bmatrix}$$

Using Cramer's rule gives

$$\Delta = \begin{vmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{vmatrix} = -18, \qquad \Delta_1 = \begin{vmatrix} 0 & -1 & -2 \\ 80 & -1 & -2 \\ 40 & -5 & -16 \end{vmatrix} = -480,$$

$$\Delta_3 = \begin{vmatrix} 3 & 0 & -2 \\ 6 & 80 & -2 \\ 6 & 40 & -16 \end{vmatrix} = -3120, \qquad \Delta_4 = \begin{vmatrix} 3 & -1 & 0 \\ 6 & -1 & 80 \\ 6 & -5 & 40 \end{vmatrix} = 840$$

Thus, we arrive at the node voltages as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{-480}{-18} = 26.67 \text{ V}, \qquad v_3 = \frac{\Delta_3}{\Delta} = \frac{-3120}{-18} = 173.33 \text{ V},$$

$$v_4 = \frac{\Delta_4}{\Delta} = \frac{840}{-18} = -46.67 \text{ V}$$

and $v_2 = v_1 - 20 = 6.667$ V. We have not used Eq. (3.4.5); it can be used to cross check results.

Example 3.7:

For the circuit in Fig. 3.24, find i_1 to i_4 using mesh analysis

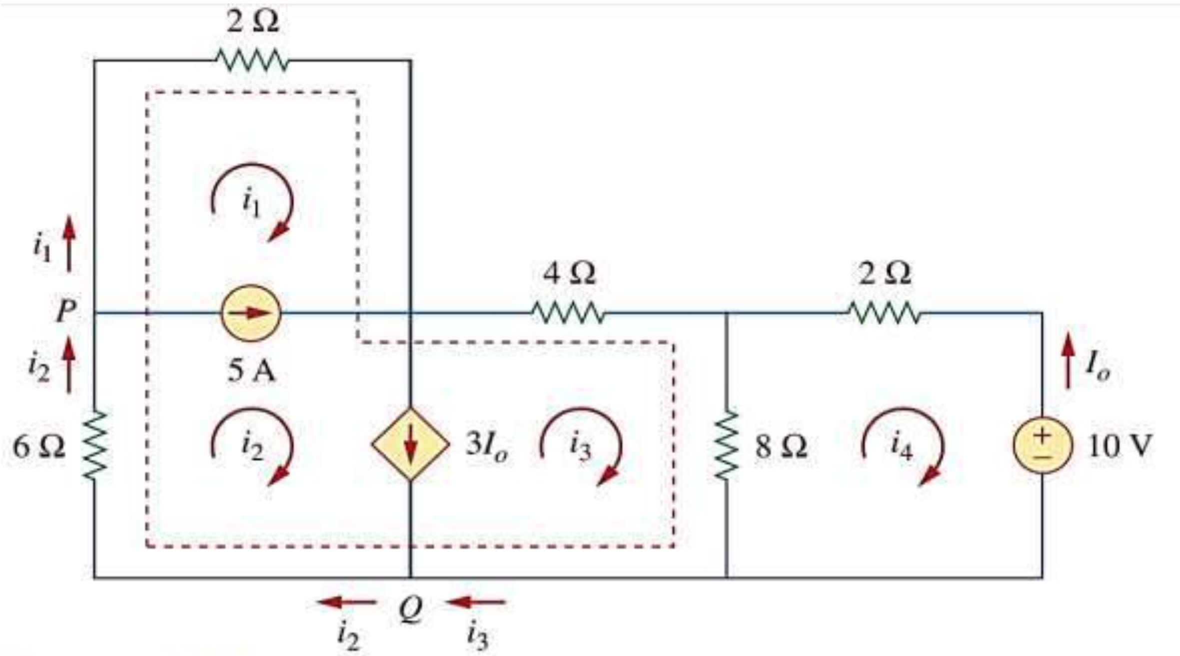


Figure 3.24
For Example 3.7.

Solution:

Note that meshes 1 and 2 form a supermesh since they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common. The two supermeshes intersect and form a larger supermesh as shown. Applying KVL to the larger supermesh,

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

or

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \tag{3.7.1}$$

For the independent current source, we apply KCL to node P:

$$i_2 = i_1 + 5 \tag{3.7.2}$$

For the dependent current source, we apply KCL to node Q:

$$i_2 = i_3 + 3I_o$$

But $I_o = -i_4$, hence,

$$i_2 = i_3 - 3i_4 \tag{3.7.3}$$

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

or

$$5i_4 - 4i_3 = -5 \tag{3.7.4}$$

From Eqs. (3.7.1) to (3.7.4),

$$i_1 = -7.5\text{ A}, \qquad i_2 = -2.5\text{ A}, \qquad i_3 = 3.93\text{ A}, \qquad i_4 = 2.143\text{ A}$$

Example 4.12:

Using Norton's theorem, find R_N and I_N of the circuit in Fig. 4.43 at terminals *a-b*.

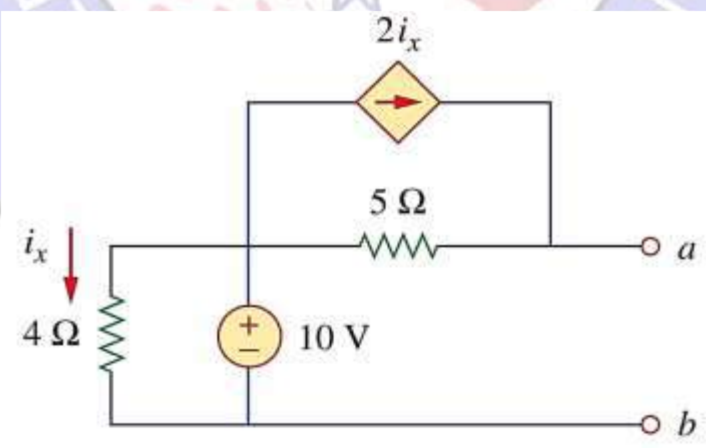


Figure 4.43
For Example 4.12.

Solution:

To find R_N, we set the independent voltage source equal to zero and connect a voltage source of $v_o = 1$ V (or any unspecified voltage v_o) to the terminals. We obtain the circuit in Fig. 4.44(a). We ignore the 4-Ω resistor because it is short-circuited. Also due to the short circuit, the 5-Ω resistor, the voltage source, and the dependent current source are all in parallel. Hence, $i_x = 0$. At node a, $i_o = \frac{1v}{5\Omega} = 0.2$ A, and

$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

To find I_N, we short-circuit terminals a and b and find the current i_{sc}, as indicated in Fig. 4.44(b). Note from this figure that the 4-Ω resistor, the 10-V voltage source, the 5-Ω resistor, and the dependent current source are all in parallel. Hence,

$$i_x = \frac{10}{4} = 2.5\ \text{A}$$

At node a, KCL gives

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\ \text{A}$$

Thus,

$$I_N = 7\ \text{A}$$

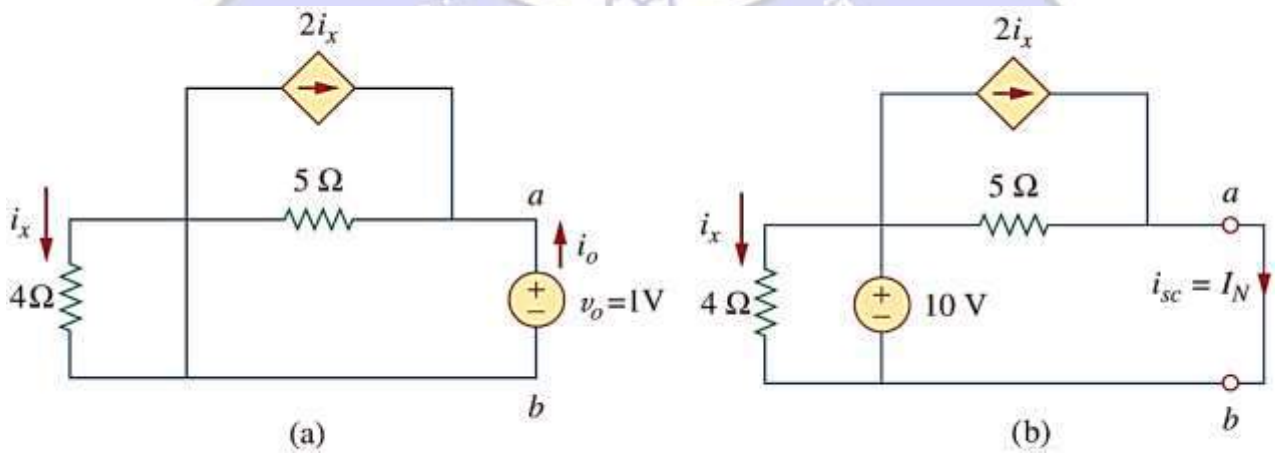


Figure 4.44
For Example 4.12: (a) finding R_N, (b) finding I_N.

1.6- Capacitors and Inductors

- A capacitor is a passive element designed to store energy in its electric field.
- The amount of charge stored, represented by q, is directly proportional to the applied voltage v as shown in figure 6.2.

$$\boxed{q = Cv} \qquad (6.1)$$

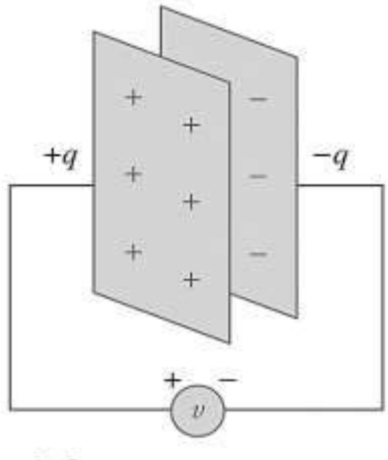


Figure 6.2
A capacitor with applied voltage v.

- To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$i = \frac{dq}{dt} \qquad (6.3)$$

differentiating both sides of Eq. (6.1) gives

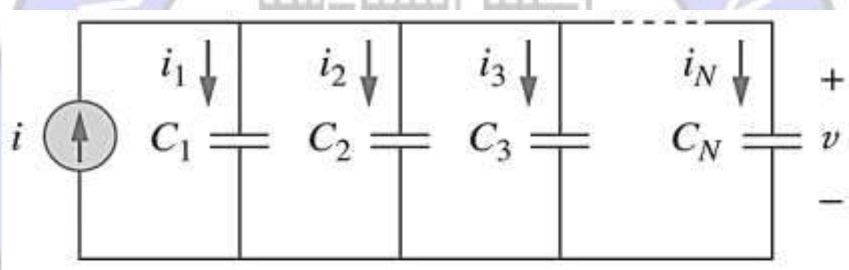
$$\boxed{i = C\frac{dv}{dt}} \qquad (6.4)$$

- The energy stored in the capacitor is

$$w = \frac{1}{2}Cv^2 \qquad (6.9)$$

- A capacitor is an open circuit to dc.
- The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances.

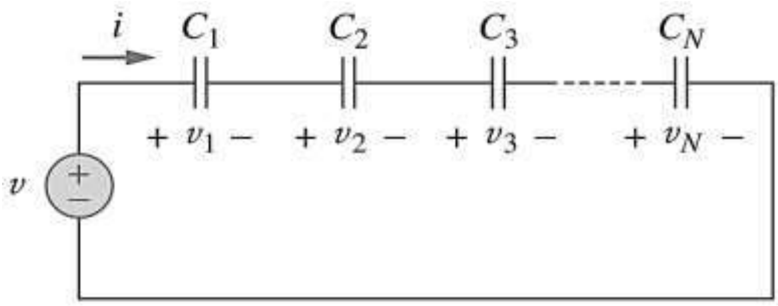
$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N \qquad (6.13)$$



(a)

- The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \qquad (6.17)$$



(a)

- An inductor is a passive element designed to store energy in its magnetic field.
- The voltage across the inductor is directly proportional to the time rate of change of the current.

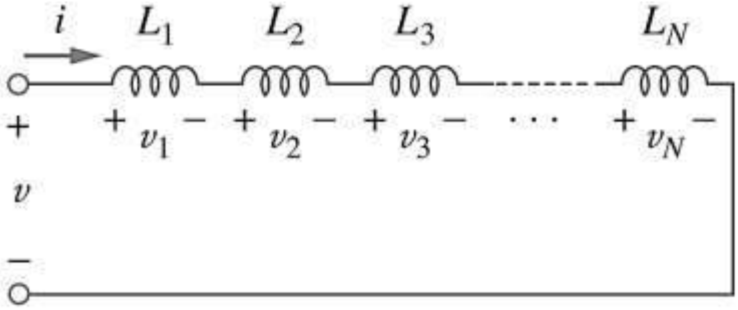
$$\boxed{v = L\frac{di}{dt}} \qquad (6.18)$$

- The energy stored in the inductor is

$$\boxed{w = \frac{1}{2}Li^2} \qquad (6.24)$$

- The equivalent inductance of a series-connected is:

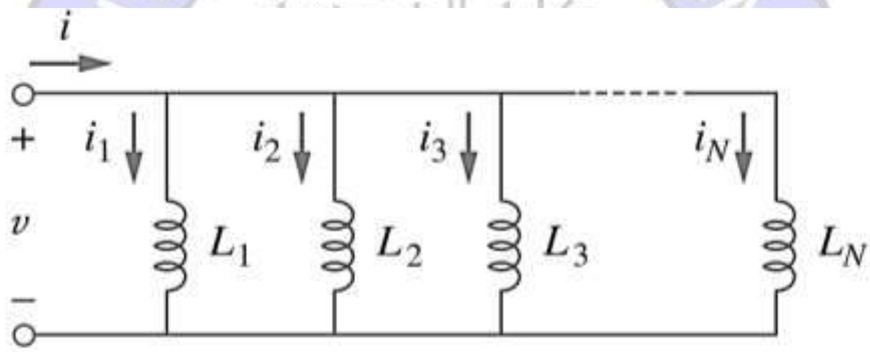
$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N} \qquad (6.27)$$



(a)

- The equivalent inductance of a parallel-connected is:

$$L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \qquad \textbf{(6.31)}$$



(a)

- The summary is given in Table 6.1.

TABLE 6.1

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit

Chapter Two

Operational Amplifiers (Op Amp)

2.1- Introduction to Op. Amp.

- An op amp is an active circuit element designed to perform mathematical operations of addition, subtraction, multiplication, division, differentiation, and integration.

- A typical Op amp is the eight-pin dual in-line package (or DIP), shown in Fig. 2.1(a). Pin or terminal 8 is unused, and terminals 1 and 5 are of little concern to us. The five important terminals are:
  1- The inverting input, pin 2.
  2- The noninverting input, pin 3.
  3- The output, pin 6.
  4- The positive power supply V^{+}, pin 7.
  5- The negative power supply V^{-}, pin 4.

- The circuit symbol for the op amp is the triangle in Fig. 2.1(b); as shown, the op amp has two inputs and one output. The inputs are marked with minus (-) and plus (+) to specify inverting and noninverting inputs, respectively. An input applied to the noninverting terminal will appear with the same polarity at the output, while an input applied to the inverting terminal will appear inverted at the output.

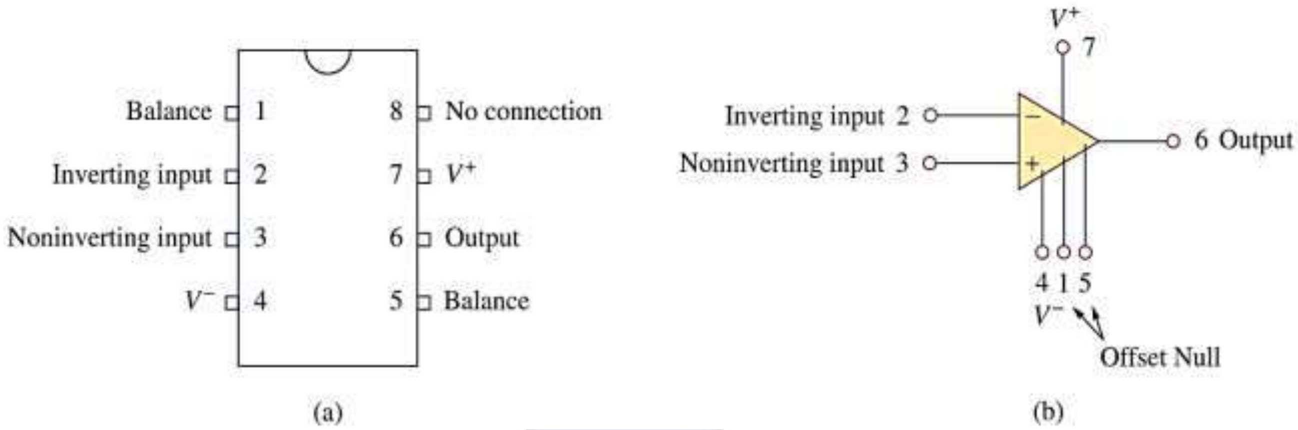


Figure 2.1: A typical op amp: (a) pin configuration, (b) circuit symbol

- As an active element, the op amp must be powered by a voltage supply as typically shown in Fig. 2.2. The power supply currents must not be overlooked. By KCL:

$$i_o = i_1 + i_2 + i_+ + i_- \tag{2.1}$$

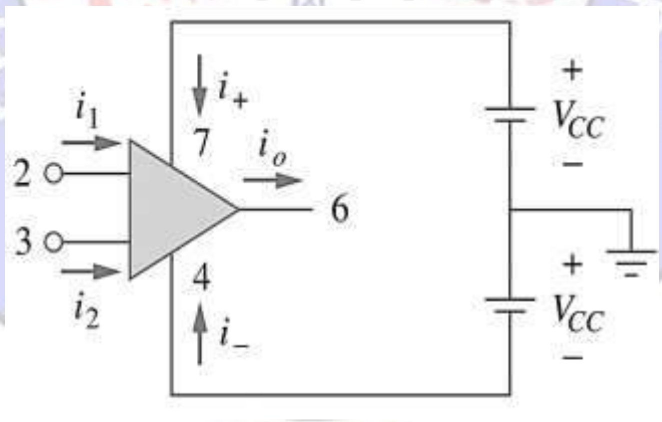


Figure 2.2: Powering the op amp

- The equivalent circuit model of an op amp is shown in Fig. 2.3.
- The output section consists of a voltage-controlled source in series with the output resistance R_o.
- It is evident from Fig. 2.3 that the input resistance R_i is the Thevenin equivalent resistance seen at the input terminals, while the output resistance R_o is the Thevenin equivalent resistance seen at the output.
- The differential input voltage V_d is given by:

$$v_d = v_2 - v_1 \tag{2.2}$$

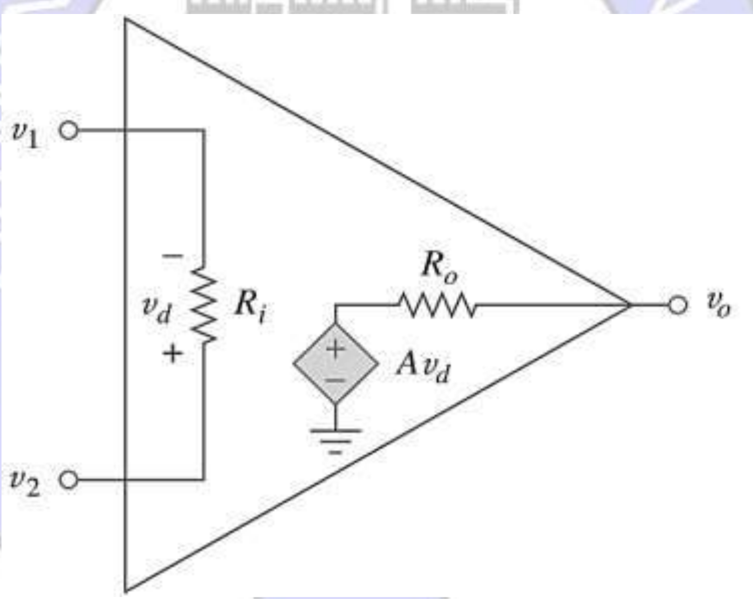


Figure 2.3: The equivalent circuit of the nonideal op amp.

- Where v_1 is the voltage between the inverting terminal and ground, and v_2 is the voltage between the noninverting terminal and ground.

- The op amp senses the difference between the two inputs, multiplies it by the gain A, and causes the resulting voltage to appear at the output. Thus, the output v_o is given by

$$v_o = Av_d = A(v_2 - v_1) \tag{2.3}$$

- A is called the open-loop voltage gain because it is the gain of the op amp without any external feedback from output to input. Table 2.1 shows typical values of voltage gain A, input resistance R_i, output resistance R_o, and supply voltage V_{cc}.

Table 2.1
Typical ranges for op amp parameters.

Parameter	Typical range	Ideal values
Open-loop gain, A	10^5 to 10^8	∞
Input resistance, R_i	10^5 to 10^{13} Ω	∞ Ω
Output resistance, R_o	10 to 100 Ω	0 Ω
Supply voltage, V_{CC}	5 to 24 V	

- A practical limitation of the op amp is that the magnitude of its output voltage cannot exceed $|V_{cc}|$.
- In other words, the output voltage is dependent on and is limited by the power supply voltage.
- Figure 2.4 illustrates that the op amp can operate in three modes, depending on the differential input voltage v_d:

1- Positive saturation, $v_o = V_{cc}$.

2- Linear region, $-V_{cc} \le v_o = Av_d \le V_{cc}$.

3- Negative saturation, $v_o = -V_{cc}$

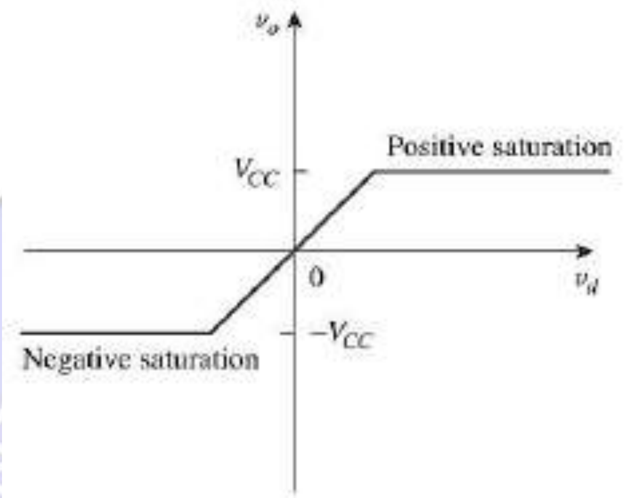
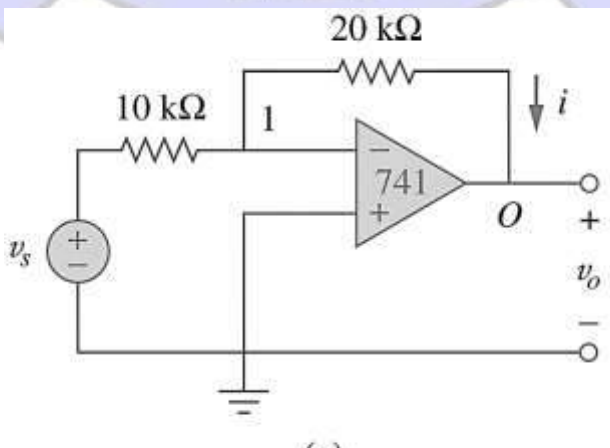


Figure 2.4: Op amp output voltage v_o as a function of the differential input voltage v_d

Example 5.1:

A 741 op amp has an open-loop voltage gain of 2×10^5, input resistance of 2 MΩ, and output resistance of 50 Ω. The op amp is used in the circuit of Fig. 5.6(a). Find the closed-loop gain v_o/v_s. Determine current i when $v_s = 2$ V.



(a)

Solution:

Using the op amp model in Fig. 5.4, we obtain the equivalent circuit of Fig. 5.6(a) as shown in Fig. 5.6(b). We now solve the circuit in Fig. 5.6(b) by using nodal analysis. At node 1, KCL gives

$$\frac{v_s - v_1}{10 \times 10^3} = \frac{v_1}{2000 \times 10^3} + \frac{v_1 - v_o}{20 \times 10^3}$$

Multiplying through by 2000×10^3, we obtain

$$200v_s = 301v_1 - 100v_o$$

or

$$2v_s \simeq 3v_1 - v_o \quad \Rightarrow \quad v_1 = \frac{2v_s + v_o}{3} \tag{5.1.1}$$

At node O,

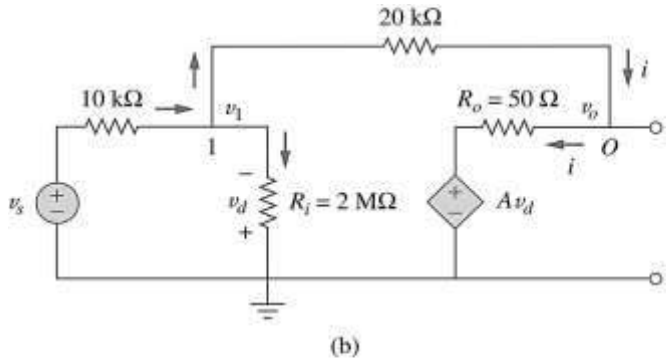


Figure 5.6
For Example 5.1: (a) original circuit, (b) the equivalent circuit

$$\frac{v_1 - v_o}{20 \times 10^3} = \frac{v_o - Av_d}{50}$$

But $v_d = -v_1$ and $A = 200{,}000$. Then

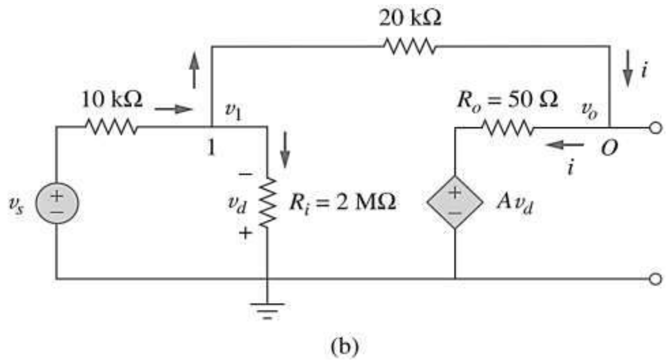
$$v_1 - v_o = 400(v_o + 200{,}000v_1) \tag{5.1.2}$$

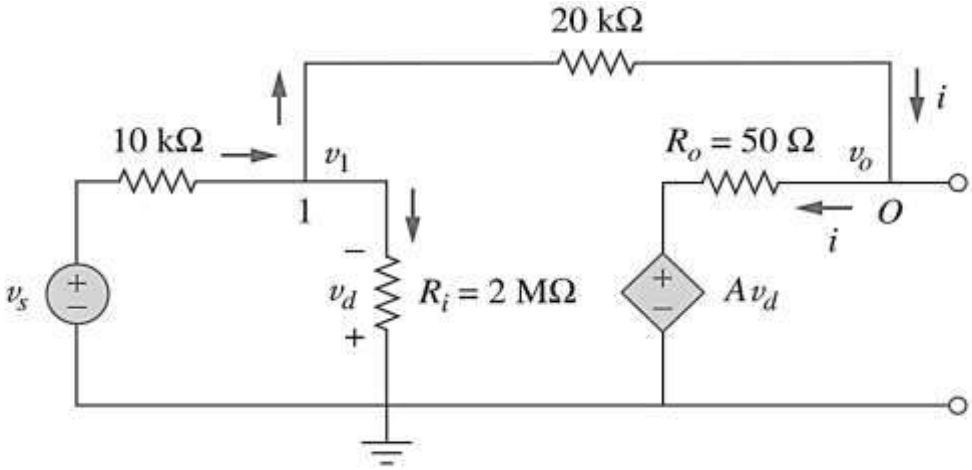
Substituting v_1 from Eq. (5.1.1) into Eq. (5.1.2) gives

$$0 \simeq 26{,}667{,}067v_o + 53{,}333{,}333v_s \quad \Rightarrow \quad \frac{v_o}{v_s} = -1.9999699$$

This is closed-loop gain, because the 20-kΩ feedback resistor closes the loop between the output and input terminals. When $v_s = 2$ V, $v_o = -3.9999398$ V. From Eq. (5.1.1), we obtain $v_1 = 20.066667\ \mu$V. Thus,

$$i = \frac{v_1 - v_o}{20 \times 10^3} = 0.19999 \text{ mA}$$





(b)

Figure 5.6
For Example 5.1: (a) original circuit, (b) the equivalent circuit

Solution:
Using the op amp model in Fig. 5.4, we obtain the equivalent circuit of Fig. 5.6(a) as shown in Fig. 5.6(b). We now solve the circuit in Fig. 5.6(b) by using nodal analysis. At node 1, KCL gives

$$\frac{v_s - v_1}{10 \times 10^3} = \frac{v_1}{2000 \times 10^3} + \frac{v_1 - v_o}{20 \times 10^3}$$

Multiplying through by 2000×10^3, we obtain

$$200v_s = 301v_1 - 100v_o$$

or

$$2v_s \simeq 3v_1 - v_o \quad \Rightarrow \quad v_1 = \frac{2v_s + v_o}{3} \qquad \textbf{(5.1.1)}$$

At node O,

$$\frac{v_1 - v_o}{20 \times 10^3} = \frac{v_o - Av_d}{50}$$

But $v_d = -v_1$ and $A = 200{,}000$. Then

$$v_1 - v_o = 400(v_o + 200{,}000v_1) \qquad \textbf{(5.1.2)}$$

Substituting v_1 from Eq. (5.1.1) into Eq. (5.1.2) gives

$$0 \simeq 26{,}667{,}067v_o + 53{,}333{,}333v_s \quad \Rightarrow \quad \frac{v_o}{v_s} = -1.9999699$$

This is closed-loop gain, because the 20-kΩ feedback resistor closes the loop between the output and input terminals. When $v_s = 2$ V, $v_o = -3.9999398$ V. From Eq. (5.1.1), we obtain $v_1 = 20.066667\ \mu$V. Thus,

$$i = \frac{v_1 - v_o}{20 \times 10^3} = 0.19999 \text{ mA}$$

2.2- Ideal Op. Amp.

- An ideal op amp is an amplifier with infinite open-loop gain, infinite input resistance, and zero output resistance.

  1- Infinite open-loop gain, $A = \infty$

  2- Infinite input resistance, $R_i = \infty$

  3- Zero output resistance, $R_o = 0$

- For circuit analysis, the ideal op amp is illustrated in Fig. 5.8.

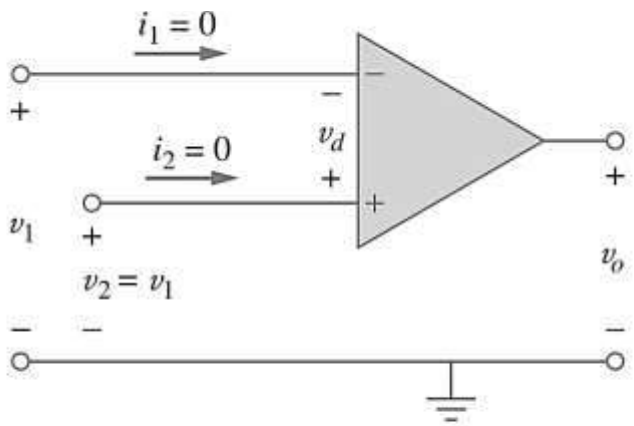


Figure 5.8
Ideal op amp model.

- Two important characteristics of the ideal op amp are:

 1- The currents into both input terminals are zero:

$$\boxed{i_1 = 0, \qquad i_2 = 0} \tag{5.5}$$

 2- The voltage across the input terminals is equal to zero; i.e.,

$$v_d = v_2 - v_1 = 0 \tag{5.6}$$

or

$$\boxed{v_1 = v_2} \tag{5.7}$$

Example 5.2:

Rework Practice Prob. 5.1 using the ideal op amp model.

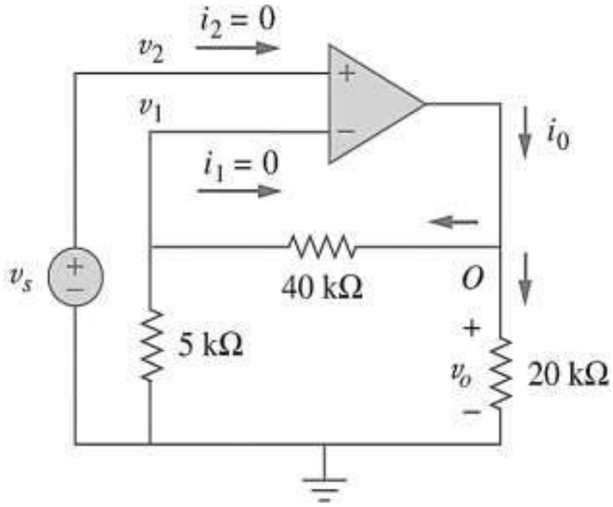
If the same 741 op amp in Example 5.1 is used in the circuit of Fig. 5.7, calculate the closed-loop gain v_o/v_s. Find i_o when $v_s = 1$ V.

Solution:
We may replace the op amp in Fig. 5.7 by its equivalent model in Fig. 5.9 as we did in Example 5.1. But we do not really need to do this. We just need to keep Eqs. (5.5) and (5.7) in mind as we analyze the circuit in Fig. 5.7. Thus, the Fig. 5.7 circuit is presented as in Fig. 5.9. Notice that

$$v_2 = v_s \tag{5.2.1}$$

Since $i_1 = 0$, the 40-kΩ and 5-kΩ resistors are in series; the same current flows through them. v_1 is the voltage across the 5-kΩ resistor. Hence, using the voltage division principle,

$$v_1 = \frac{5}{5+40}v_o = \frac{v_o}{9} \tag{5.2.2}$$



According to Eq. (5.7),

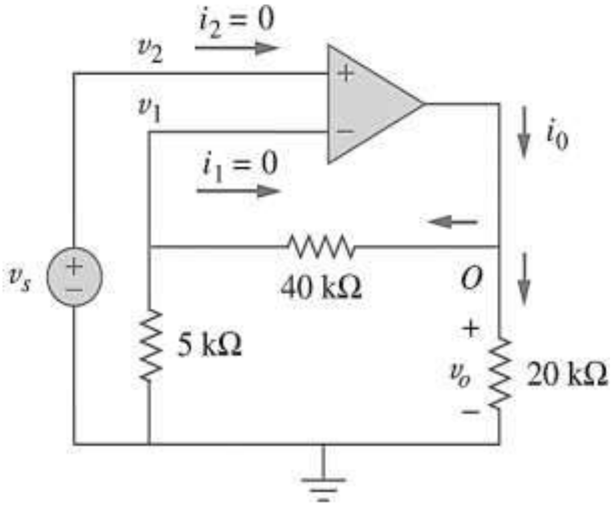
$$v_2 = v_1 \tag{5.2.3}$$

Substituting Eqs. (5.2.1) and (5.2.2) into Eq. (5.2.3) yields the closed-loop gain,

$$v_s = \frac{v_o}{9} \quad \Rightarrow \quad \frac{v_o}{v_s} = 9 \tag{5.2.4}$$

which is very close to the value of 9.00041 obtained with the nonideal model in Practice Prob. 5.1. This shows that negligibly small error results from assuming ideal op amp characteristics.

At node O,

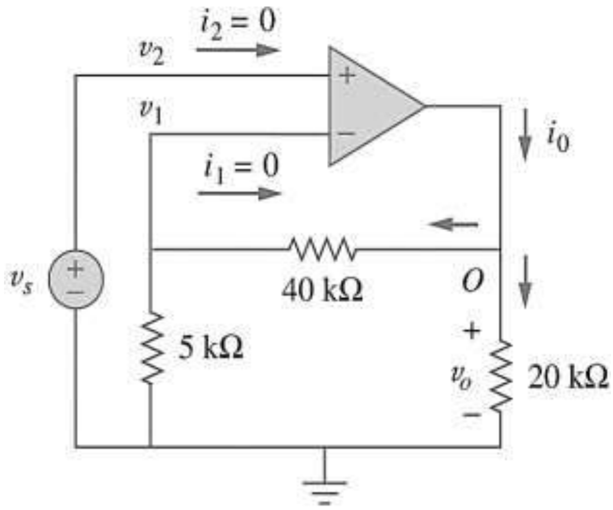


$$i_o = \frac{v_o}{40+5} + \frac{v_o}{20}\text{mA} \tag{5.2.5}$$

From Eq. (5.2.4), when $v_s = 1$ V, $v_o = 9$ V. Substituting for $v_o = 9$ V in Eq. (5.2.5) produces

$$i_o = 0.2 + 0.45 = 0.65 \text{ mA}$$

This, again, is close to the value of 0.657 mA obtained in Practice Prob. 5.1 with the nonideal model.



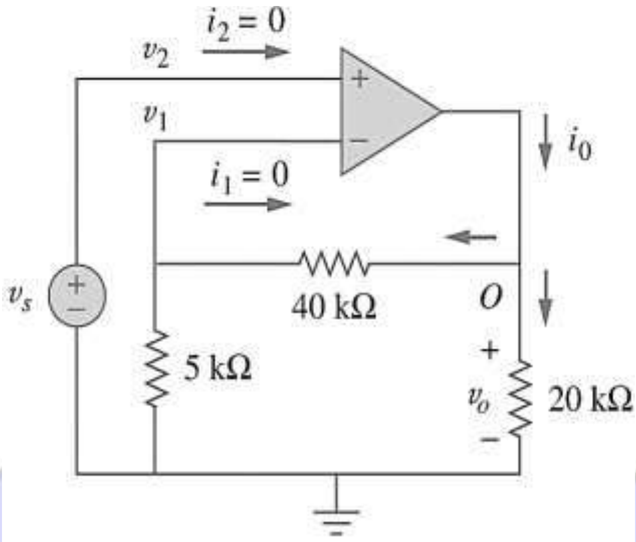


Figure 5.9
For Example 5.2.

Solution:

We may replace the op amp in Fig. 5.7 by its equivalent model in Fig. 5.9 as we did in Example 5.1. But we do not really need to do this. We just need to keep Eqs. (5.5) and (5.7) in mind as we analyze the circuit in Fig. 5.7. Thus, the Fig. 5.7 circuit is presented as in Fig. 5.9. Notice that

$$v_2 = v_s \tag{5.2.1}$$

Since $i_1 = 0$, the 40-kΩ and 5-kΩ resistors are in series; the same current flows through them. v_1 is the voltage across the 5-kΩ resistor. Hence, using the voltage division principle,

$$v_1 = \frac{5}{5 + 40} v_o = \frac{v_o}{9} \tag{5.2.2}$$

According to Eq. (5.7),

$$v_2 = v_1 \tag{5.2.3}$$

Substituting Eqs. (5.2.1) and (5.2.2) into Eq. (5.2.3) yields the closed-loop gain,

$$v_s = \frac{v_o}{9} \qquad \Rightarrow \qquad \frac{v_o}{v_s} = 9 \tag{5.2.4}$$

which is very close to the value of 9.00041 obtained with the nonideal model in Practice Prob. 5.1. This shows that negligibly small error results from assuming ideal op amp characteristics.

At node O,

$$i_o = \frac{v_o}{40 + 5} + \frac{v_o}{20} \text{mA} \tag{5.2.5}$$

From Eq. (5.2.4), when $v_s = 1$ V, $v_o = 9$ V. Substituting for $v_o = 9$ V in Eq. (5.2.5) produces

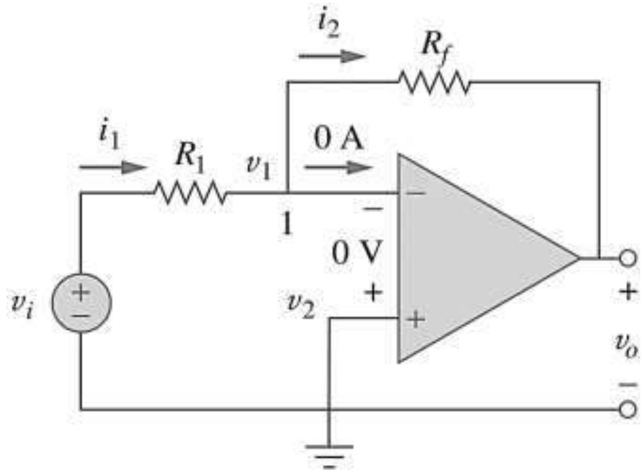
$$i_o = 0.2 + 0.45 = 0.65 \text{ mA}$$

This, again, is close to the value of 0.657 mA obtained in Practice Prob. 5.1 with the nonideal model.

2.3- Inverting Op. Amp.

- The first of op amp circuits is the inverting amplifier shown in Fig. 5.10.

- In this circuit, the noninverting input is grounded, v_i is connected to the inverting input through R_1, and the feedback resistor R_f is connected between the inverting input and output.
- Our goal is to obtain the relationship between the input voltage v_i and the output voltage v_o.



- Applying KCL at node 1,

$$i_1 = i_2 \quad \Rightarrow \quad \frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_f} \tag{5.8}$$

But $v_1 = v_2 = 0$ for an ideal op amp, since the noninverting terminal is grounded. Hence,

$$\frac{v_i}{R_1} = -\frac{v_o}{R_f}$$

or

$$\boxed{v_o = -\frac{R_f}{R_1} v_i} \tag{5.9}$$

The voltage gain is $A_v = v_o/v_i = -R_f/R_1$. The designation of the circuit in Fig. 5.10 as an *inverter* arises from the negative sign.

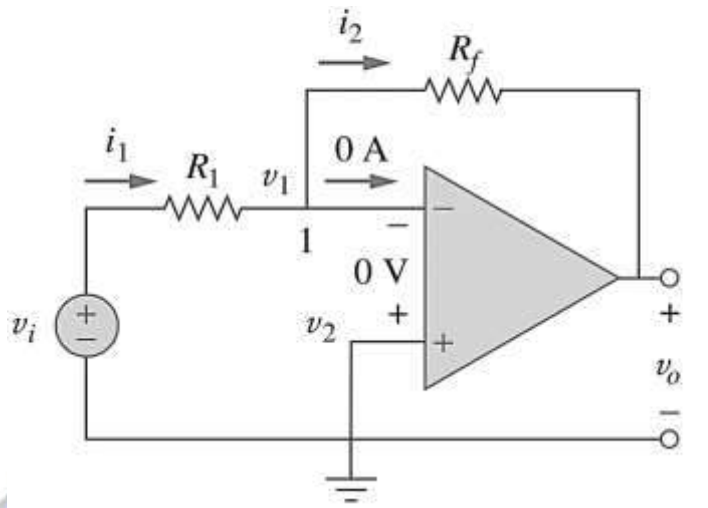


Figure 5.10
The inverting amplifier.

- Applying KCL at node 1,

$$i_1 = i_2 \quad \Rightarrow \quad \frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_f} \tag{5.8}$$

But $v_1 = v_2 = 0$ for an ideal op amp, since the noninverting terminal is grounded. Hence,

$$\frac{v_i}{R_1} = -\frac{v_o}{R_f}$$

or

$$\boxed{v_o = -\frac{R_f}{R_1} v_i} \tag{5.9}$$

The voltage gain is $A_v = v_o/v_i = -R_f/R_1$. The designation of the circuit in Fig. 5.10 as an *inverter* arises from the negative sign.

- Notice that the gain is the feedback resistance divided by the input resistance which means that the gain depends only on the external elements connected to the op amp as shown in Eq. (5.9) and figure 5.11.

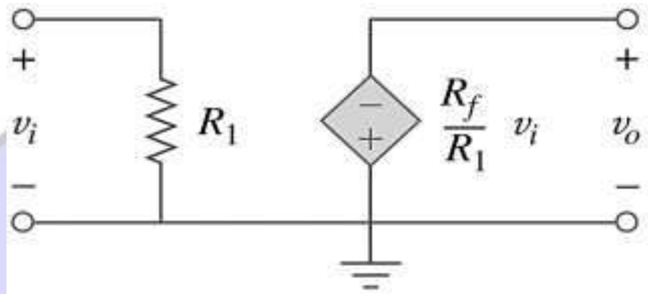


Figure 5.11
An equivalent circuit for the inverter in Fig. 5.10.

Example 5.3:

Refer to the op amp in Fig. 5.12. If $v_i = 0.5$ V, calculate: (a) the output voltage v_o, and (b) the current in the 10-kΩ resistor.

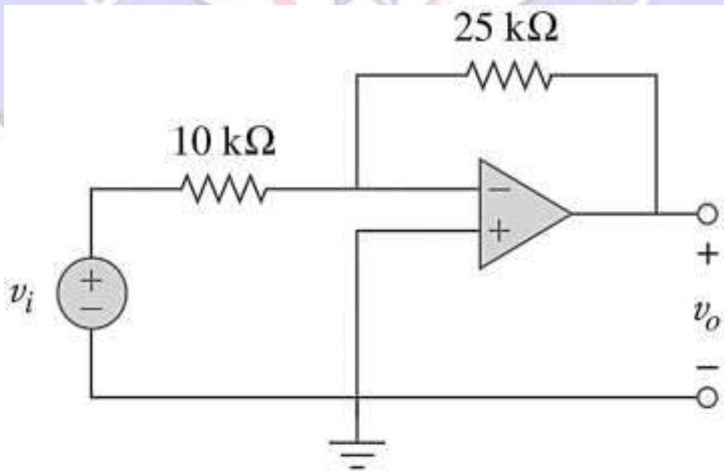


Figure 5.12
For Example 5.3.

Solution:

(a) Using Eq. (5.9),

$$\frac{v_o}{v_i} = -\frac{R_f}{R_1} = -\frac{25}{10} = -2.5$$

$$v_o = -2.5v_i = -2.5(0.5) = -1.25 \text{ V}$$

(b) The current through the 10-kΩ resistor is

$$i = \frac{v_i - 0}{R_1} = \frac{0.5 - 0}{10 \times 10^3} = 50 \ \mu\text{A}$$

Solution:

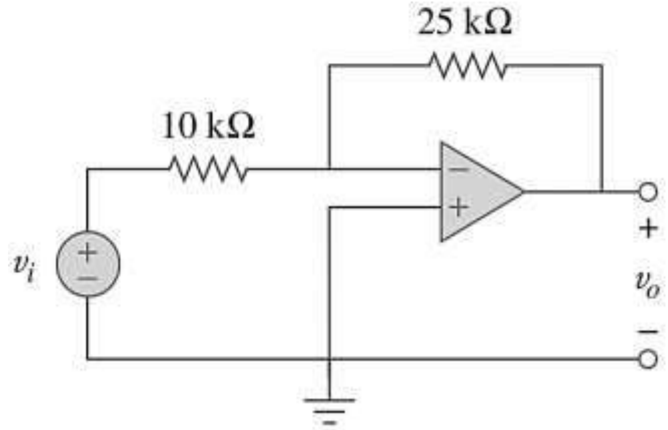
(a) Using Eq. (5.9),

$$\frac{v_o}{v_i} = -\frac{R_f}{R_1} = -\frac{25}{10} = -2.5$$

$$v_o = -2.5v_i = -2.5(0.5) = -1.25 \text{ V}$$

(b) The current through the 10-kΩ resistor is

$$i = \frac{v_i - 0}{R_1} = \frac{0.5 - 0}{10 \times 10^3} = 50 \ \mu\text{A}$$



Example 5.4:

Determine v_o in the op amp circuit shown in Fig. 5.14.

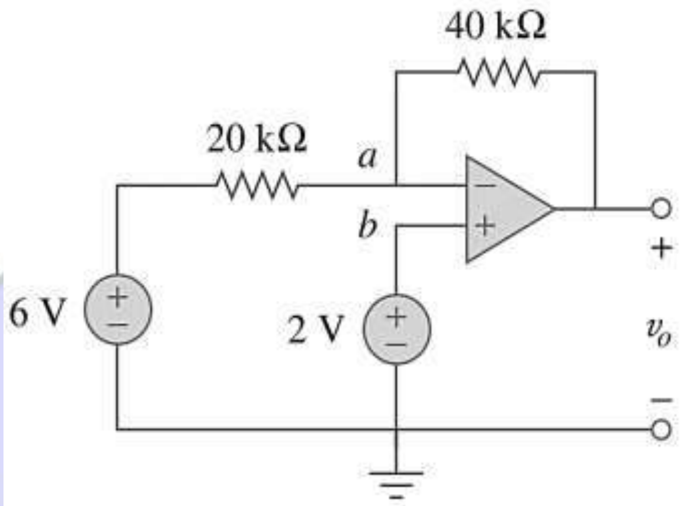


Figure 5.14
For Example 5.4.

Solution:
Applying KCL at node a,

$$\frac{v_a - v_o}{40\ \text{k}\Omega} = \frac{6 - v_a}{20\ \text{k}\Omega}$$

$$v_a - v_o = 12 - 2v_a \quad \Rightarrow \quad v_o = 3v_a - 12$$

But $v_a = v_b = 2$ V for an ideal op amp, because of the zero voltage drop across the input terminals of the op amp. Hence,

$$v_o = 6 - 12 = -6\ \text{V}$$

Notice that if $v_b = 0 = v_a$, then $v_o = -12$, as expected from Eq. (5.9).

Solution:
Applying KCL at node a,

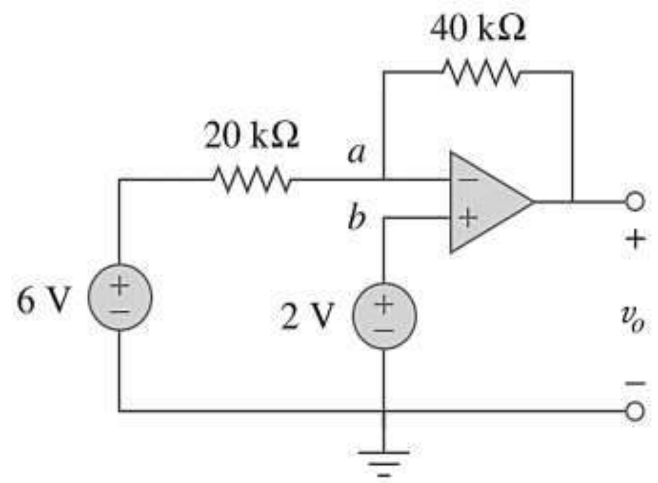
$$\frac{v_a - v_o}{40\text{ k}\Omega} = \frac{6 - v_a}{20\text{ k}\Omega}$$

$$v_a - v_o = 12 - 2v_a \quad \Rightarrow \quad v_o = 3v_a - 12$$

But $v_a = v_b = 2$ V for an ideal op amp, because of the zero voltage drop across the input terminals of the op amp. Hence,

$$v_o = 6 - 12 = -6\text{ V}$$

Notice that if $v_b = 0 = v_a$, then $v_o = -12$, as expected from Eq. (5.9).



2.4- Non-inverting Op. Amp.

- Another important application of the op amp is the noninverting amplifier shown in Fig. 5.16.

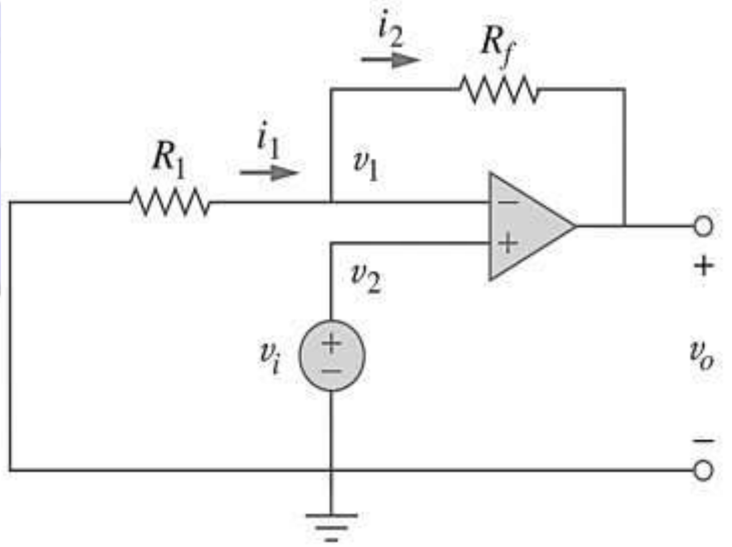


Fig. 5.16: The noninverting amplifier

- In this case, the input voltage v_i is applied directly at the noninverting input terminal, and resistor R_1 is connected between the ground and the inverting terminal.
- Application of KCL at the inverting terminal gives:

$$i_1 = i_2 \quad \Rightarrow \quad \frac{0 - v_1}{R_1} = \frac{v_1 - v_o}{R_f} \qquad \textbf{(5.10)}$$

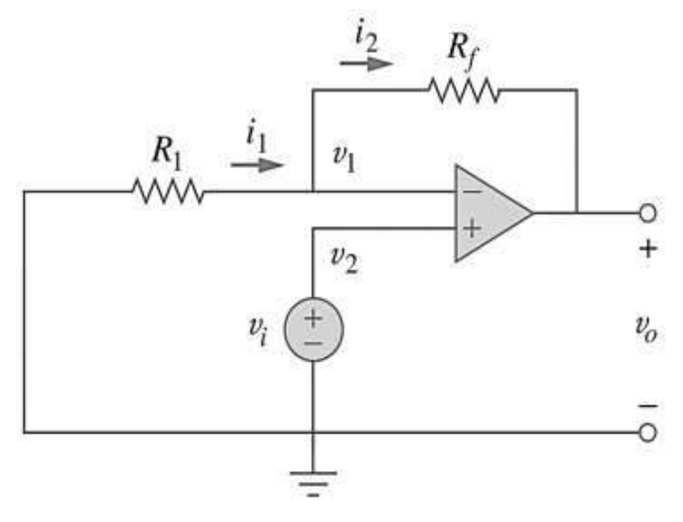
But $v_1 = v_2 = v_i$. Equation (5.10) becomes

$$\frac{-v_i}{R_1} = \frac{v_i - v_o}{R_f}$$

or

$$\boxed{v_o = \left(1 + \frac{R_f}{R_1}\right)v_i} \qquad \textbf{(5.11)}$$

The voltage gain is $A_v = v_o/v_i = 1 + R_f/R_1$, which does not have a negative sign. Thus, the output has the same polarity as the input.



$$i_1 = i_2 \quad \Rightarrow \quad \frac{0 - v_1}{R_1} = \frac{v_1 - v_o}{R_f} \qquad \textbf{(5.10)}$$

But $v_1 = v_2 = v_i$. Equation (5.10) becomes

$$\frac{-v_i}{R_1} = \frac{v_i - v_o}{R_f}$$

or

$$\boxed{v_o = \left(1 + \frac{R_f}{R_1}\right)v_i} \qquad \textbf{(5.11)}$$

The voltage gain is $A_v = v_o/v_i = 1 + R_f/R_1$, which does not have a negative sign. Thus, the output has the same polarity as the input.

- A noninverting amplifier is an op amp circuit designed to provide a positive voltage gain.
- Notice that,

Notice that if feedback resistor $R_f = 0$ (short circuit) or $R_1 = \infty$ (open circuit) or both, the gain becomes 1. Under these conditions ($R_f = 0$ and $R_1 = \infty$), the circuit in Fig. 5.16 becomes that shown in Fig. 5.17, which is called a *voltage follower* (or *unity gain amplifier*) because the output follows the input. Thus, for a voltage follower

$$\boxed{v_o = v_i} \tag{5.12}$$

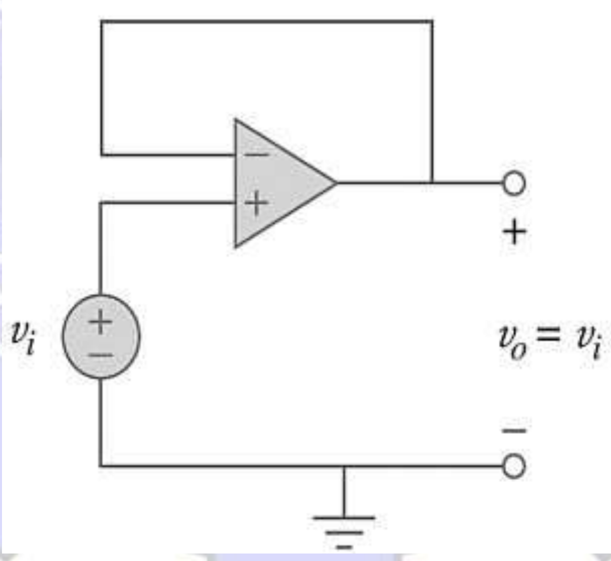


Figure 5.17
The voltage follower.

- Such a circuit has a very high input impedance and is therefore useful as an intermediate-stage (or buffer) amplifier to isolate one circuit from another, as portrayed in Fig. 5.18.

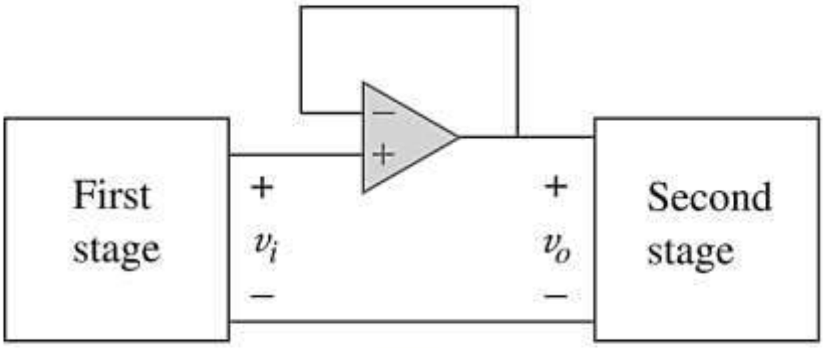


Figure 5.18
A voltage follower used to isolate two cascaded stages of a circuit.

Example 5.5:

For the op amp circuit in Fig. 5.19, calculate the output voltage v_o.

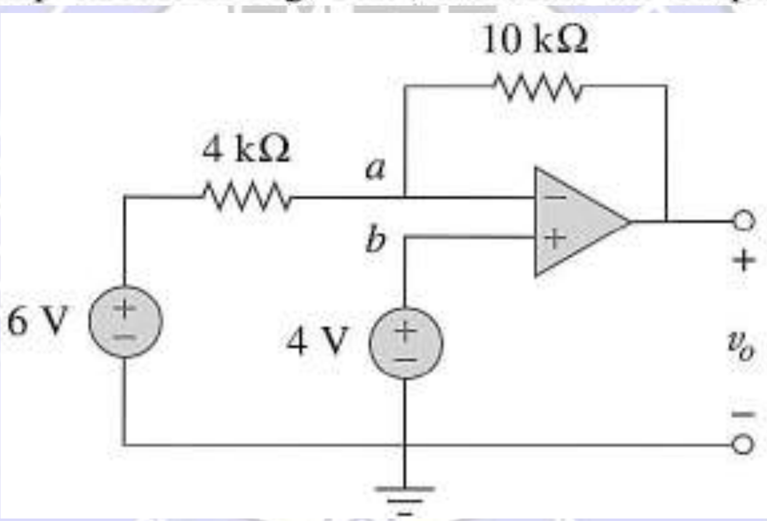


Figure 5.19
For Example 5.5

Solution:
We may solve this in two ways: using superposition and using nodal analysis.

■ **METHOD 1** Using superposition, we let

$$v_o = v_{o1} + v_{o2}$$

where v_{o1} is due to the 6-V voltage source, and v_{o2} is due to the 4-V input. To get v_{o1}, we set the 4-V source equal to zero. Under this condition, the circuit becomes an inverter. Hence Eq. (5.9) gives

$$v_{o1} = -\frac{10}{4}(6) = -15 \text{ V}$$

To get v_{o2}, we set the 6-V source equal to zero. The circuit becomes a noninverting amplifier so that Eq. (5.11) applies.

$$v_{o2} = \left(1 + \frac{10}{4}\right)4 = 14 \text{ V}$$

Thus,

$$v_o = v_{o1} + v_{o2} = -15 + 14 = -1 \text{ V}$$

METHOD 2 Applying KCL at node a,

$$\frac{6 - v_a}{4} = \frac{v_a - v_o}{10}$$

But $v_a = v_b = 4$, and so

$$\frac{6 - 4}{4} = \frac{4 - v_o}{10} \quad \Rightarrow \quad 5 = 4 - v_o$$

or $v_o = -1$ V, as before.

Solution:
We may solve this in two ways: using superposition and using nodal analysis.

METHOD 1 Using superposition, we let

$$v_o = v_{o1} + v_{o2}$$

where v_{o1} is due to the 6-V voltage source, and v_{o2} is due to the 4-V input. To get v_{o1}, we set the 4-V source equal to zero. Under this condition, the circuit becomes an inverter. Hence Eq. (5.9) gives

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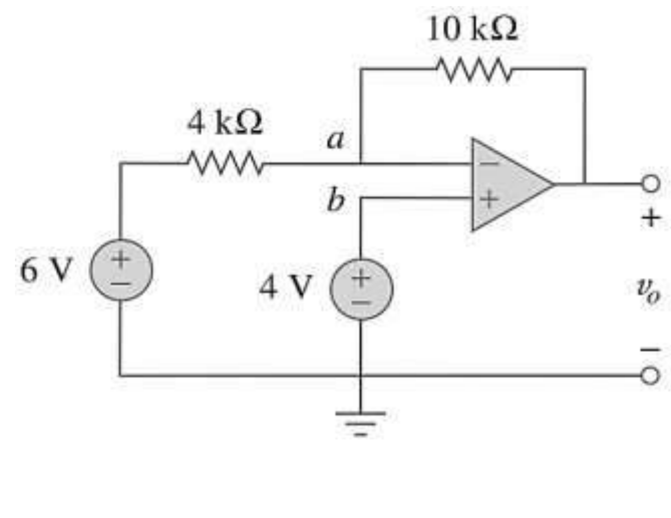
Thus,

$$v_o = v_{o1} + v_{o2} = -15 + 14 = -1 \text{ V}$$



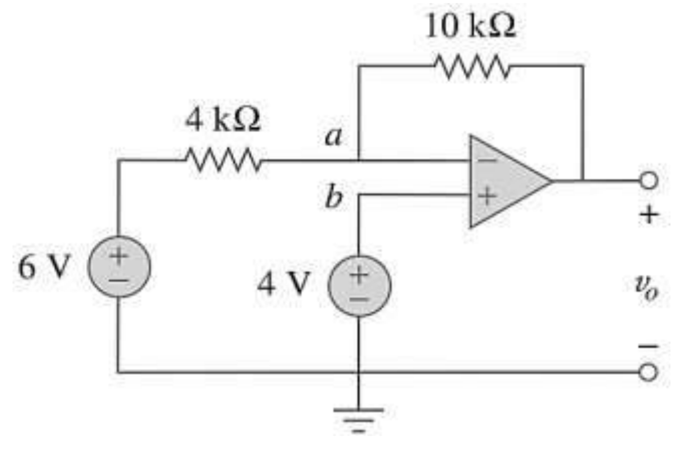
METHOD 2 Applying KCL at node a,

$$\frac{6 - v_a}{4} = \frac{v_a - v_o}{10}$$

But $v_a = v_b = 4$, and so

$$\frac{6-4}{4} = \frac{4 - v_o}{10} \quad \Rightarrow \quad 5 = 4 - v_o$$

or $v_o = -1$ V, as before.



2.5- Summing Op. Amp.

- A summing amplifier is an op amp circuit that combines several inputs and produces an output that is the weighted sum of the inputs as shown in figure 5.21.

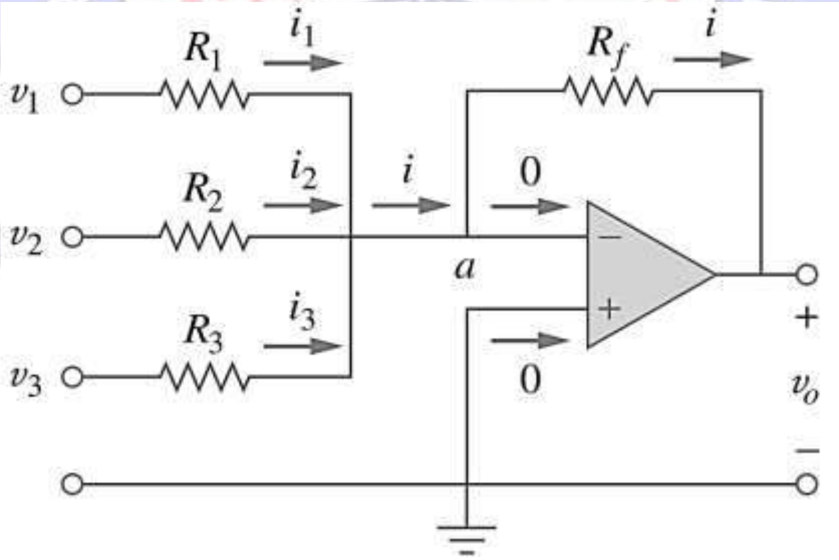


Figure 5.21
The summing amplifier.

- Applying KCL at node a give:

$$i = i_1 + i_2 + i_3 \tag{5.13}$$

But

$$i_1 = \frac{v_1 - v_a}{R_1}, \quad i_2 = \frac{v_2 - v_a}{R_2}$$
$$i_3 = \frac{v_3 - v_a}{R_3}, \quad i = \frac{v_a - v_o}{R_f} \tag{5.14}$$

We note that $v_a = 0$ and substitute Eq. (5.14) into Eq. (5.13). We get

$$\boxed{v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)} \tag{5.15}$$

- Applying KCL at node a give:

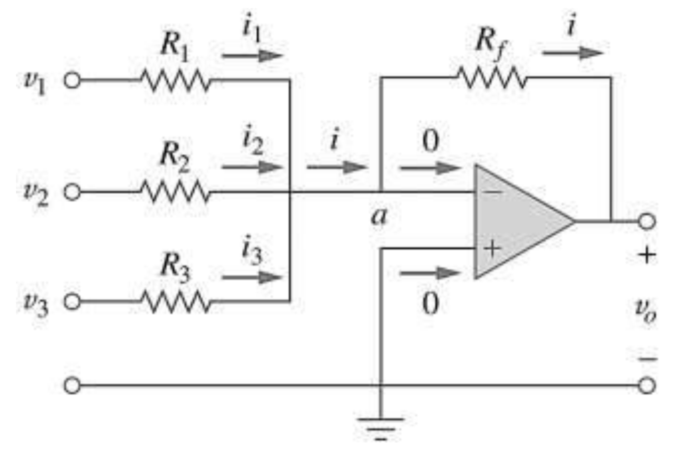
$$i = i_1 + i_2 + i_3 \tag{5.13}$$

But

$$i_1 = \frac{v_1 - v_a}{R_1}, \quad i_2 = \frac{v_2 - v_a}{R_2}$$
$$i_3 = \frac{v_3 - v_a}{R_3}, \quad i = \frac{v_a - v_o}{R_f} \tag{5.14}$$

We note that $v_a = 0$ and substitute Eq. (5.14) into Eq. (5.13). We get

$$\boxed{v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)} \tag{5.15}$$



Example 5.6:

Calculate v_o and i_o in the op amp circuit in Fig. 5.22.

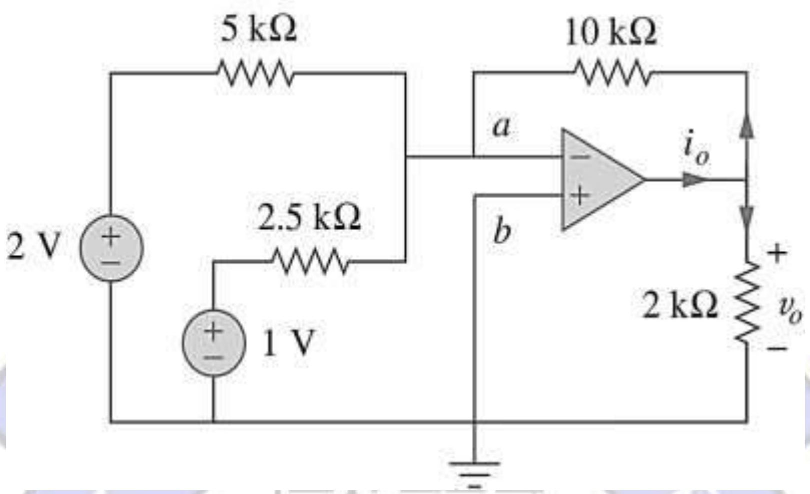


Figure 5.22
For Example 5.6

This is a summer with two inputs. Using Eq. (5.15) gives

$$v_o = -\left[\frac{10}{5}(2) + \frac{10}{2.5}(1)\right] = -(4 + 4) = -8 \text{ V}$$

The current i_o is the sum of the currents through the 10-kΩ and 2-kΩ resistors. Both of these resistors have voltage $v_o = -8$ V across them, since $v_a = v_b = 0$. Hence,

$$i_o = \frac{v_o - 0}{10} + \frac{v_o - 0}{2} \text{mA} = -0.8 - 4 = -4.8 \text{ mA}$$

2.6- Subtracting (Difference) Op. Amp.

- A difference amplifier is a device that amplifies the difference between two inputs.

- Consider the op amp circuit shown in Fig. 5.24.

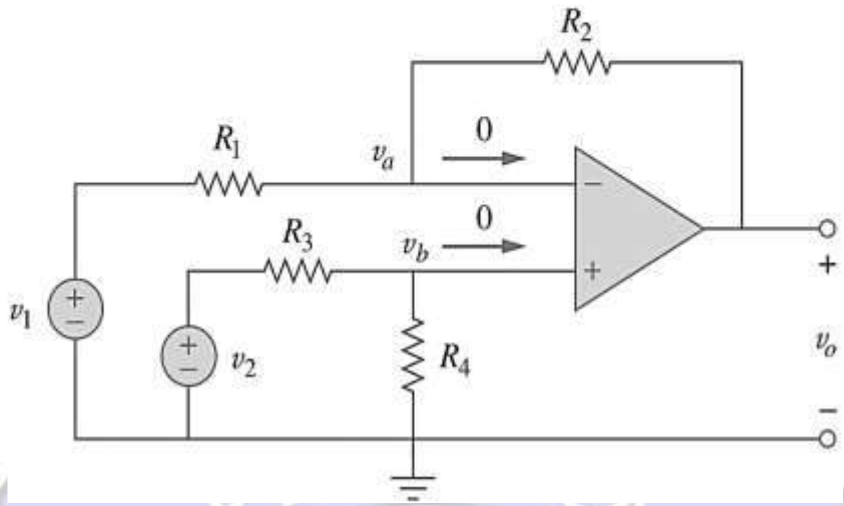


Figure 5.24
Difference amplifier.

- Keep in mind that zero currents enter the op amp terminals. Applying KCL to node a,

$$\frac{v_1 - v_a}{R_1} = \frac{v_a - v_o}{R_2}$$

or

$$v_o = \left(\frac{R_2}{R_1} + 1\right)v_a - \frac{R_2}{R_1}v_1 \tag{5.16}$$

Applying KCL to node b,

$$\frac{v_2 - v_b}{R_3} = \frac{v_b - 0}{R_4}$$

or

$$v_b = \frac{R_4}{R_3 + R_4}v_2 \tag{5.17}$$

But $v_a = v_b$. Substituting Eq. (5.17) into Eq. (5.16) yields

$$v_o = \left(\frac{R_2}{R_1} + 1\right)\frac{R_4}{R_3 + R_4}v_2 - \frac{R_2}{R_1}v_1$$

or

$$\boxed{v_o = \frac{R_2(1 + R_1/R_2)}{R_1(1 + R_3/R_4)}v_2 - \frac{R_2}{R_1}v_1} \tag{5.18}$$

Since a difference amplifier must reject a signal common to the two inputs, the amplifier must have the property that $v_o = 0$ when $v_1 = v_2$. This property exists when

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \tag{5.19}$$

Thus, when the op amp circuit is a difference amplifier, Eq. (5.18) becomes

$$v_o = \frac{R_2}{R_1}(v_2 - v_1) \tag{5.20}$$

If $R_2 = R_1$ and $R_3 = R_4$, the difference amplifier becomes a *subtractor*, with the output

$$v_o = v_2 - v_1 \tag{5.21}$$

Example 5.6:

Design an op amp circuit with inputs v_1 and v_2 such that $v_o = -5v_1 + 3v_2$.

Solution:
The circuit requires that

$$v_o = 3v_2 - 5v_1 \tag{5.7.1}$$

This circuit can be realized in two ways.

Design 1 If we desire to use only one op amp, we can use the op amp circuit of Fig. 5.24. Comparing Eq. (5.7.1) with Eq. (5.18), we see

$$\frac{R_2}{R_1} = 5 \qquad \Rightarrow \qquad R_2 = 5R_1 \tag{5.7.2}$$

Also,

$$5\frac{(1 + R_1/R_2)}{(1 + R_3/R_4)} = 3 \qquad \Rightarrow \qquad \frac{\frac{6}{5}}{1 + R_3/R_4} = \frac{3}{5}$$

or

$$2 = 1 + \frac{R_3}{R_4} \qquad \Rightarrow \qquad R_3 = R_4 \tag{5.7.3}$$

If we choose $R_1 = 10\ \text{k}\Omega$ and $R_3 = 20\ \text{k}\Omega$, then $R_2 = 50\ \text{k}\Omega$ and $R_4 = 20\ \text{k}\Omega$.

Design 2 If we desire to use more than one op amp, we may cascade an inverting amplifier and a two-input inverting summer, as shown in Fig. 5.25. For the summer,

$$v_o = -v_a - 5v_1 \tag{5.7.4}$$

and for the inverter,

$$v_a = -3v_2 \tag{5.7.5}$$

Combining Eqs. (5.7.4) and (5.7.5) gives

$$v_o = 3v_2 - 5v_1$$

which is the desired result. In Fig. 5.25, we may select $R_1 = 10\ \text{k}\Omega$ and $R_3 = 20\ \text{k}\Omega$ or $R_1 = R_3 = 10\ \text{k}\Omega$.

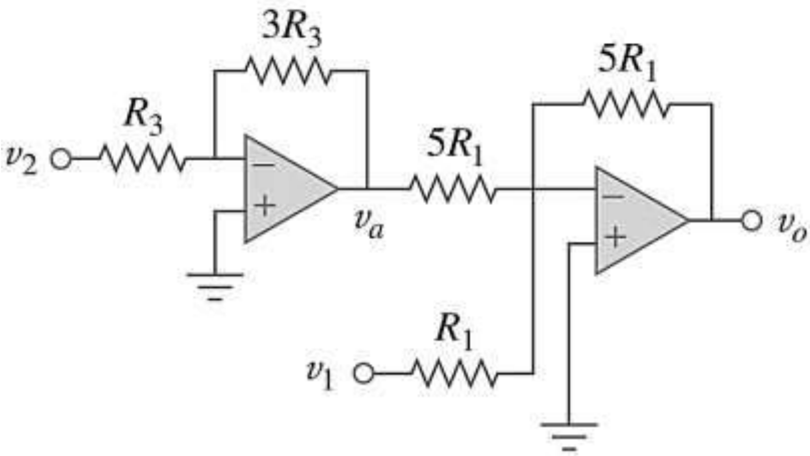


Figure 5.25
For Example 5.7

2.6- Cascaded Op Amp:

- A cascade connection is a head-to-tail arrangement of two or more op amp circuits such that the output of one is the input of the next.
- Figure 5.28 displays a block diagram representation of three op amp circuits in cascade.
- The output of one stage is the input to the next stage, the overall gain of the cascade connection is the product of the gains of the individual op amp circuits.

$$A = A_1 A_2 A_3 \qquad (5.22)$$

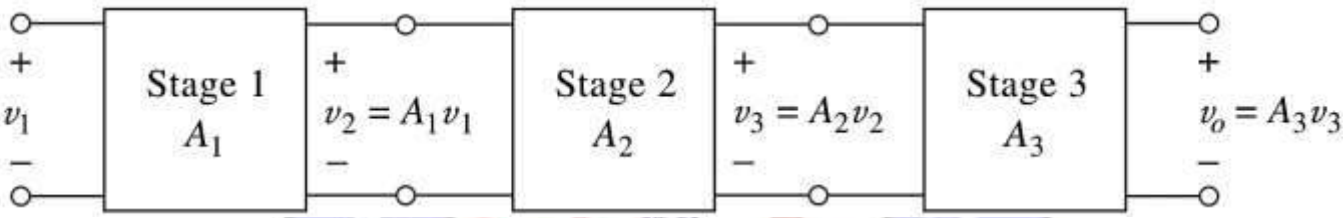


Figure 5.28
A three-stage cascaded connection.

Example 5.9:

Find v_o and i_o in the circuit in Fig. 5.29.

Solution:
This circuit consists of two noninverting amplifiers cascaded. At the output of the first op amp,

$$v_a = \left(1 + \frac{12}{3}\right)(20) = 100 \text{ mV}$$

At the output of the second op amp,

$$v_o = \left(1 + \frac{10}{4}\right)v_a = (1 + 2.5)100 = 350\ \text{mV}$$

The required current i_o is the current through the 10-kΩ resistor.

$$i_o = \frac{v_o - v_b}{10}\ \text{mA}$$

But $v_b = v_a = 100$ mV. Hence,

$$i_o = \frac{(350 - 100) \times 10^{-3}}{10 \times 10^3} = 25\ \mu\text{A}$$

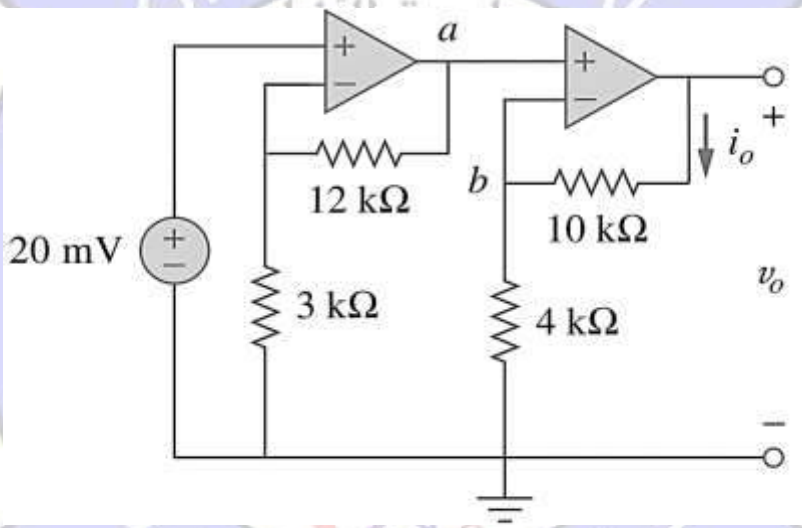


Figure 5.29
For Example 5.9.

2.7- Integrator Op Amp:

- An integrator is an op amp circuit whose output is proportional to the integral of the input signal.
- If the feedback resistor R_f in the familiar inverting amplifier of Fig. 6.35(a) is replaced by a capacitor, we obtain an ideal integrator, as shown in Fig. 6.35(b).

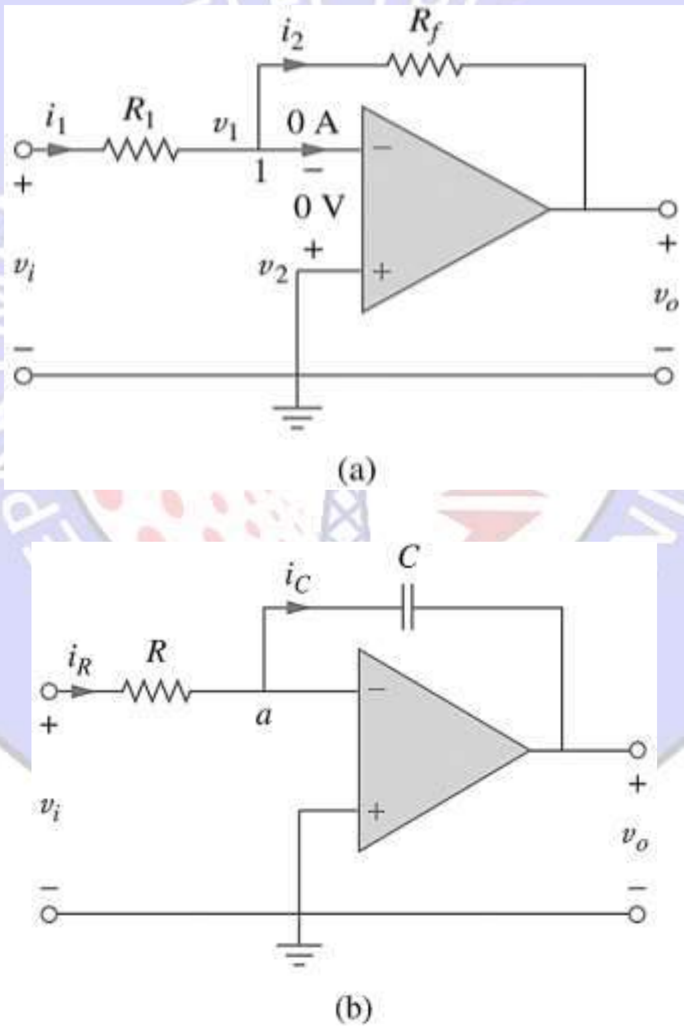


Figure 6.35
Replacing the feedback resistor in the inverting amplifier in (a) produces an integrator in (b).

- We can obtain a mathematical representation of integration. At node a in Fig. 6.35(b),

$$i_R = i_C \tag{6.32}$$

But

$$i_R = \frac{v_i}{R}, \qquad i_C = -C\frac{dv_o}{dt}$$

Substituting these in Eq. (6.32), we obtain

$$\frac{v_i}{R} = -C\frac{dv_o}{dt} \tag{6.33a}$$

$$dv_o = -\frac{1}{RC}v_i\,dt \tag{6.33b}$$

Integrating both sides gives

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau \tag{6.34}$$

To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrator's capacitor prior to the application of a signal.

$$\boxed{v_o = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau} \tag{6.35}$$

- Assuming $v_o(0) = 0$ which shows that the circuit in Fig. 6.35(b) provides an output voltage proportional to the integral of the input.
- In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation.

Example 6.13:

If $v_1 = 10\cos 2t$ mV and $v_2 = 0.5t$ mV, find v_o in the op amp circuit in Fig. 6.36. Assume that the voltage across the capacitor is initially zero.

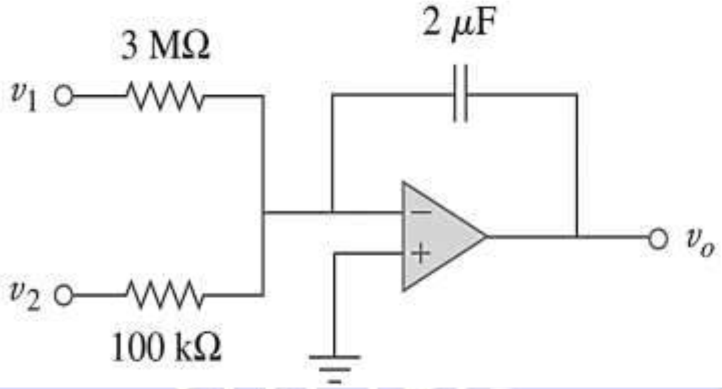


Figure 6.36
For Example 6.13.

Solution:

This is a summing integrator, and

$$v_o = -\frac{1}{R_1C}\int v_1\,dt - \frac{1}{R_2C}\int v_2\,dt$$

$$= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos(2\tau)d\tau$$

$$-\frac{1}{100\times 10^3\times 2\times 10^{-6}}\int_0^t 0.5\tau\,d\tau$$

$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2 \text{ mV}$$

2.8- Differentiator Op Amp:

- A differentiator is an op amp circuit whose output is proportional to the rate of change of the input signal.
- In Fig. 6.35(a), if the input resistor is replaced by a capacitor, the resulting circuit is a differentiator, shown in Fig. 6.37.

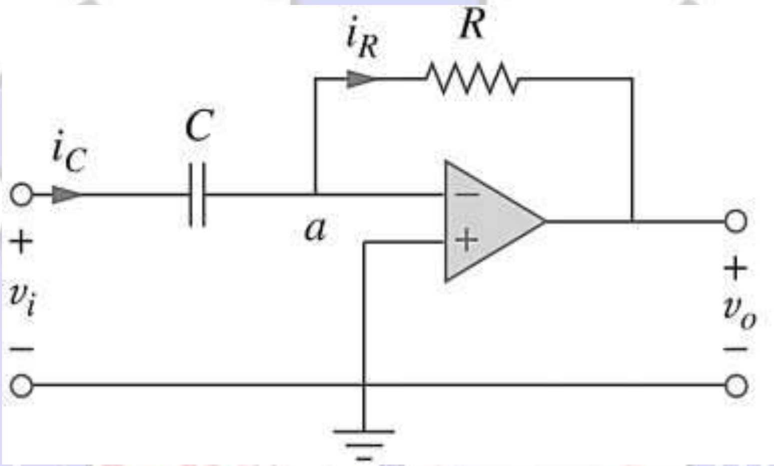


Figure 6.37
An op amp differentiator.

Applying KCL at node a,

$$i_R = i_C \tag{6.36}$$

But

$$i_R = -\frac{v_o}{R}, \qquad i_C = C\frac{dv_i}{dt}$$

Substituting these in Eq. (6.36) yields

$$v_o = -RC\frac{dv_i}{dt} \tag{6.37}$$

- Above equation shows that the output is the derivative of the input.

Chapter Three

First-Order Circuits

3.1- Introduction

3.2- Source-free RC circuit

3.3- Source-free RL circuit

3.4- Step response of RC circuit

3.5- Step response of RL circuit

3.6- First-order Op. Amp. circuit

3.7- Examples

3.1- Introduction:

- The three passive elements (resistors, capacitors, and inductors) and one active element (the op amp) have been considered.
- Two types of simple circuits have been considered such as a circuit which consists of resistor and capacitor called RC, another circuit called RL which consists of resistor and an inductor.
- RC and RL circuits can be analyzed by applying Kirchhoff's laws.
- Applying the Kirchhoff's laws to RC and RL circuits produces differential equations.
- The differential equations resulting from analyzing RC and RL circuits are called first order circuits.
- A first-order circuit is characterized by a first-order differential equation.
- There are two ways to excite RC and RL circuits.
- The first way is by initial conditions of the storage elements in the circuits which called source-free circuits. Assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors.
- The second way of exciting first-order circuits is by independent sources. the independent sources such as dc sources have been considered.

3.2- Source-free RC circuit:

- A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.
- Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 7.1.

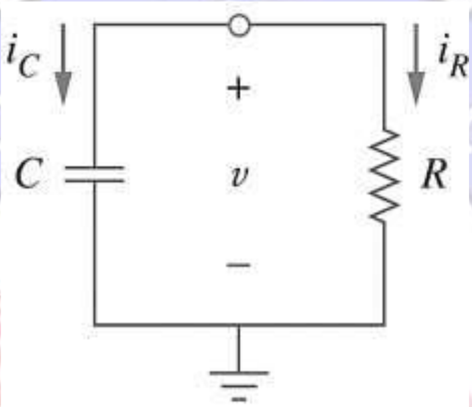


Figure 7.1
A source-free *RC* circuit.

- The objective is to determine the circuit response.
- Assume that at time t=0, initial voltage across the capacitor is

$$v(0) = V_0 \qquad (7.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad (7.2)$$

Applying KCL at the top node of the circuit in Fig. 7.1 yields

$$i_C + i_R = 0 \qquad (7.3)$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \tag{7.4a}$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \tag{7.4b}$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}\,dt \tag{7.5}$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC} \tag{7.6}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0 e^{-t/RC} \tag{7.7}$$

- This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. The response is due to the initial energy stored which is called the natural response of the circuit.

- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- The natural response is illustrated graphically in Fig. 7.2. As t increases, the voltage decreases toward zero.

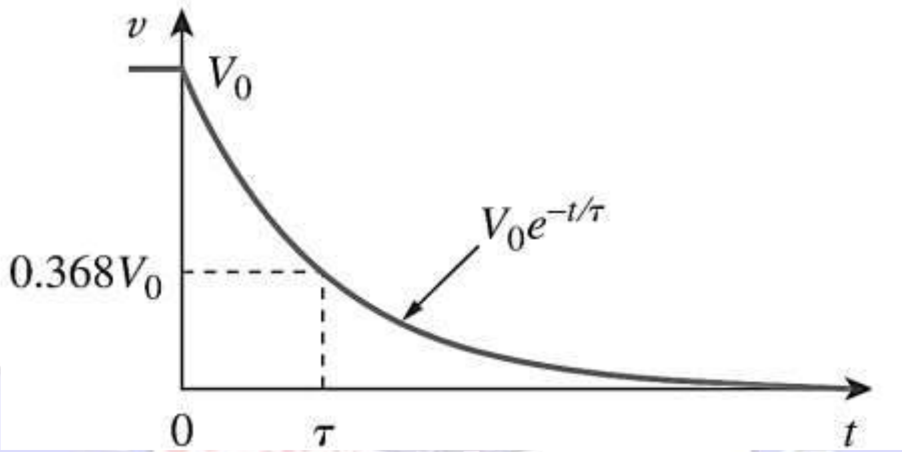


Figure 7.2
The voltage response of the *RC* circuit.

- The voltage decreasing is expressed in terms of the time constant, denoted by τ,
- The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

This implies that at $t = \tau$, Eq. (7.7) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

or

$$\boxed{\tau = RC} \qquad (7.8)$$

In terms of the time constant, Eq. (7.7) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \qquad (7.9)$$

- With a calculator it is easy to show that the value of is as shown in Table 7.1.

TABLE 7.1

Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

- Observe from Eq. (7.8) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.
- This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state)

quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state.

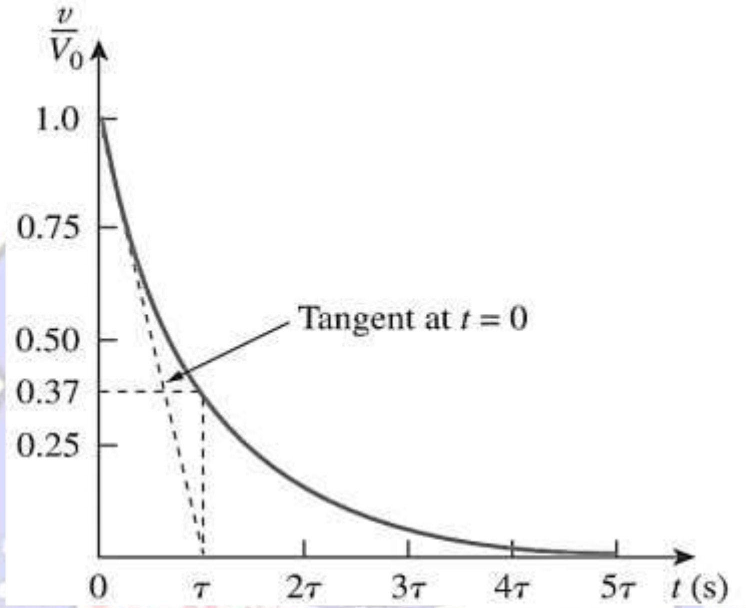


Figure 7.3
Graphical determination of the time constant τ from the response curve.

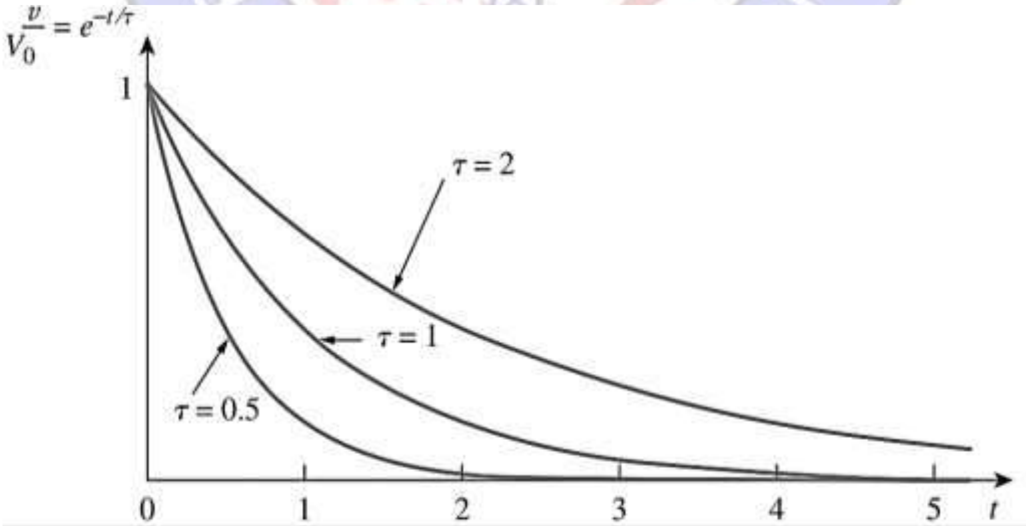


Figure 7.4
Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant

With the voltage $v(t)$ in Eq. (7.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \tag{7.10}$$

The Key to Working with a Source-Free *RC* Circuit Is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

Example 7.1:

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

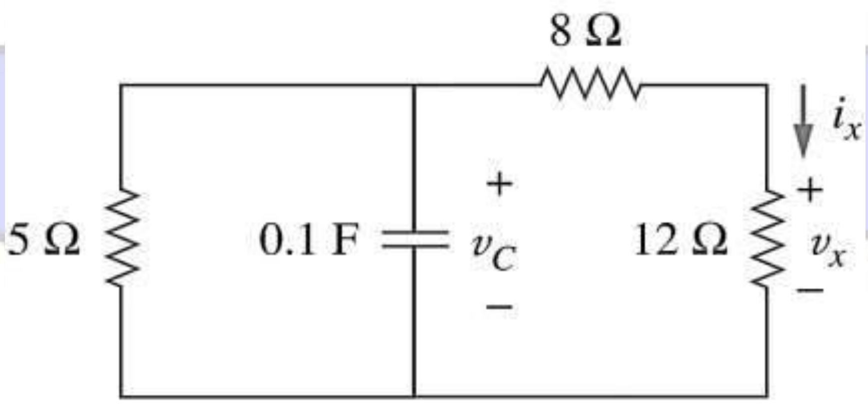


Figure 7.5
For Example 7.1.

- We first need to make the circuit in Fig. 7.5 conform with the standard RC circuit in Fig. 7.1.
- We find the equivalent resistance or the Thevenin resistance at the capacitor terminals.

- The objective is always to first obtain capacitor voltage v_c. From this, we can determine v_x and i_x.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

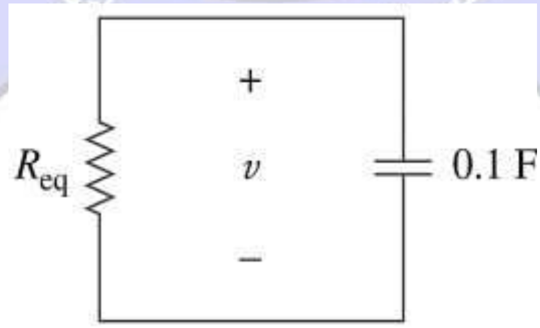


Figure 7.6
Equivalent circuit for the circuit in Fig. 7.5.

Example 7.2:

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

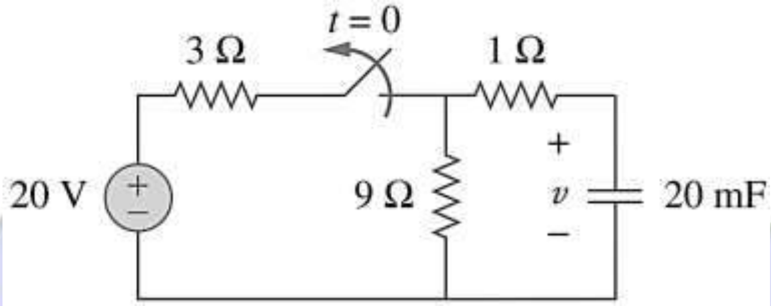


Figure 7.8
For Example 7.2

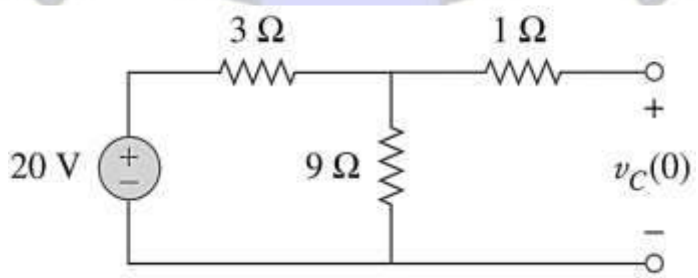
Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$



(a)

Figure 7.9
For Example 7.2: (a) $t < 0$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or

$$v(t) = 15e^{-5t}\ \text{V}$$

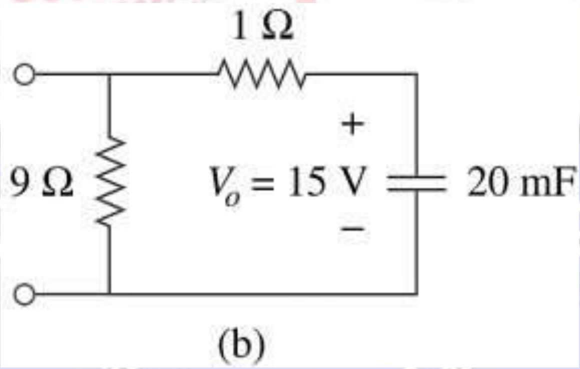


Figure 7.9
For Example 7.2: (b) $t > 0$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ \text{J}$$

3.3- Source-free RL circuit:

- Consider the series connection of a resistor and an inductor, as shown in Fig. 7.11.

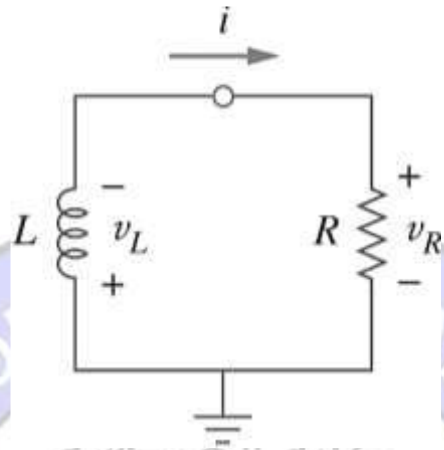


Figure 7.11
A source-free *RL* circuit

- The goal is to determine the circuit response, assume that the current $i(t)$ will through the inductor.
- The idea that the inductor current cannot change instantaneously. At $t = 0$, it is assumed that the inductor has an initial current I_o, or

$$i(0) = I_0 \tag{7.13}$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2 \tag{7.14}$$

Applying KVL around the loop in Fig. 7.11,

$$v_L + v_R = 0 \tag{7.15}$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \tag{7.16}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}dt$$

$$\ln i\Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{7.17}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{7.18}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig. 7.12. It is evident from Eq. (7.18) that the time constant for the RL circuit is

$$\boxed{\tau = \frac{L}{R}} \tag{7.19}$$

with τ again having the unit of seconds. Thus, Eq. (7.18) may be written as

$$\boxed{i(t) = I_0 e^{-t/\tau}} \tag{7.20}$$

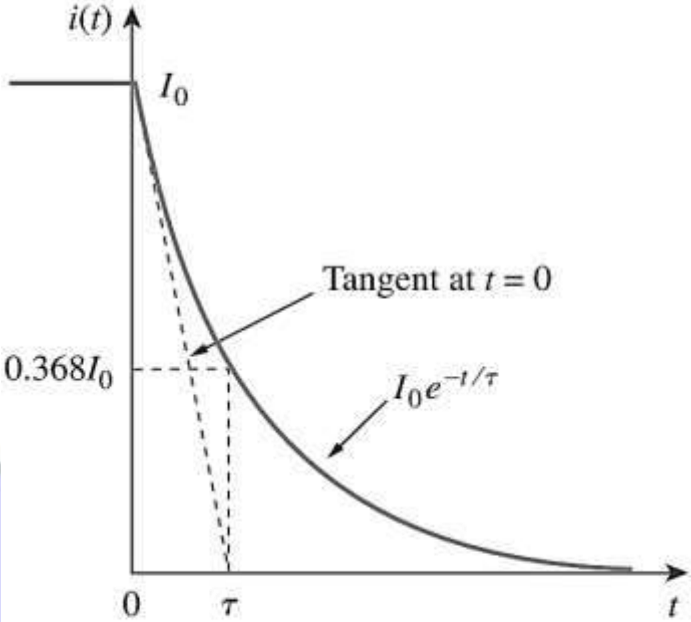


Figure 7.12
The current response of the *RL* circuit.

With the current in Eq. (7.20), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{7.21}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{7.22}$$

The Key to Working with a Source-Free *RL* Circuit Is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

Example 7.3:

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig. 7.13.

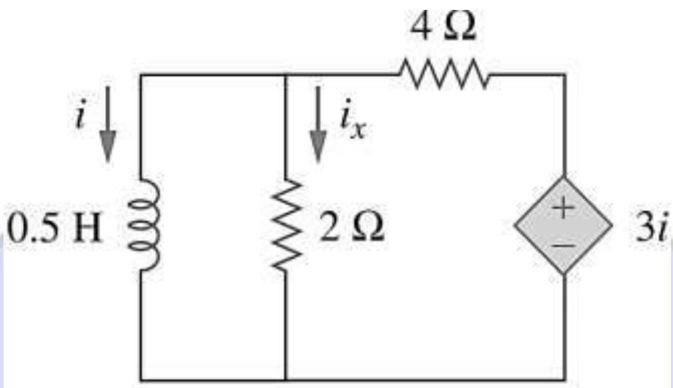


Figure 7.13
For Example 7.3

Solution:

There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (7.20). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

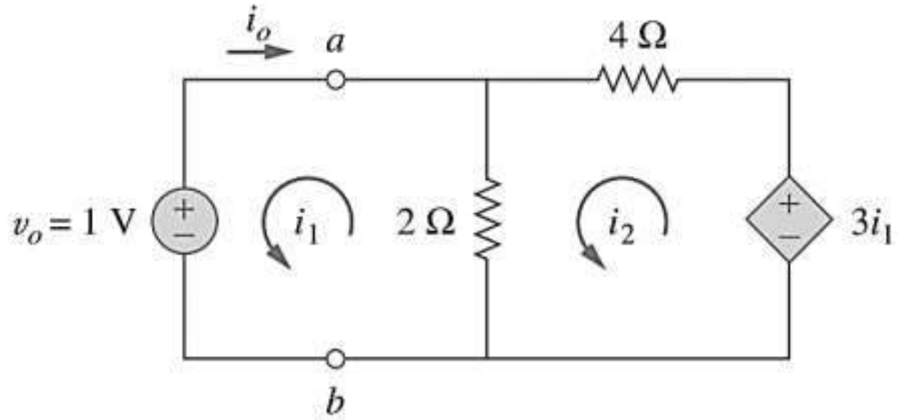
■ **METHOD 1** The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals *a-b*, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2} \qquad \textbf{(7.3.1)}$$

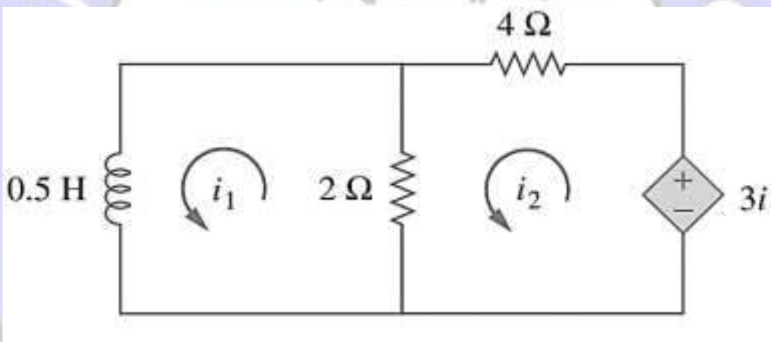
$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \qquad \textbf{(7.3.2)}$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives

$$i_1 = -3\text{ A}, \qquad i_o = -i_1 = 3\text{ A}$$



(a)



(b)

Figure 7.14
Solving the circuit in Fig. 7.13.

Hence,

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\ \text{A}, \qquad t > 0$$

METHOD 2 We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \tag{7.3.3}$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \qquad \Rightarrow \qquad i_2 = \frac{5}{6}i_1 \tag{7.3.4}$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i\Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_{0}^{t}$$

or

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t}\text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t}\text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t}\text{ A}, \qquad t > 0$$

Example 7.4:

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

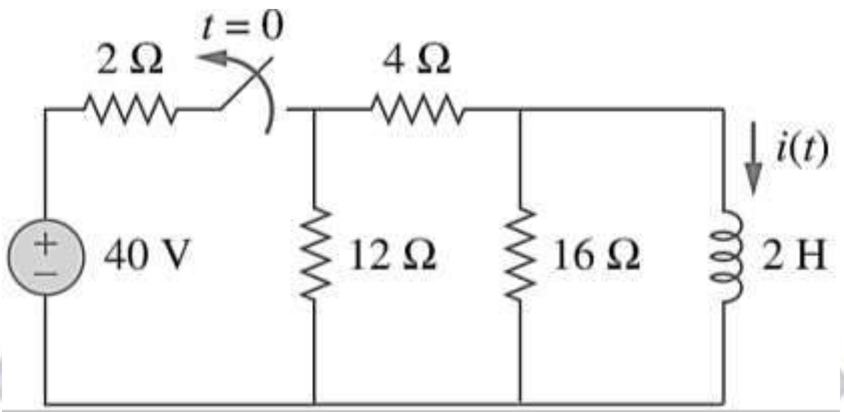


Figure 7.16
For Example 7.4.

Solution:

When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig. 7.17(a). To get i_1 in Fig. 7.17(a), we combine the 4-Ω and 12-Ω resistors in parallel to get

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

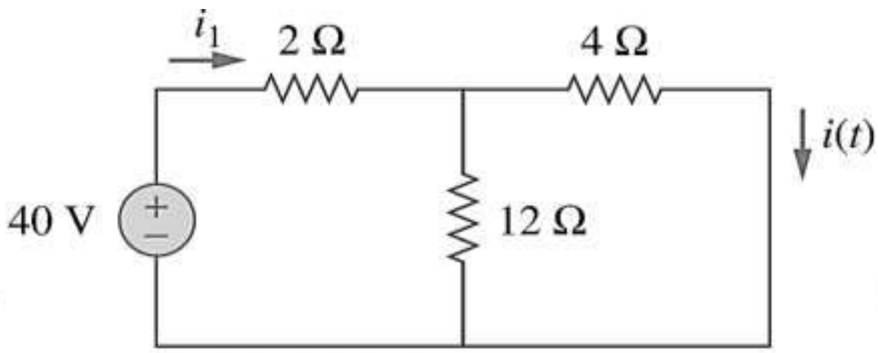
$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

We obtain $i(t)$ from i_1 in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12+4} i_1 = 6 \text{ A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6 \text{ A}$$



(a)

Figure 7.17
Fig. 7.16: (a) for $t < 0$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free *RL* circuit in Fig. 7.17(b). Combining the resistors, we have

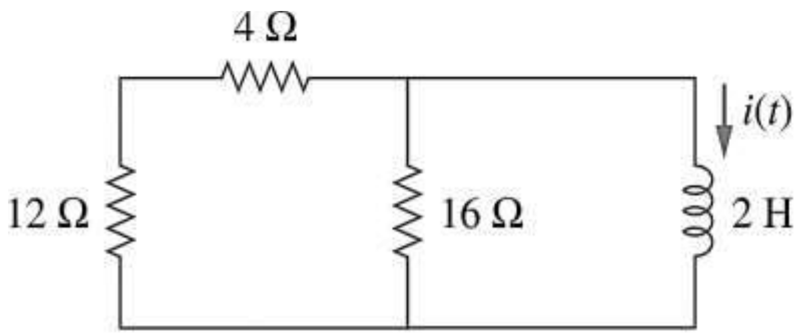
$$R_{eq} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} \text{ s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t} \text{ A}$$



(b)

Figure 7.17
Fig. 7.16: (b) for $t > 0$

Example 7.5:

In the circuit shown in Fig. 7.19, find i_o, v_o, and i for all time, assuming that the switch was open for a long time.

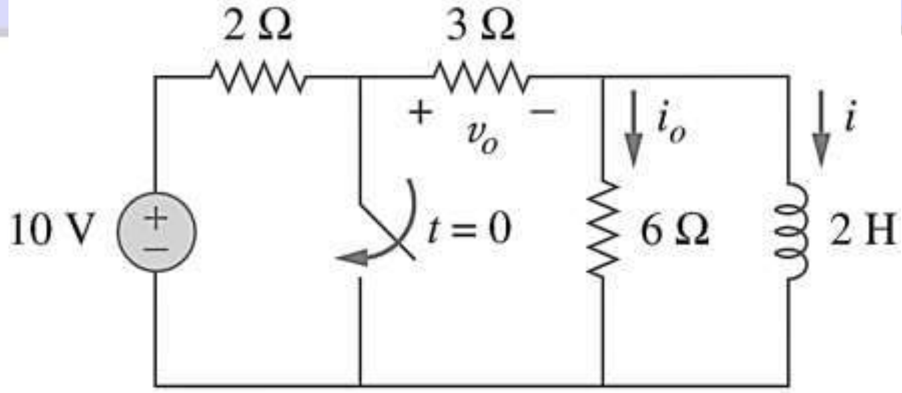


Figure 7.19
For Example 7.5.

Solution:

It is better to first find the inductor current i and then obtain other quantities from it.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

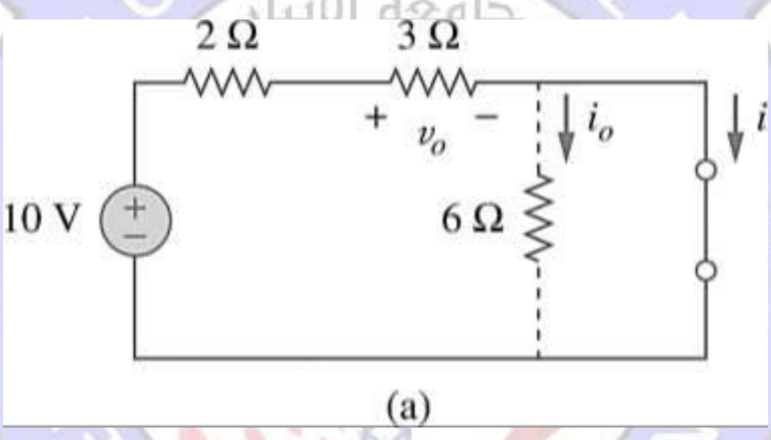


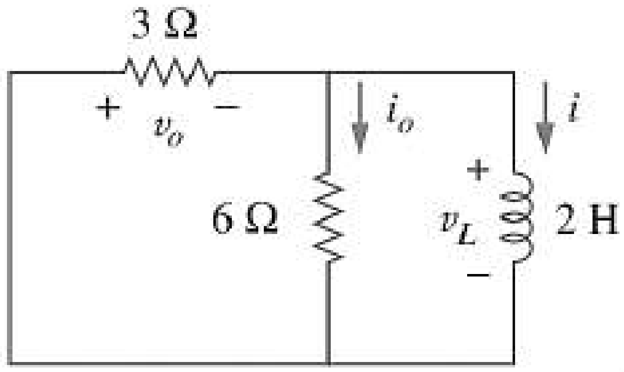
Figure 7.20
Fig. 7.19 for: (a) $t < 0$

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals,

$$R_{\text{Th}} = 3 \,\|\, 6 = 2\,\Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{\text{Th}}} = 1\text{ s}$$



(b)

Figure 7.20
Fig. 7.19 (b) for $t > 0$

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

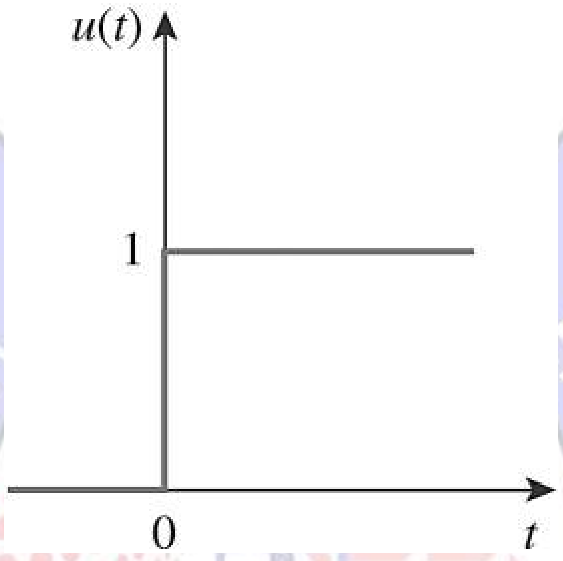
Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

and

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$

- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.
- The unit step function u(t) is 0 for negative values of t and 1 for positive values of t as shown below



In mathematical terms,

$$\boxed{u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}} \qquad \textbf{(7.24)}$$

If the change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \qquad \textbf{(7.25)}$$

which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in Fig. 7.24(a).

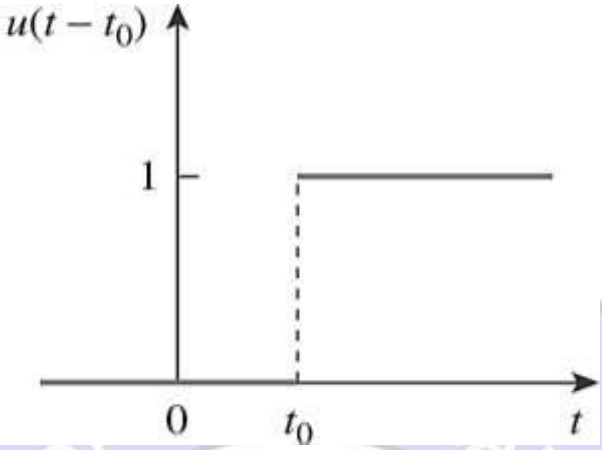


Figure 7.24
(a) The unit step function delayed by t_0,

If the change is at $t = -t_0$, the unit step function becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \quad \textbf{(7.26)}$$

meaning that $u(t)$ is advanced by t_0 seconds, as shown in Fig. 7.24(b).

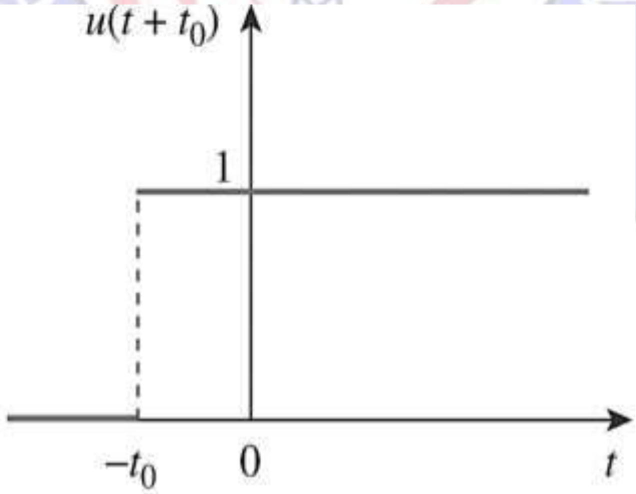


Figure 7.24
(b) the unit step advanced by t_0.

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \tag{7.27}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \tag{7.28}$$

If we let $t_0 = 0$, then $v(t)$ is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig. 7.25(a); its equivalent circuit is shown in Fig. 7.25(b). It is evident in Fig. 7.25(b) that terminals a-b are short-circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals

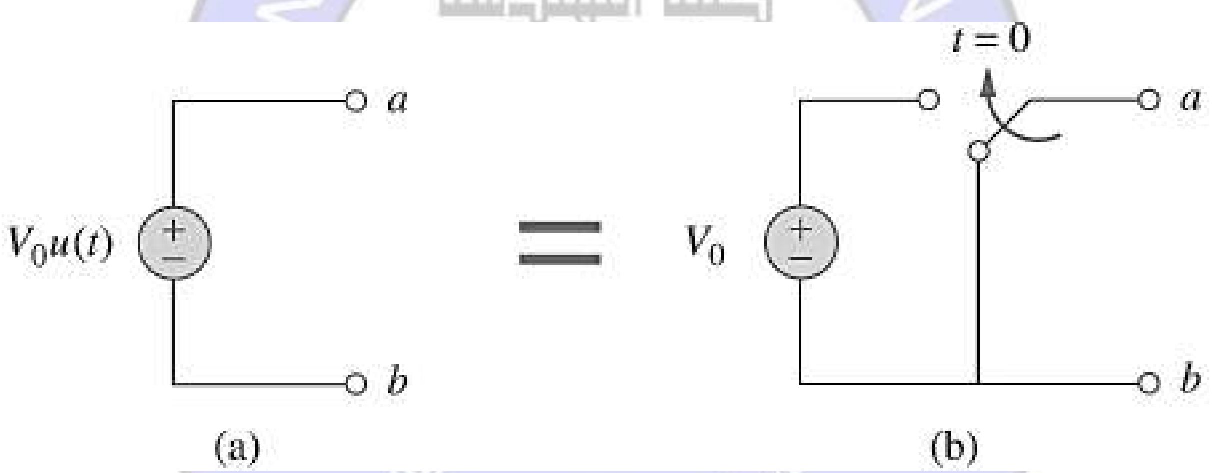


Figure 7.25
(a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

for $t > 0$. Similarly, a current source of $I_0 u(t)$ is shown in Fig. 7.26(a), while its equivalent circuit is in Fig. 7.26(b). Notice that for $t < 0$, there is an open circuit ($i = 0$), and that $i = I_0$ flows for $t > 0$.

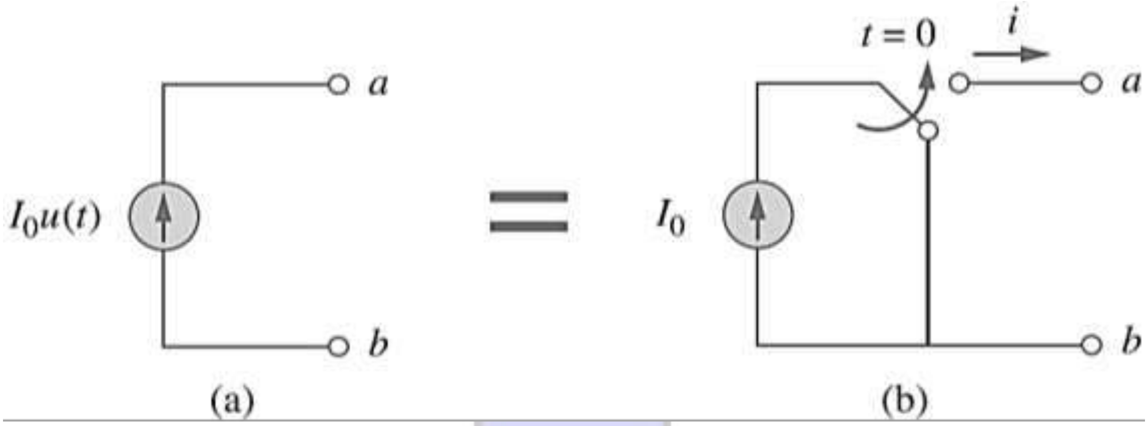
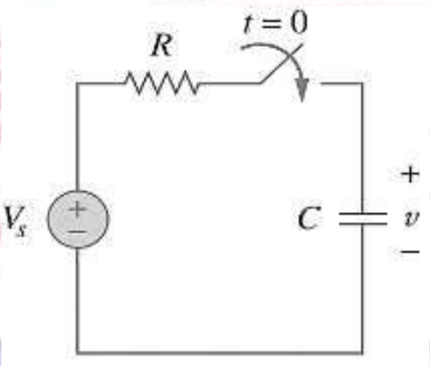


Figure 7.26
(a) Current source of $I_0u(t)$, (b) its equivalent circuit.

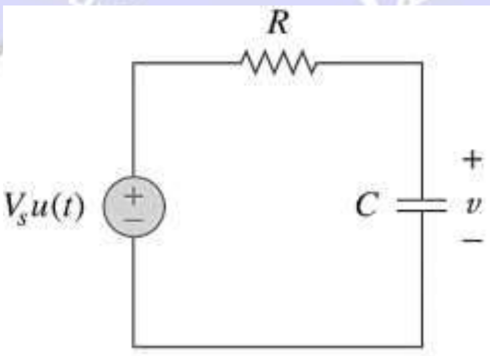
3.4- Step response of RC circuit:

- When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a step response.
- The step response of a circuit is its behavior when the excitation is the step function which may be a voltage or a current source.

Consider the RC circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where V_s is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined.



(a)



(b)

Figure 7.40
An RC circuit with voltage step input.

We assume an initial voltage V_0 on the capacitor,
Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad \textbf{(7.40)}$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad \textbf{(7.41)}$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (7.41) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad \textbf{(7.42)}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \qquad \textbf{(7.43)}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \qquad \textbf{(7.44)}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \tag{7.45}$$

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{7.46}$$

This is known as the *complete response* (or total response) of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig. 7.41.

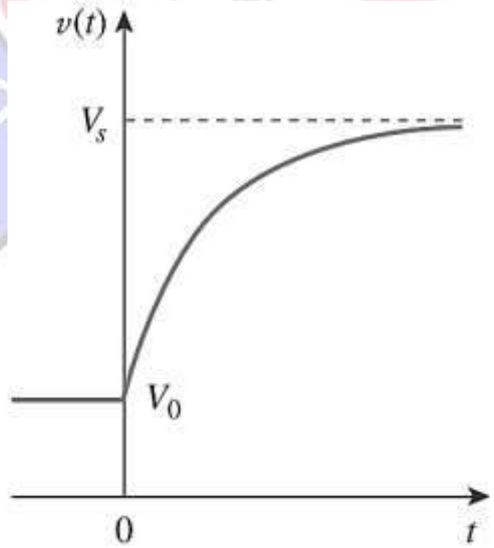


Figure 7.41
Response of an *RC* circuit with initially charged capacitor.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (7.46) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{7.47}$$

which can be written alternatively as

$$v(t) = V_s(1 - e^{-t/\tau})u(t) \tag{7.48}$$

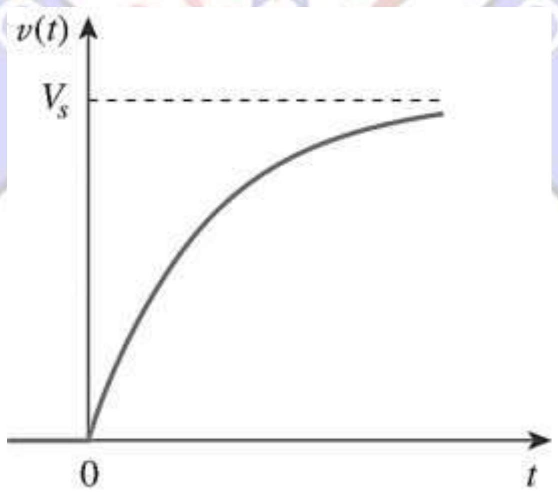
This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (7.47) using $i(t) = C\,dv/dt$. We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \qquad t > 0$$

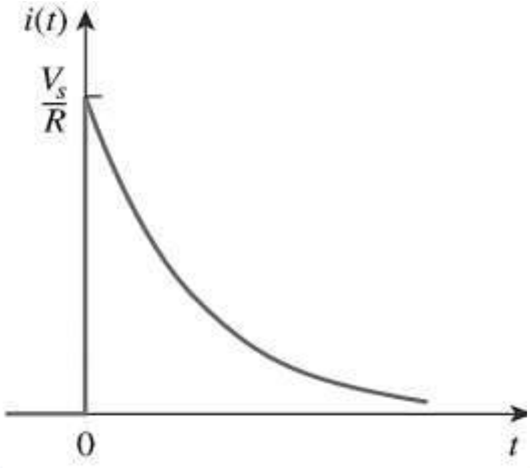
or

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \tag{7.49}$$

Figure 7.42 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.



(a)



(b)

Figure 7.42
Step response of an *RC* circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

$$\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}} + \underbrace{\text{forced response}}_{\text{independent source}}$$

or

$$v = v_n + v_f \tag{7.50}$$

where

$$v_n = V_o e^{-t/\tau}$$

and

$$v_f = V_s(1 - e^{-t/\tau})$$

We are familiar with the natural response v_n of the circuit, as discussed in Section 7.2. v_f is known as the *forced* response because it is produced by the circuit when an external "force" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady-state component of the forced response.

The complete response in Eq. (7.45) may be written as

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}} \qquad \textbf{(7.53)}$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady-state value. Thus, to find the step response of an *RC* circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

Example 7.10:

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.

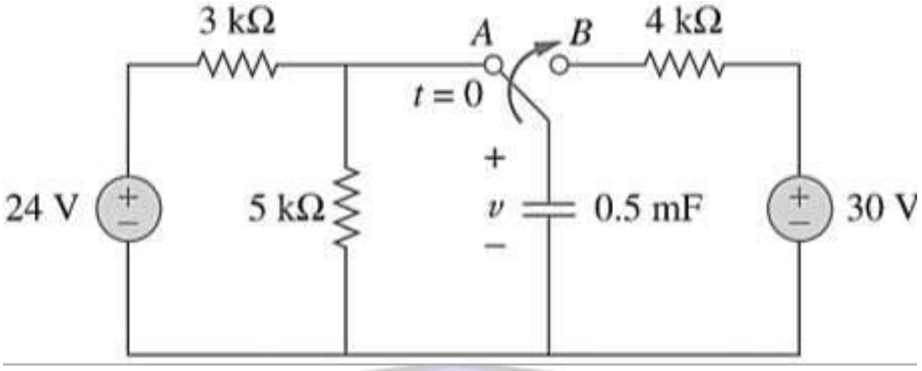


Figure 7.43
For Example 7.10.

Solution:

For $t < 0$, the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the 5-kΩ resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4$ kΩ, and the time constant is

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\ \text{V}$$

At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.9\ \text{V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97\ \text{V}$$

Example 7.11:

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

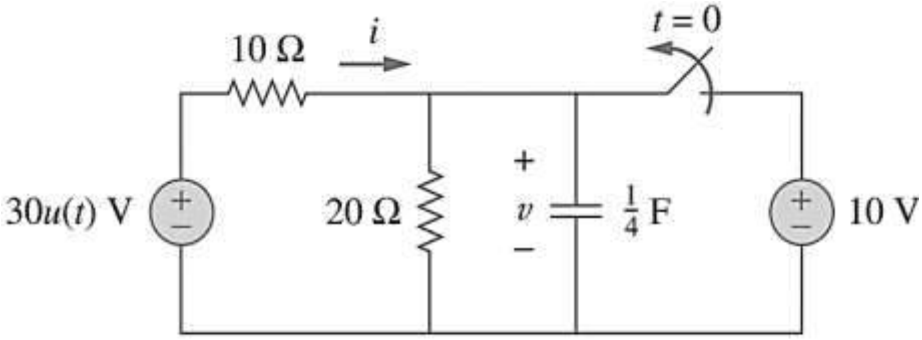


Figure 7.45
For Example 7.11.

Solution:
The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

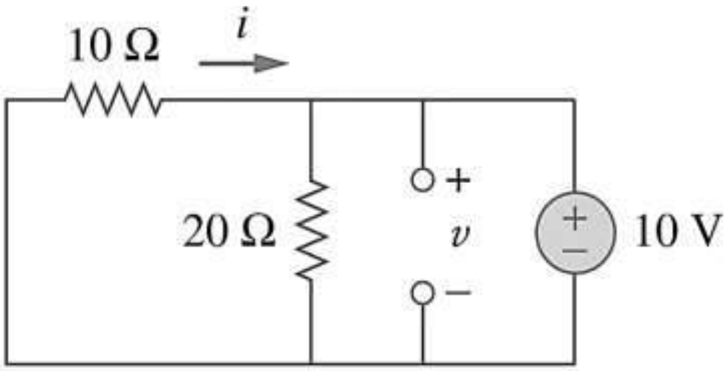
By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig. 7.46(a) for $t < 0$. From this circuit we obtain

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10 \text{ V}$$



(a)

Figure 7.46
Example 7.11: (a) for $t < 0$.

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \,\|\, 20 = \frac{10 \times 20}{30} = \frac{20}{3}\,\Omega$$

and the time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\text{ s}$$

Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V}$$

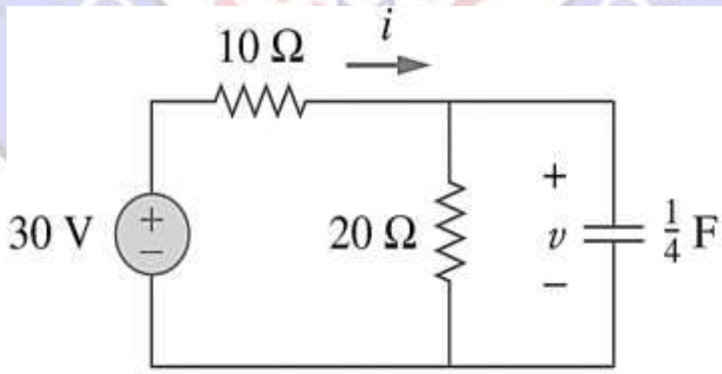
To obtain i, we notice from Fig. 7.46(b) that i is the sum of the currents through the 20-Ω resistor and the capacitor; that is,

$$i = \frac{v}{20} + C\frac{dv}{dt}$$
$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A}$$

Notice from Fig. 7.46(b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10\text{ V}, & t < 0 \\ (20 - 10e^{-0.6t})\text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1\text{ A}, & t < 0 \\ (1 + e^{-0.6t})\text{ A}, & t > 0 \end{cases}$$



(b)

Figure 7.46
Example 7.11: (b) for $t > 0$

3.5- Step response of RL circuit:

Consider the *RL* circuit in Fig. 7.48(a), which may be replaced by the circuit in Fig. 7.48(b). Again, our goal is to find the inductor current i as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (7.50) through (7.53). Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \tag{7.55}$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{7.56}$$

where A is a constant to be determined.

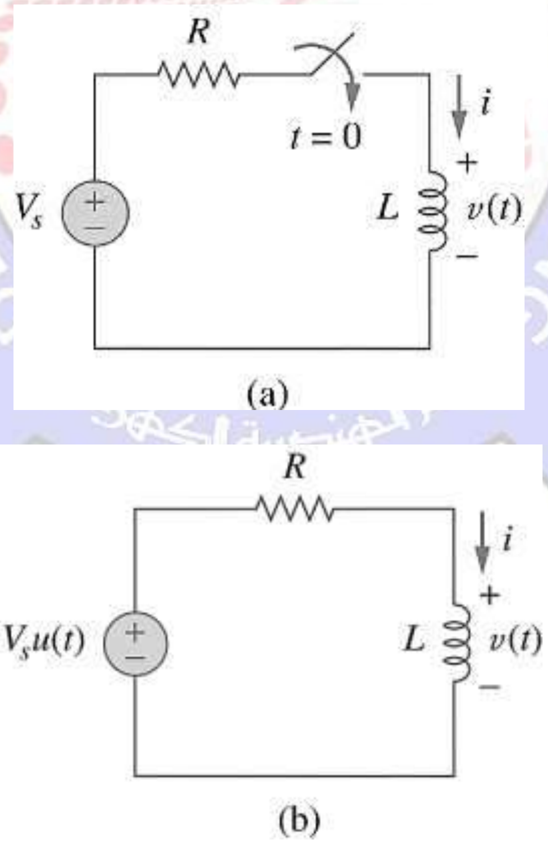


Figure 7.48
An *RL* circuit with a step input voltage.

The steady-state response is the value of the current a long time after the switch in Fig. 7.48(a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R} \tag{7.57}$$

Substituting Eqs. (7.56) and (7.57) into Eq. (7.55) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \tag{7.58}$$

We now determine the constant A from the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{7.59}$$

Thus, at $t = 0$, Eq. (7.58) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Eq. (7.58), we get

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \tag{7.60}$$

This is the complete response of the *RL* circuit.
The response in Eq. (7.60) may be written as

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}} \tag{7.61}$$

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an *RL* circuit requires three things:

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

Example 7.12:

Find $i(t)$ in the circuit of Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.

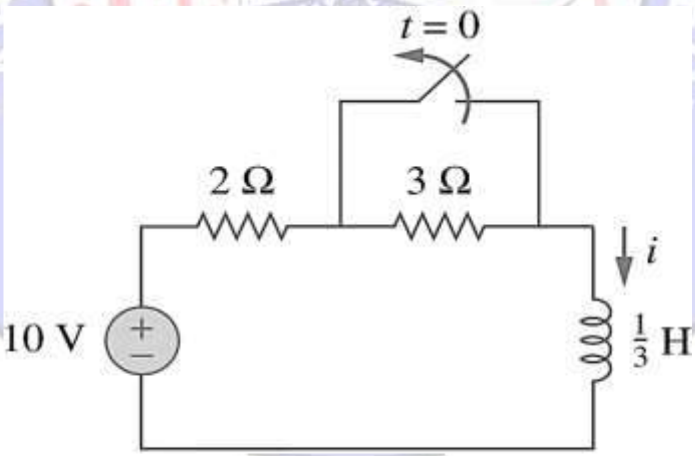


Figure 7.51
For Example 7.12

Solution:

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2 + 3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{\text{Th}} = 2 + 3 = 5\ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

Check: In Fig. 7.51, for $t > 0$, KVL must be satisfied; that is,

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + \left[\frac{1}{3}(3)(-15)e^{-15t}\right] = 10$$

This confirms the result.

Problems: Section 7.2 The Source-Free RC Circuit

7.1 In the circuit shown in Fig. 7.81

$$v(t) = 56e^{-200t}\ \text{V}, \quad t > 0$$
$$i(t) = 8e^{-200t}\ \text{mA}, \quad t > 0$$

(a) Find the values of R and C.

(b) Calculate the time constant τ.

(c) Determine the time required for the voltage to decay half its initial value at $t = 0$.

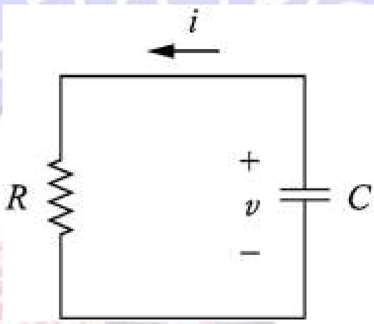


Figure 7.81
For Prob. 7.1.

(a) $\tau = RC = 1/200$

For the resistor, $V = iR = 56e^{-200t} = 8Re^{-200t} \times 10^{-3} \longrightarrow R = \frac{56}{8} = \underline{7\ k\Omega}$

$$C = \frac{1}{200R} = \frac{1}{200X7X10^3} = \underline{0.7143\mu F}$$

(b) $\tau = 1/200 = \underline{\mathbf{5\ ms}}$

(c) If value of the voltage at = 0 is 56 .

$$\frac{1}{2}x56 = 56e^{-200t} \longrightarrow e^{200t} = 2$$

$$200t_o = \ln 2 \longrightarrow t_o = \frac{1}{200}\ln 2 = \underline{3.466\ ms}$$

7.4 The switch in Fig. 7.84 has been in position A for a long time. Assume the switch moves instantaneously from A to B at $t = 0$. Find v for $t > 0$.

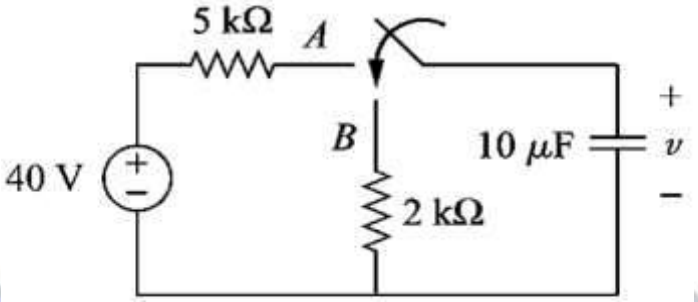


Figure 7.84
For Prob. 7.4.

For $t<0$, $v(0^-)=40$ V.

For $t>0$. we have a source-free RC circuit.

$$\tau = RC = 2x10^3 x10x10^{-6} = 0.02$$

$$v(t) = v(0)e^{-t/\tau} = \underline{40e^{-50t}\ \text{V}}$$

Section 7.3 The Source-Free RL Circuit:

7.11 For the circuit in Fig. 7.91, find i_o for $t > 0$.

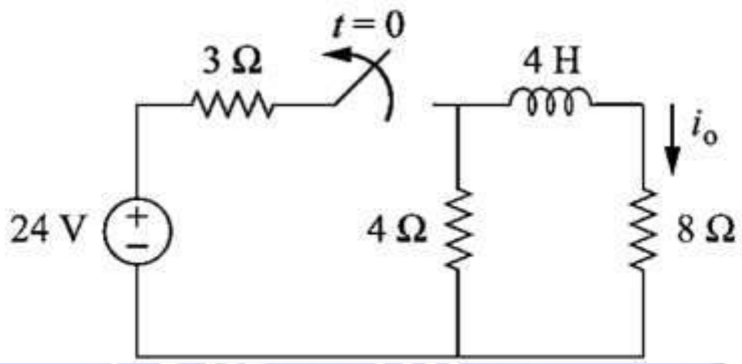
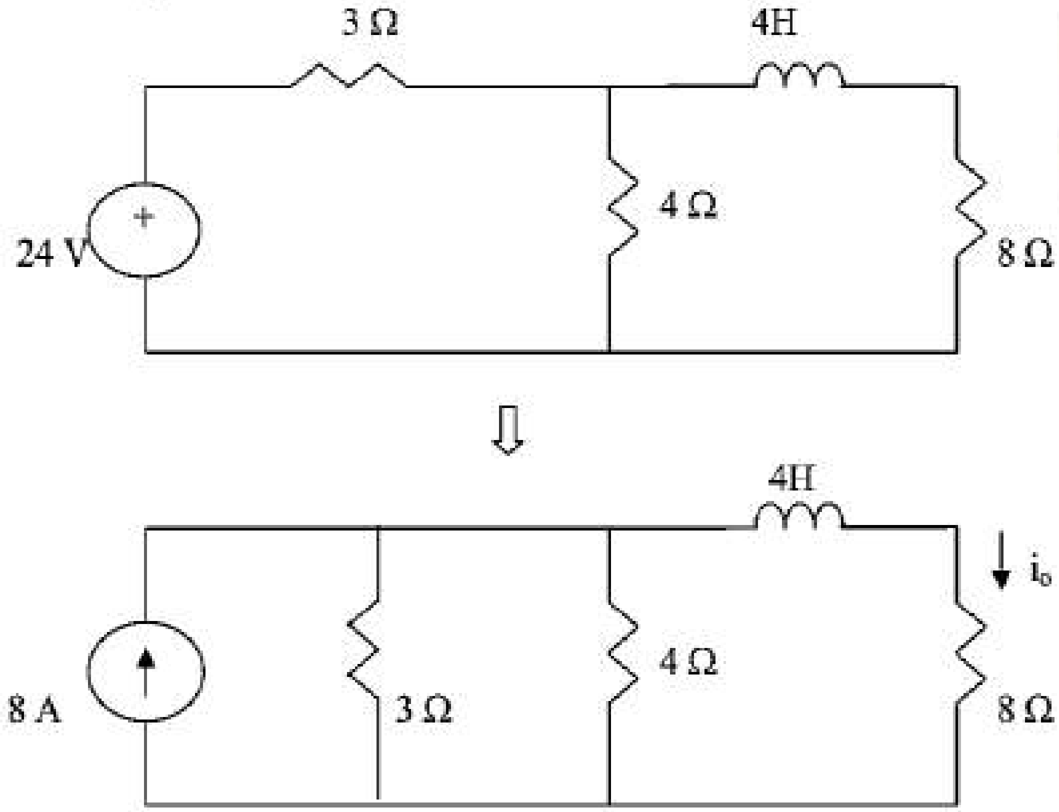


Figure 7.91
For Prob. 7.11.

For t<0, we have the circuit shown below.



3//4= 4x3/7=1.7143

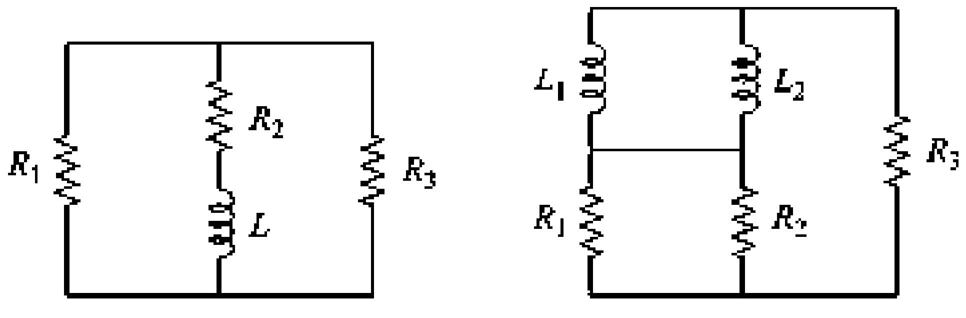
$$i_o(0^-) = \frac{1.7143}{1.7143+8}(8) = 1.4118\text{ A}$$

For t >0, we have a source-free RL circuit.

$$\tau = \frac{L}{R} = \frac{4}{4+8} = 1/3$$

$$i_o(t) = i_o(0)e^{-t/\tau} = \underline{1.4118e^{-3t}\text{ A}}$$

7.16 Determine the time constant for each of the circuits in Fig. 7.96.



$$\tau = \frac{L_{eq}}{R_{eq}}$$

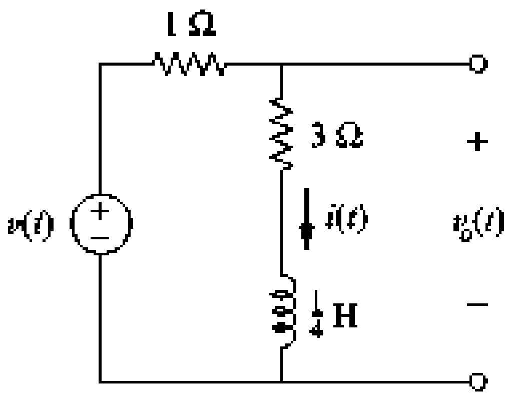
(a) $L_{eq} = L$ and $R_{eq} = R_2 + \dfrac{R_1R_3}{R_1+R_3} = \dfrac{R_2(R_1+R_3)+R_1R_3}{R_1+R_3}$

$$\tau = \underline{\frac{\mathbf{L(R_1+R_3)}}{\mathbf{R_2(R_1+R_3)+R_1R_3}}}$$

(b) where $L_{eq} = \dfrac{L_1L_2}{L_1+L_2}$ and $R_{eq} = R_3 + \dfrac{R_1R_2}{R_1+R_2} = \dfrac{R_3(R_1+R_2)+R_1R_2}{R_1+R_2}$

$$\tau = \underline{\frac{\mathbf{L_1L_2(R_1+R_2)}}{\mathbf{(L_1+L_2)(R_3(R_1+R_2)+R_1R_2)}}}$$

7.17 Consider the circuit of Fig. 7.97. Find $v_o(t)$ if $i(0) = 2$ A and $v(t) = 0$.



$$i(t) = i(0)e^{-t/\tau}, \qquad \tau = \frac{L}{R_{eq}} = \frac{1/4}{4} = \frac{1}{16}$$

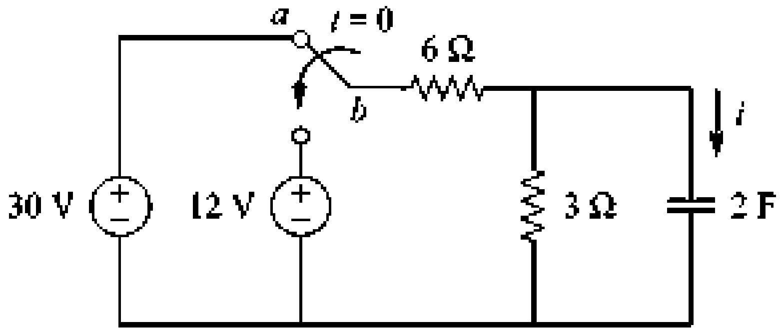
$$i(t) = 2e^{-16t}$$

$$v_o(t) = 3i + L\frac{di}{dt} = 6e^{-16t} + (1/4)(-16)2e^{-16t}$$

$$v_o(t) = \underline{-2e^{-16t}\ u(t)V}$$

Section 7.5 Step Response of an RC Circuit:

7.44 The switch in Fig. 7.111 has been in position a for a long time. At $t = 0$, it moves to position b. Calculate $i(t)$ for all $t > 0$.



$$R_{eq} = 6 \parallel 3 = 2\,\Omega, \qquad \tau = RC = 4$$

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right] e^{-t/\tau}$$

Using voltage division,

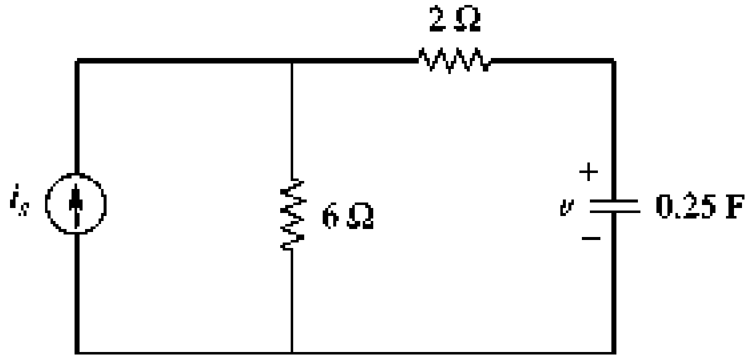
$$v(0) = \frac{3}{3+6}(30) = 10\text{ V}, \qquad v(\infty) = \frac{3}{3+6}(12) = 4\text{ V}$$

Thus,

$$v(t) = 4 + (10-4)\,e^{-t/4} = 4 + 6\,e^{-t/4}$$

$$i(t) = C\frac{dv}{dt} = (2)(6)\left(\frac{-1}{4}\right)e^{-t/4} = \underline{-3e^{-0.25t}\text{ A}}$$

7.46 For the circuit in Fig. 7.113, $i_s(t) = 5u(t)$. Find $v(t)$.



$$\tau = R_{Th}C = (2+6)x0.25 = 2s, \qquad v(0) = 0, \qquad v(\infty) = 6i_s = 6x5 = 30$$

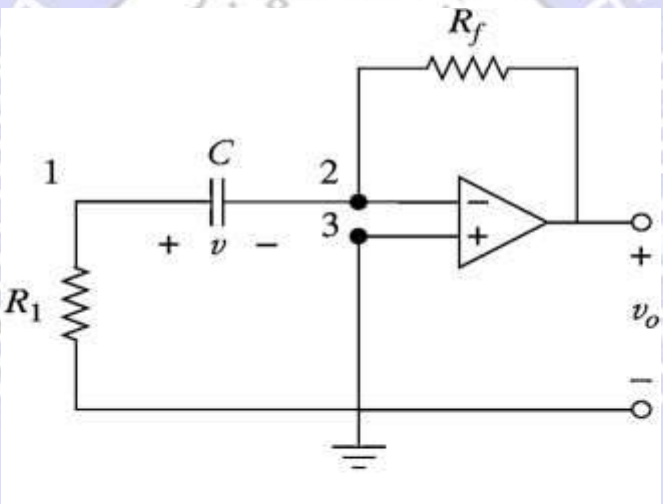
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} = 30(1 - e^{-t/2})\ \text{V}$$

3.6- First-order Op Amp circuit:

- Differentiators and integrators are examples of first-order op amp circuits.
- RC circuit will be considered in Op Amp representation.
- Nodal analysis will be used to analyze Op Amp circuits.

Example 7.14:

For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5\ \mu$F.



(a)

Figure 7.55
For Example 7.14

Consider the circuit in Fig. 7.55(a). Let us derive the appropriate differential equation using nodal analysis. If v_1 is the voltage at node 1, at that node, KCL gives

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt} \qquad \textbf{(7.14.1)}$$

Since nodes 2 and 3 must be at the same potential, the potential at node 2 is zero. Thus, $v_1 - 0 = v$ or $v_1 = v$ and Eq. (7.14.1) becomes

$$\frac{dv}{dt} + \frac{v}{CR_1} = 0 \tag{7.14.2}$$

This is similar to Eq. (7.4b) so that the solution is obtained the same way as in Section 7.2, i.e.,

$$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C \tag{7.14.3}$$

where V_0 is the initial voltage across the capacitor. But $v(0) = 3 = V_0$ and $\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1$. Hence,

$$v(t) = 3e^{-10t} \tag{7.14.4}$$

Applying KCL at node 2 gives

$$C\frac{dv}{dt} = \frac{0 - v_o}{R_f}$$

or

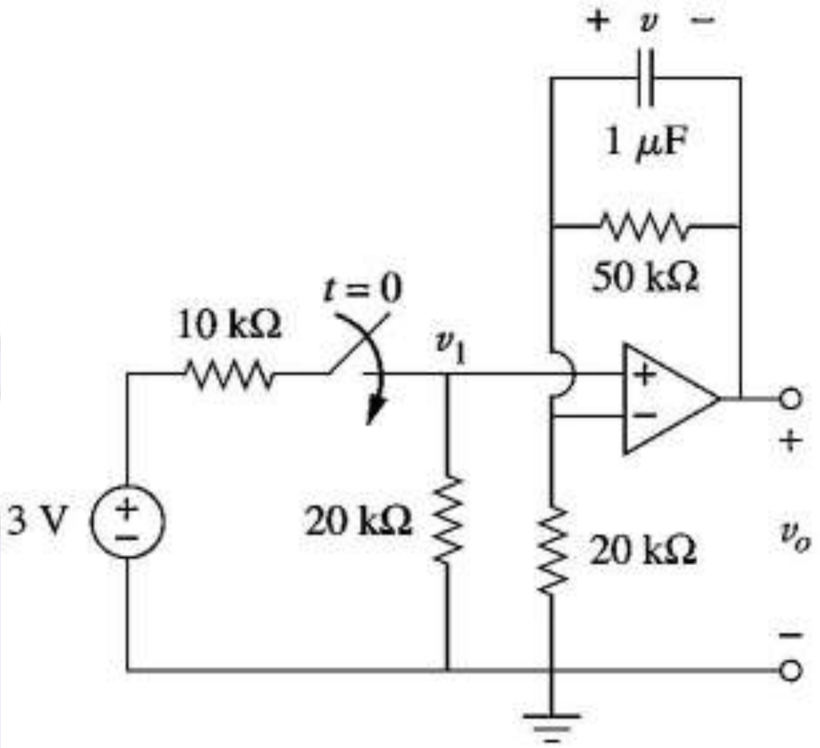
$$v_o = -R_f C\frac{dv}{dt} \tag{7.14.5}$$

Now we can find v_0 as

$$v_o = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}\,\text{V}, \qquad t > 0$$

Example 7.15:

Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.



Solution:

This problem can be solved in two ways, just like the previous example. However, we will apply only the second method. Since what we are looking for is the step response, we can apply Eq. (7.53) and write

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}, \qquad t > 0 \qquad \textbf{(7.15.1)}$$

where we need only find the time constant τ, the initial value $v(0)$, and the final value $v(\infty)$. Notice that this applies strictly to the capacitor voltage due a step input. Since no current enters the input terminals of the op amp, the elements on the feedback loop of the op amp constitute an RC circuit, with

$$\tau = RC = 50 \times 10^3 \times 10^{-6} = 0.05 \qquad \textbf{(7.15.2)}$$

For $t < 0$, the switch is open and there is no voltage across the capacitor. Hence, $v(0) = 0$. For $t > 0$, we obtain the voltage at node 1 by voltage division as

$$v_1 = \frac{20}{20 + 10} 3 = 2 \text{ V} \qquad \textbf{(7.15.3)}$$

Since there is no storage element in the input loop, v_1 remains constant for all t. At steady state, the capacitor acts like an open circuit so that the op amp circuit is a noninverting amplifier. Thus,

$$v_o(\infty) = \left(1 + \frac{50}{20}\right) v_1 = 3.5 \times 2 = 7 \text{ V} \qquad \textbf{(7.15.4)}$$

But

$$v_1 - v_o = v \qquad \textbf{(7.15.5)}$$

so that

$$v(\infty) = 2 - 7 = -5 \text{ V}$$

Substituting $\tau, v(0)$, and $v(\infty)$ into Eq. (7.15.1) gives

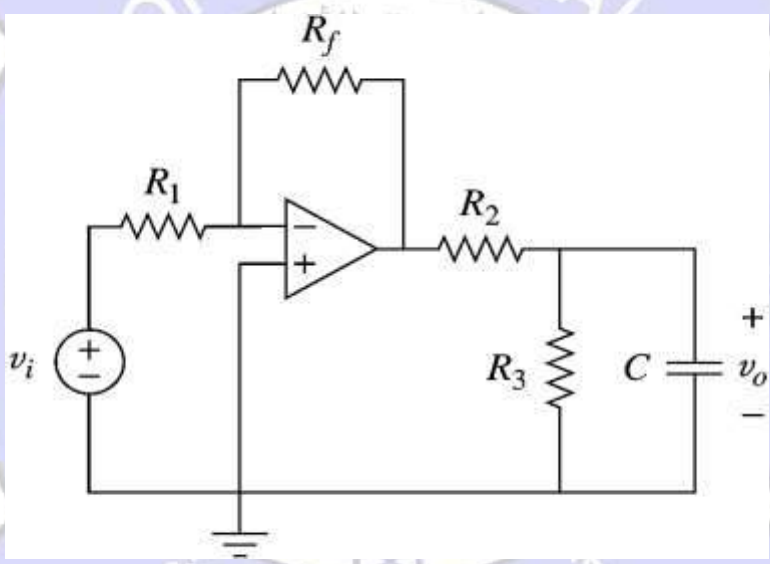
$$v(t) = -5 + [0 - (-5)]e^{-20t} = 5(e^{-20t} - 1)\text{ V}, \qquad t > 0 \qquad \textbf{(7.15.6)}$$

From Eqs. (7.15.3), (7.15.5), and (7.15.6), we obtain

$$v_o(t) = v_1(t) - v(t) = 7 - 5e^{-20t}\text{ V}, \qquad t > 0 \qquad \textbf{(7.15.7)}$$

Example 7.16:

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit of Fig. 7.59. Let $v_i = 2u(t)$ V, $R_1 = 20\text{ k}\Omega$, $R_f = 50\text{ k}\Omega$, $R_2 = R_3 = 10\text{ k}\Omega$, $C = 2\ \mu\text{F}$.



Solution:

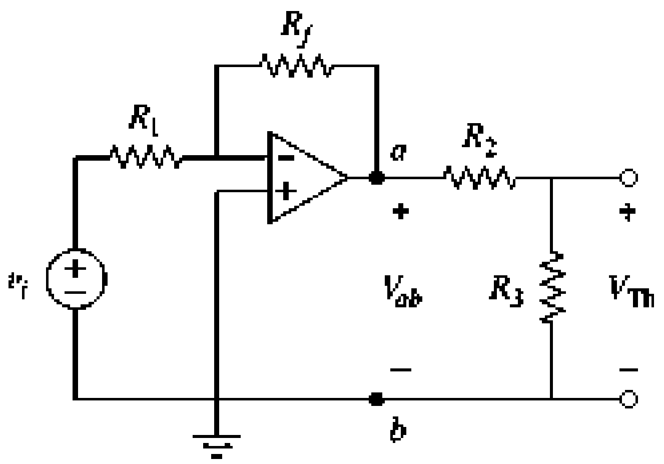
Notice that the capacitor in Example 7.14 is located in the input loop, while the capacitor in Example 7.15 is located in the feedback loop. In this example, the capacitor is located in the output of the op amp. Again, we can solve this problem directly using nodal analysis. However, using the Thevenin equivalent circuit may simplify the problem.

We temporarily remove the capacitor and find the Thevenin equivalent at its terminals. To obtain V_{Th}, consider the circuit in Fig. 7.60(a). Since the circuit is an inverting amplifier,

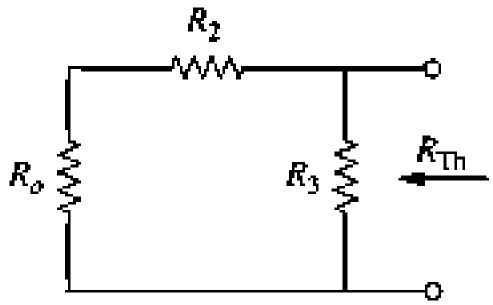
$$V_{ab} = -\frac{R_f}{R_1} v_i$$

By voltage division,

$$V_{\text{Th}} = \frac{R_3}{R_2 + R_3} V_{ab} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i$$



(a)



(b)

To obtain R_{Th}, consider the circuit in Fig. 7.60(b), where R_o is the output resistance of the op amp. Since we are assuming an ideal op amp, $R_o = 0$, and

$$R_{Th} = R_2 \| R_3 = \frac{R_2 R_3}{R_2 + R_3}$$

Substituting the given numerical values,

$$V_{Th} = -\frac{R_3}{R_2 + R_3}\frac{R_f}{R_1}v_i = -\frac{10}{20}\frac{50}{20}2u(t) = -2.5u(t)$$

$$R_{Th} = \frac{R_2 R_3}{R_2 + R_3} = 5\text{ k}\Omega$$

The Thevenin equivalent circuit is shown in Fig. 7.61, which is similar to Fig. 7.40. Hence, the solution is similar to that in Eq. (7.48); that is,

$$v_o(t) = -2.5(1 - e^{-t/\tau})u(t)$$

where $\tau = R_{Th}C = 5 \times 10^3 \times 2 \times 10^{-6} = 0.01$. Thus, the step response for $t > 0$ is

$$v_o(t) = 2.5(e^{-100t} - 1)u(t)\text{ V}$$

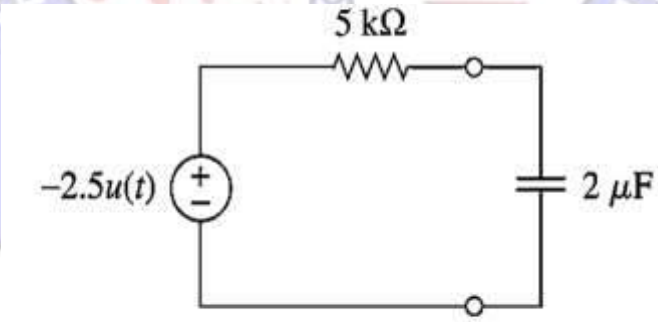


Figure 7.61

Chapter Four

Second-Order Circuits

4.1- Introduction

4.2- Source-free series RLC circuit.

4.3- Source-free parallel RLC circuit.

4.4- Step response of series RLC circuit.

4.5- Step response of parallel RLC circuit.

4.6- Second-order Op. Amp. circuit.

4.7- Examples.

4.1- Introduction:

- In the previous chapter we considered circuits with a single storage element (a capacitor or an inductor).
- In this chapter we will consider circuits containing two storage elements.
- These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.
- Typical examples of second-order circuits are series and parallel RLC circuits as shown in Fig. 8.1(a) and (b) respectively.

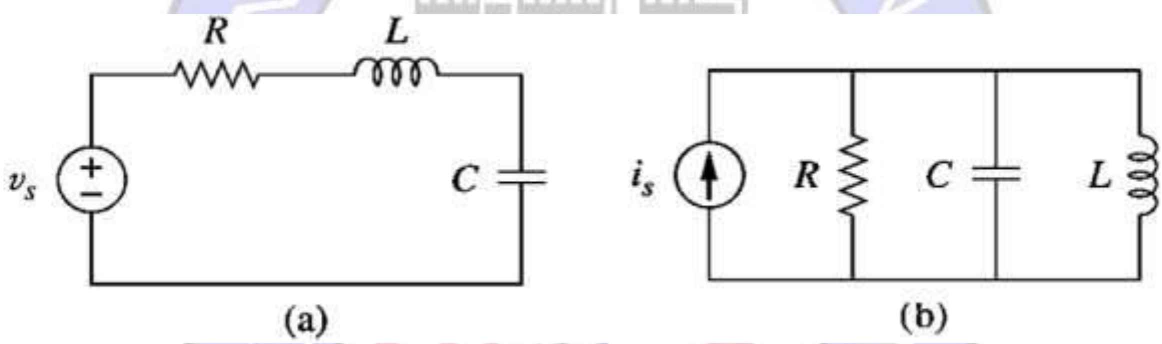


Fig. 8.1:

- A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.
- Two ways are used to excite second-order circuits. First way is by the initial conditions of the storage elements and the second way is by independent sources (dc sources or step inputs).

- The major problem students face in handling second-order circuits is finding the initial and final conditions on circuit variables.
- Students are usually comfortable getting the initial and final values of v and i but often have difficulty finding the initial values of their derivatives dv/dt and di/dt.
- Therefore, $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$ and $v(\infty)$ must be find, where v denotes capacitor voltage while i is the inductor current.
- There are two key points to keep in mind in determining the initial conditions.
  - ✓ First, we must carefully handle the polarity of voltage across the capacitor and the direction of the current through the inductor.
  - ✓ Second, keep in mind that the capacitor voltage is always continuous so that;

$$v(0^+) = v(0^-) \tag{8.1a}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \tag{8.1b}$$

where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

Example 8.1:

The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+), v(0^+)$, (b) $di(0^+)/dt, dv(0^+)/dt$, (c) $i(\infty), v(\infty)$.

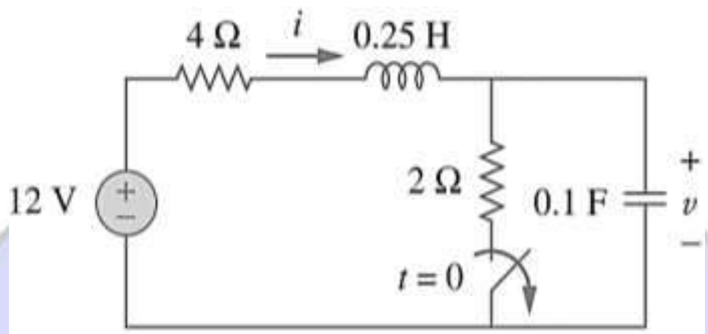


Figure 8.2
For Example 8.1.

Solution:

(a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$. At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig. 8.3(a) at $t = 0^-$. Thus,

$$i(0^-) = \frac{12}{4+2} = 2 \text{ A}, \qquad v(0^-) = 2i(0^-) = 4 \text{ V}$$

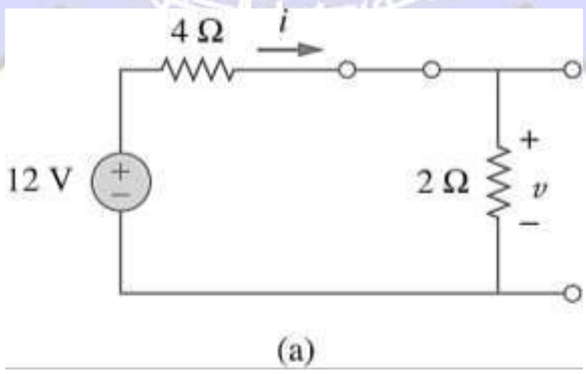


Figure 8.3
Fig. 8.2 for: (a) $t = 0^-$

As the inductor current and the capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2 \text{ A}, \qquad v(0^+) = v(0^-) = 4 \text{ V}$$

(b) At $t = 0^+$, the switch is open; the equivalent circuit is as shown in Fig. 8.3(b). The same current flows through both the inductor and capacitor. Hence,

$$i_C(0^+) = i(0^+) = 2 \text{ A}$$

Since $C\,dv/dt = i_C$, $dv/dt = i_C/C$, and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20 \text{ V/s}$$

Similarly, since $L\,di/dt = v_L$, $di/dt = v_L/L$. We now obtain v_L by applying KVL to the loop in Fig. 8.3(b). The result is

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

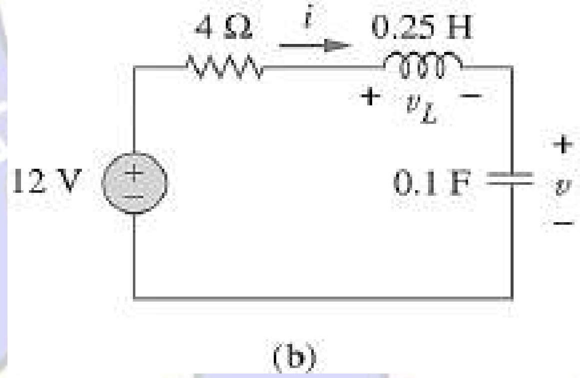


Figure 8.3
Fig. 8.2 for: (b) $t = 0^+$

or

$$v_L(0^+) = 12 - 8 - 4 = 0$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0 \text{ A/s}$$

(c) For $t > 0$, the circuit undergoes transience. But as $t \rightarrow \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit in Fig. 8.3(b) becomes that shown in Fig. 8.3(c), from which we have

$$i(\infty) = 0 \text{ A}, \qquad v(\infty) = 12 \text{ V}$$

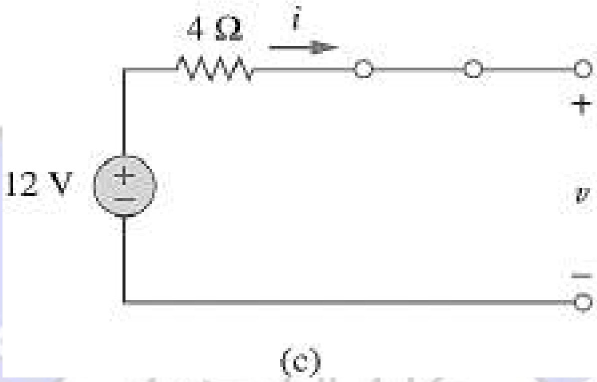


Figure 8.3
Fig. 8.2 for: (c) $t \rightarrow \infty$

Example 8.2:

In the circuit of Fig. 8.5, calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

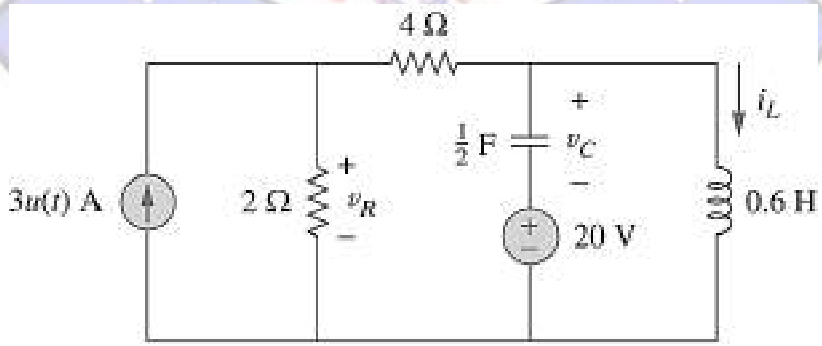


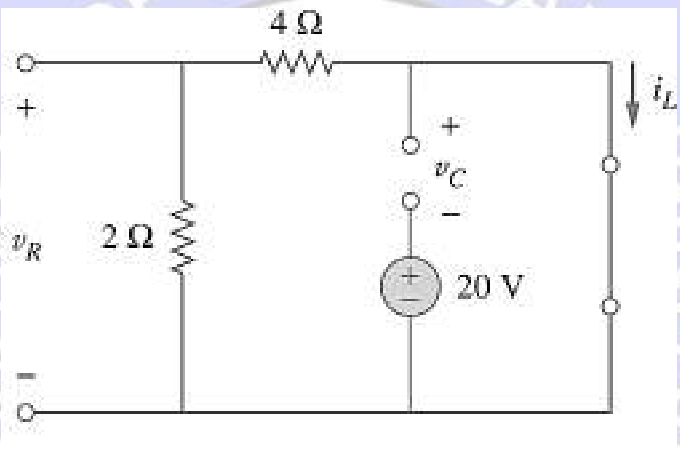
Figure 8.5
For Example 8.2.

Solution:

(a) For $t < 0, 3u(t) = 0$. At $t = 0^-$, since the circuit has reached steady state, the inductor can be replaced by a short circuit, while the capacitor is replaced by an open circuit as shown in Fig. 8.6(a). From this figure we obtain

$$i_L(0^-) = 0, \qquad v_R(0^-) = 0, \qquad v_C(0^-) = -20\text{ V} \qquad \textbf{(8.2.1)}$$

Although the derivatives of these quantities at $t = 0^-$ are not required, it is evident that they are all zero, since the circuit has reached steady state and nothing changes.



(a)

Figure 8.6

Fig. 8.5 for: (a) $t = 0^-$

4.2- Source-free series RLC circuit:

- Consider the natural response of series RLC circuit shown in Fig. 8.8.

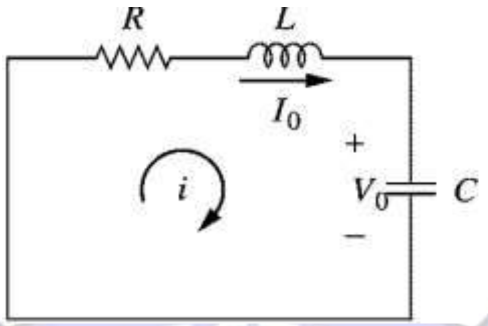


Figure 8.8
A source-free series *RLC* circuit.

- The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_o and initial inductor current I_o (at $t = 0$).

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \qquad \textbf{(8.2a)}$$

$$i(0) = I_0 \qquad \textbf{(8.2b)}$$

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0 \qquad \textbf{(8.3)}$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \qquad \textbf{(8.4)}$$

- To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial value v
- Substitute equations 8.2a and 8.2b into equation 8.3, we get;

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{8.5}$$

- Based on the first-order circuit, the solution of equation 8.5 is the exponential form as;

$$i = Ae^{st} \tag{8.6}$$

where A and s are constants to be determined.

- By substitution equation 8.6 into equation 8.4, we get;

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \tag{8.7}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{8.8}$$

This quadratic equation is known as the *characteristic equation*

- The two roots of equation 8.8 are;

$$S = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4(1)\left(\frac{1}{LC}\right)}}{2}$$

$$S = \frac{-\frac{2R}{2L} \pm \sqrt{\frac{4R^2}{4L^2} - \frac{4}{LC}}}{2} = \frac{-\frac{2R}{2L} \pm 2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}{2}$$

$$\boxed{S = -\left(\frac{R}{2L}\right) \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}}$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \textbf{(8.9a)}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \textbf{(8.9b)}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \qquad \textbf{(8.10)}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \textbf{(8.11)}$$

The roots s_1 and s_2 are called *natural frequencies*,

In terms of α and ω_0, Eq. (8.8) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad \textbf{(8.8a)}$$

The two values of s in Eq. (8.10) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (8.6); that is,

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t} \tag{8.12}$$

Since Eq. (8.4) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of Eq. (8.4).
Thus, the natural response of the series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.13}$$

where the constants A_1 and A_2 are determined

From Eq. (8.10), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.14}$$

which decays and approaches zero as t increases. Figure 8.9(a) illustrates a typical overdamped response.

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \quad \textbf{(8.22a)}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \quad \textbf{(8.22b)}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \quad \textbf{(8.23)}$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \quad \textbf{(8.24)}$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \quad \textbf{(8.25)}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)} \quad \textbf{(8.26)}$$

Example 8.3:

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

Solution:

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

Example 8.4:

Find $i(t)$ in the circuit of Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

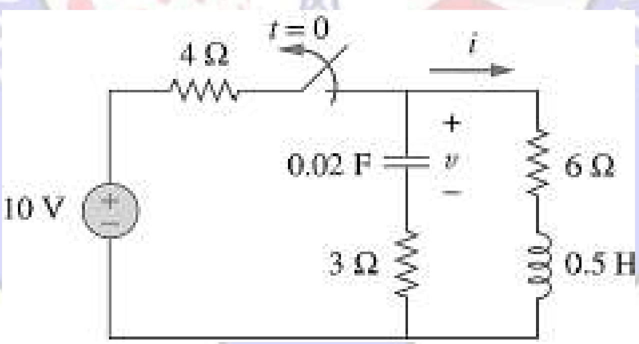


Fig. 8.10

Solution:

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4+6} = 1 \text{ A}, \qquad v(0) = 6i(0) = 6 \text{ V}$$

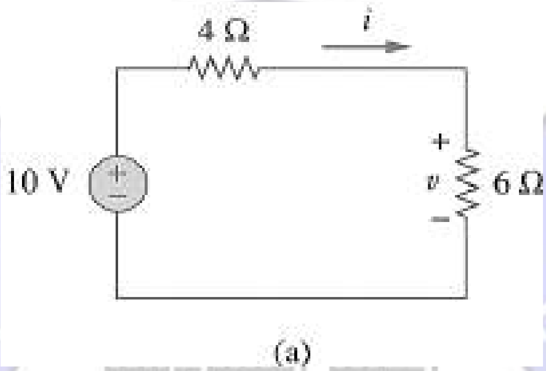


Fig. 8.10a: for $t < 0$.

where $i(0)$ is the initial current through the inductor and $v(0)$ is the initial voltage across the capacitor.

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig. 8.11(b), which is a source-free series RLC circuit. Notice that the 3-Ω and 6-Ω resistors, which are in series in Fig. 8.10 when the switch is opened, have been combined to give $R = 9\ \Omega$ in Fig. 8.11(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

or

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \qquad \textbf{(8.4.1)}$$

We now obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \qquad \textbf{(8.4.2)}$$

From Eq. (8.5),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s} \qquad \textbf{(8.4.3)}$$

Note that $v(0) = V_0 = -6$ V is used, because the polarity of v in Fig. 8.11(b) is opposite that in Fig. 8.8. Taking the derivative of $i(t)$ in Eq. (8.4.1),

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$
$$+ e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \qquad \Rightarrow \qquad A_2 = 0.6882$$

Substituting the values of A_1 and A_2 in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

4.3- Source-free parallel RLC circuit:

- Consider the parallel RLC circuit shown in Fig. 8.13. Assume initial inductor current I_o and initial capacitor voltage V_o.

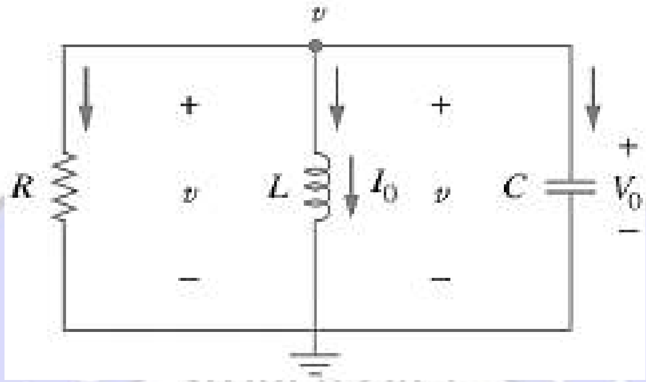


Figure 8.13
A source-free parallel *RLC* circuit.

- Since the three elements are in parallel, they have the same voltage v across them. Thus, applying KCL at the top node gives;

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \qquad \textbf{(8.28)}$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad \textbf{(8.29)}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. The characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad \textbf{(8.30)}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$

Overdamped Case ($\alpha > \omega_0$)

From Eq. (8.32), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.33}$$

Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$\boxed{v(t) = (A_1 + A_2 t)e^{-\alpha t}} \tag{8.34}$$

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{8.35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8.36}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{8.37}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$.

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \tag{8.38}$$

Example 8.5:

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

Solution:

■ **CASE 1** If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \tag{8.5.1}$$

We now apply the initial conditions to get A_1 and A_2.

$$v(0) = 5 = A_1 + A_2 \tag{8.5.2}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating Eq. (8.5.1),

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2 \tag{8.5.3}$$

From Eqs. (8.5.2) and (8.5.3), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$. Substituting A_1 and A_2 in Eq. (8.5.1) yields

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t} \tag{8.5.4}$$

CASE 2 When $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \tag{8.5.5}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1 \tag{8.5.6}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. (8.5.5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2 \tag{8.5.7}$$

From Eqs. (8.5.6) and (8.5.7), $A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t}\ \text{V} \tag{8.5.8}$$

CASE 3 When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \tag{8.5.9}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1 \tag{8.5.10}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

But differentiating Eq. (8.5.9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2 \tag{8.5.11}$$

From Eqs. (8.5.10) and (8.5.11), $A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t} \tag{8.5.12}$$

4.4- Step response of series RLC circuit:

- The step response is obtained by the sudden application of a dc source.
- Consider the series RLC circuit shown in Fig. 8.18. Applying KVL around the loop for $t > 0$.

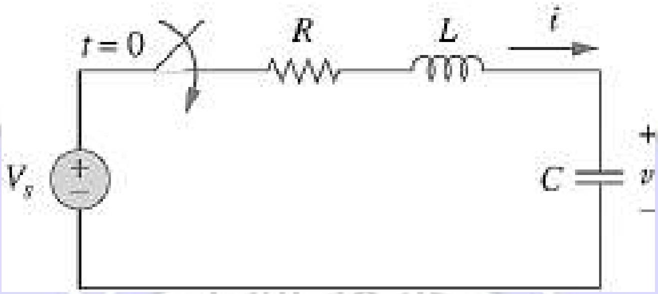


Figure 8.18
Step voltage applied to a series *RLC* circuit.

- Applying KVL around the loop for $t > 0$.

$$L\frac{di}{dt} + Ri + v = V_s \tag{8.39}$$

But

$$i = C\frac{dv}{dt}$$

Substituting for i in Eq. (8.39) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \tag{8.40}$$

which has the same form as Eq. (8.4).
Hence, the characteristic equation for the series *RLC* circuit is not affected by the presence of the dc source.

The solution to Eq. (8.40) has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$; that is,

$$v(t) = v_t(t) + v_{ss}(t) \tag{8.41}$$

The transient response $v_t(t)$ is the component of the total response that dies out with time. Therefore, the transient response
$v_t(t)$ for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)} \tag{8.42a}$$

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)} \tag{8.42b}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \tag{8.42c}$$

The steady-state response is the final value of $v(t)$. In the circuit in Fig. 8.18, the final value of the capacitor voltage is the same as the source voltage V_s. Hence,

$$v_{ss}(t) = v(\infty) = V_s \tag{8.43}$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$\boxed{\begin{aligned} v(t) &= V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)} \\ v(t) &= V_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)} \\ v(t) &= V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)} \end{aligned}} \qquad \begin{aligned} &\textbf{(8.44a)} \\ &\textbf{(8.44b)} \\ &\textbf{(8.44c)} \end{aligned}$$

The values of the constants A_1 and A_2 are obtained from the initial conditions: $v(0)$ and $dv(0)/dt$.

Example 8.7:

For the circuit in Fig. 8.19, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.

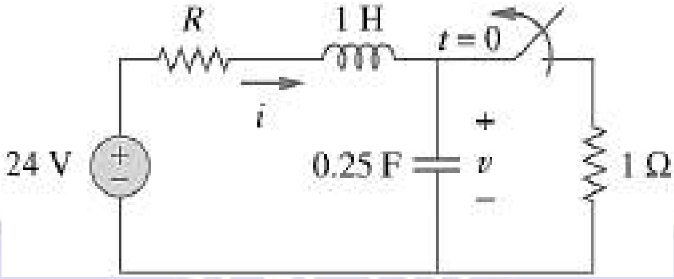


Figure 8.19
For Example 8.7.

Solution:

CASE 1 When $R = 5\ \Omega$. For $t < 0$, the switch is closed for a long time. The capacitor behaves like an open circuit while the inductor acts like a short circuit. The initial current through the inductor is

$$i(0) = \frac{24}{5 + 1} = 4\text{ A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor; that is,

$$v(0) = 1i(0) = 4\text{ V}$$

For $t > 0$, the switch is opened, so that we have the 1-Ω resistor disconnected. What remains is the series *RLC* circuit with the voltage source. The characteristic roots are determined as follows:

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$

where v_{ss} is the steady-state response. It is the final value of the capacitor voltage. In Fig. 8.19, $v_f = 24$ V. Thus,

$$v(t) = 24 + (A_1 e^{-t} + A_2 e^{-4t}) \tag{8.7.1}$$

We now need to find A_1 and A_2 using the initial conditions.

$$v(0) = 4 = 24 + A_1 + A_2$$

or

$$-20 = A_1 + A_2 \tag{8.7.2}$$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0^+$ because the inductor and capacitor are now in series. Hence,

$$i(0) = C\frac{dv(0)}{dt} = 4 \qquad \Rightarrow \qquad \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

Before we use this condition, we need to take the derivative of v in Eq. (8.7.1).

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t} \tag{8.7.3}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \tag{8.7.4}$$

From Eqs. (8.7.2) and (8.7.4), $A_1 = -64/3$ and $A_2 = 4/3$. Substituting A_1 and A_2 in Eq. (8.7.1), we get

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})\text{ V} \qquad \textbf{(8.7.5)}$$

Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Hence,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.3) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t})\text{ A} \qquad \textbf{(8.7.6)}$$

Note that $i(0) = 4$ A, as expected.

CASE 2 When $R = 4\ \Omega$. Again, the initial current through the inductor is

$$i(0) = \frac{24}{4+1} = 4.8\text{ A}$$

and the initial capacitor voltage is

$$v(0) = 1i(0) = 4.8\text{ V}$$

For the characteristic roots,

$$\alpha = \frac{R}{2L} = \frac{4}{2\times 1} = 2$$

while $\omega_0 = 2$ remains the same. In this case, $s_1 = s_2 = -\alpha = -2$, and we have the critically damped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$

and, as before $v_{ss} = 24$ V,

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t} \qquad \textbf{(8.7.7)}$$

To find A_1 and A_2, we use the initial conditions. We write

$$v(0) = 4.8 = 24 + A_1 \quad \Rightarrow \quad A_1 = -19.2 \qquad \textbf{(8.7.8)}$$

Since $i(0) = C\,dv(0)/dt = 4.8$ or

$$\frac{dv(0)}{dt} = \frac{4.8}{C} = 19.2$$

From Eq. (8.7.7),

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t} \qquad \textbf{(8.7.9)}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 19.2 = -2A_1 + A_2 \qquad \textbf{(8.7.10)}$$

From Eqs. (8.7.8) and (8.7.10), $A_1 = -19.2$ and $A_2 = -19.2$. Thus, Eq. (8.7.7) becomes

$$v(t) = 24 - 19.2(1 + t)e^{-2t}\text{ V} \qquad \textbf{(8.7.11)}$$

The inductor current is the same as the capacitor current; that is,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.9) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (4.8 + 9.6t)e^{-2t}\text{ A} \qquad \textbf{(8.7.12)}$$

Note that $i(0) = 4.8$ A, as expected.

CASE 3 When $R = 1\ \Omega$. The initial inductor current is

$$i(0) = \frac{24}{1+1} = 12\text{ A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor,

$$v(0) = 1i(0) = 12\text{ V}$$

$$\alpha = \frac{R}{2L} = \frac{1}{2 \times 1} = 0.5$$

Since $\alpha = 0.5 < \omega_0 = 2$, we have the underdamped response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

The total response is therefore

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \qquad \textbf{(8.7.13)}$$

We now determine A_1 and A_2. We write

$$v(0) = 12 = 24 + A_1 \quad \Rightarrow \quad A_1 = -12 \qquad \textbf{(8.7.14)}$$

Since $i(0) = C\,dv(0)/dt = 12$,

$$\frac{dv(0)}{dt} = \frac{12}{C} = 48 \qquad \textbf{(8.7.15)}$$

But

$$\begin{aligned}\frac{dv}{dt} &= e^{-0.5t}(-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t) \\ &\quad - 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)\end{aligned} \qquad \textbf{(8.7.16)}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0)$$

Substituting $A_1 = -12$ gives $A_2 = 21.694$, and Eq. (8.7.13) becomes

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t} \text{ V} \quad \textbf{(8.7.17)}$$

The inductor current is

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.16) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t} \text{ A} \qquad \textbf{(8.7.18)}$$

Note that $i(0) = 12$ A, as expected.

4.5- Step Response of Parallel RLC Circuit

- Consider the parallel RLC circuit shown in Fig. 8.22. We want to find i due to a sudden application of a dc current.

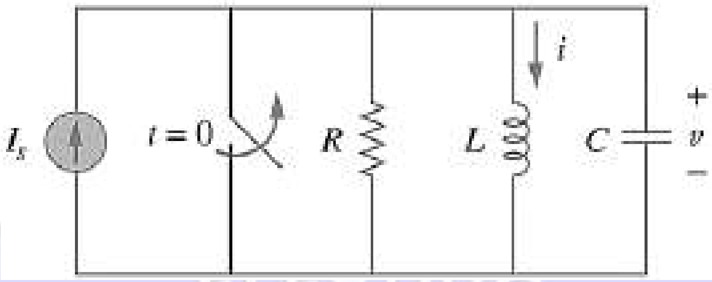


Figure 8.22
Parallel *RLC* circuit with an applied current.

- Applying KCL at the top node for $t > 0$

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \tag{8.46}$$

$$v = L\frac{di}{dt}$$

Substituting for v in Eq. (8.46) and dividing by LC, we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \tag{8.47}$$

The complete solution to Eq. (8.47) consists of the transient response $i_t(t)$ and the steady-state response i_{ss}; that is,

$$i(t) = i_t(t) + i_{ss}(t) \tag{8.48}$$

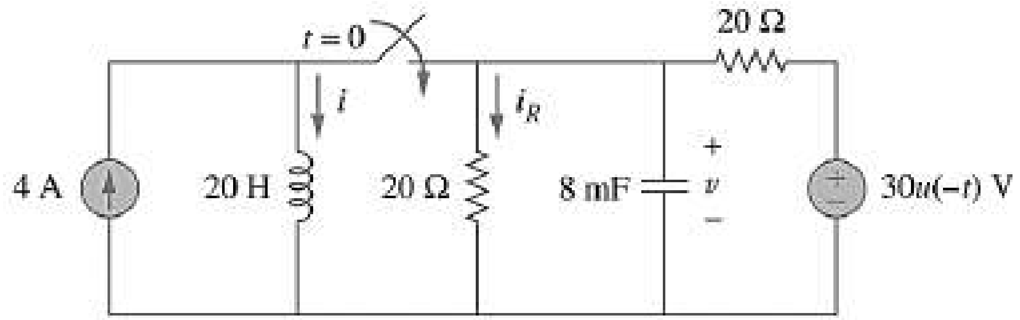
The steady-state response is the final value of i. In the circuit in Fig. 8.22, the final value of the current through the inductor is the same as the source current I_s. Thus,

$$
\boxed{\begin{aligned}
i(t) &= I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}\\
i(t) &= I_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}\\
i(t) &= I_s + (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}
\end{aligned}} \qquad \textbf{(8.49)}
$$

The constants A_1 and A_2 in each case can be determined from the initial conditions for i and di/dt. Again, we should keep in mind that Eq. (8.49) only applies for finding the inductor current i. But once the inductor current $i_L = i$ is known, we can find $v = L\,di/dt$, which is the same voltage across inductor, capacitor, and resistor. Hence, the current through the resistor is $i_R = v/R$, while the capacitor current is $i_C = C\,dv/dt$.

Example 8.8:

In the circuit of Fig. 8.23, find $i(t)$ and $i_R(t)$ for $t > 0$.



Solution:

For $t < 0$, the switch is open, and the circuit is partitioned into two independent subcircuits. The 4-A current flows through the inductor, so that

$$i(0) = 4\ \text{A}$$

Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$, the voltage source is operative for $t < 0$. The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the 20-Ω resistor connected in parallel with it. By voltage division, the initial capacitor voltage is

$$v(0) = \frac{20}{20 + 20}(30) = 15 \text{ V}$$

For $t > 0$, the switch is closed, and we have a parallel RLC circuit with a current source. The voltage source is zero which means it acts like a short-circuit. The two 20-Ω resistors are now in parallel. They are combined to give $R = 20 \| 20 = 10\ \Omega$. The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 8 \times 10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 8 \times 10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{39.0625 - 6.25}$$
$$= -6.25 \pm 5.7282$$

or

$$s_1 = -11.978, \qquad s_2 = -0.5218$$

Since $\alpha > \omega_0$, we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t} \tag{8.8.1}$$

where $I_s = 4$ is the final value of $i(t)$. We now use the initial conditions to determine A_1 and A_2. At $t = 0$,

$$i(0) = 4 = 4 + A_1 + A_2 \quad \Rightarrow \quad A_2 = -A_1 \tag{8.8.2}$$

Taking the derivative of $i(t)$ in Eq. (8.8.1),

$$\frac{di}{dt} = -11.978A_1 e^{-11.978t} - 0.5218A_2 e^{-0.5218t}$$

so that at $t = 0$,

$$\frac{di(0)}{dt} = -11.978A_1 - 0.5218A_2 \qquad \textbf{(8.8.3)}$$

But

$$L\frac{di(0)}{dt} = v(0) = 15 \quad \Rightarrow \quad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

Substituting this into Eq. (8.8.3) and incorporating Eq. (8.8.2), we get

$$0.75 = (11.978 - 0.5218)A_2 \quad \Rightarrow \quad A_2 = 0.0655$$

Thus, $A_1 = -0.0655$ and $A_2 = 0.0655$. Inserting A_1 and A_2 in Eq. (8.8.1) gives the complete solution as

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})\text{ A}$$

From $i(t)$, we obtain $v(t) = L\,di/dt$ and

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.5218t}\text{ A}$$

4.6- Second-Order Op Amp Circuit

- An op amp circuit with two storage elements that cannot be combined into a single equivalent element is second-order.
- Because inductors are bulky and heavy, they are rarely used in practical op amp circuits.
- For this reason, the RC second-order op amp circuits will be only considered.
- The analysis of a second-order op amp circuit follows the same four steps given and demonstrated in the previous section.

Example 8.11:

In the op amp circuit of Fig. 8.33, find $v_o(t)$ for $t > 0$ when $v_s = 10u(t)$ mV. Let $R_1 = R_2 = 10\ \text{k}\Omega$, $C_1 = 20\ \mu\text{F}$, and $C_2 = 100\ \mu\text{F}$.

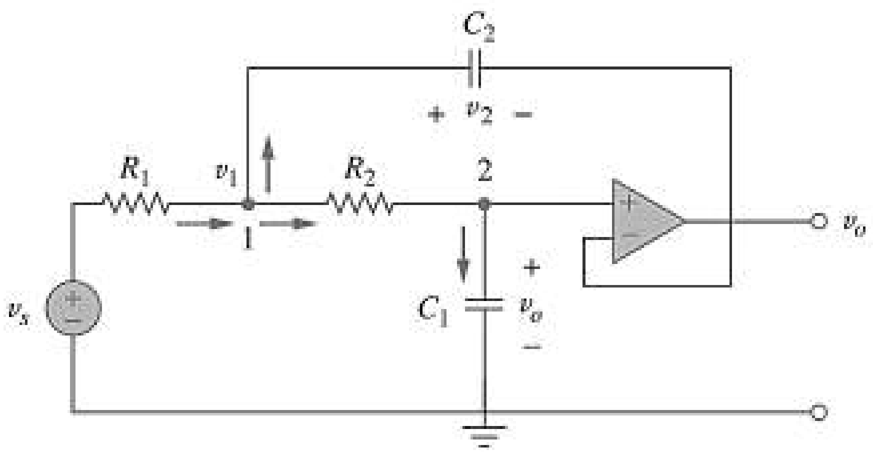


Figure 8.33
For Example 8.11

Solution:

Although we could follow the same four steps given in the previous section to solve this problem, we will solve it a little differently. Due to the voltage follower configuration, the voltage across C_1 is v_o. Applying KCL at node 1,

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2} \qquad \textbf{(8.11.1)}$$

At node 2, KCL gives

$$\frac{v_1 - v_o}{R_2} = C_1\frac{dv_o}{dt} \qquad \textbf{(8.11.2)}$$

But

$$v_2 = v_1 - v_o \qquad \textbf{(8.11.3)}$$

We now try to eliminate v_1 and v_2 in Eqs. (8.11.1) to (8.11.3). Substituting Eqs. (8.11.2) and (8.11.3) into Eq. (8.11.1) yields

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_1}{dt} - C_2\frac{dv_o}{dt} + C_1\frac{dv_o}{dt} \qquad \textbf{(8.11.4)}$$

From Eq. (8.11.2),

$$v_1 = v_o + R_2C_1\frac{dv_o}{dt} \qquad \textbf{(8.11.5)}$$

Substituting Eq. (8.11.5) into Eq. (8.11.4), we obtain

$$\frac{v_s}{R_1} = \frac{v_o}{R_1} + \frac{R_2C_1}{R_1}\frac{dv_o}{dt} + C_2\frac{dv_o}{dt} + R_2C_1C_2\frac{d^2v_o}{dt^2} - C_2\frac{dv_o}{dt} + C_1\frac{dv_o}{dt}$$

or

$$\frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_o}{dt} + \frac{v_o}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2} \qquad \textbf{(8.11.6)}$$

With the given values of R_1, R_2, C_1, and C_2, Eq. (8.11.6) becomes

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + 5v_o = 5v_s \qquad \textbf{(8.11.7)}$$

To obtain the form of the transient response, set $v_s = 0$ in Eq. (8.11.7), which is the same as turning off the source. The characteristic equation is

$$s^2 + 2s + 5 = 0$$

which has complex roots $s_{1,2} = -1 \pm j2$. Hence, the form of the transient response is

$$v_{ot} = e^{-t}(A \cos 2t + B \sin 2t) \qquad \textbf{(8.11.8)}$$

where A and B are unknown constants to be determined.

As $t \to \infty$, the circuit reaches the steady-state condition, and the capacitors can be replaced by open circuits. Since no current flows through C_1 and C_2 under steady-state conditions and no current can enter the input terminals of the ideal op amp, current does not flow through R_1 and R_2.

Thus,

$$v_o(\infty) = v_1(\infty) = v_s$$

The steady-state response is then

$$v_{oss} = v_o(\infty) = v_s = 10\text{ mV}, \qquad t > 0 \qquad \textbf{(8.11.9)}$$

The complete response is

$$v_o(t) = v_{ot} + v_{oss} = 10 + e^{-t}(A \cos 2t + B \sin 2t)\text{ mV} \qquad \textbf{(8.11.10)}$$

To determine A and B, we need the initial conditions. For $t < 0$, $v_s = 0$, so that

$$v_o(0^-) = v_2(0^-) = 0$$

For $t > 0$, the source is operative. However, due to capacitor voltage continuity,

$$v_o(0^+) = v_2(0^+) = 0 \qquad \textbf{(8.11.11)}$$

From Eq. (8.11.3),

$$v_1(0^+) = v_2(0^+) + v_o(0^+) = 0$$

and, hence, from Eq. (8.11.2),

$$\frac{dv_o(0^+)}{dt} = \frac{v_1 - v_o}{R_2C_1} = 0 \qquad \textbf{(8.11.12)}$$

We now impose Eq. (8.11.11) on the complete response in Eq. (8.11.10) at $t = 0$, for

$$0 = 10 + A \quad \Rightarrow \quad A = -10 \qquad \textbf{(8.11.13)}$$

Taking the derivative of Eq. (8.11.10),

$$\frac{dv_o}{dt} = e^{-t}(-A\cos 2t - B\sin 2t - 2A\sin 2t + 2B\cos 2t)$$

Setting $t = 0$ and incorporating Eq. (8.11.12), we obtain

$$0 = -A + 2B \qquad \textbf{(8.11.14)}$$

From Eqs. (8.11.13) and (8.11.14), $A = -10$ and $B = -5$. Thus, the step response becomes

$$v_o(t) = 10 - e^{-t}(10\cos 2t + 5\sin 2t)\text{ mV}, \qquad t > 0$$

Chapter Five

Three-Phase Circuits

5.1- Introduction

5.2- Balanced Wye-Wye connection.

5.3- Balanced Wye-Delta connection.

5.4- Balanced Delta-Delta connection.

5.5- Balanced Delta-Wye connection.

5.6- Power in balanced system.

5.7- Unbalanced three-phase systems.

5.1- Introduction:

- A single-phase A.C power system consists of a generator connected through a pair of wires (a transmission line) to a load. Figure 5.1 shows a single-phase two-wire system, where V_P is the rms magnitude of the source voltage and ∅ is the phase.

Fig. 5.1:

- Figure 5.2 shows a three-phase four-wire system. A three-phase system is produced by a generator consisting of three sources having the same amplitude and frequency but out of phase with each other by 120°.

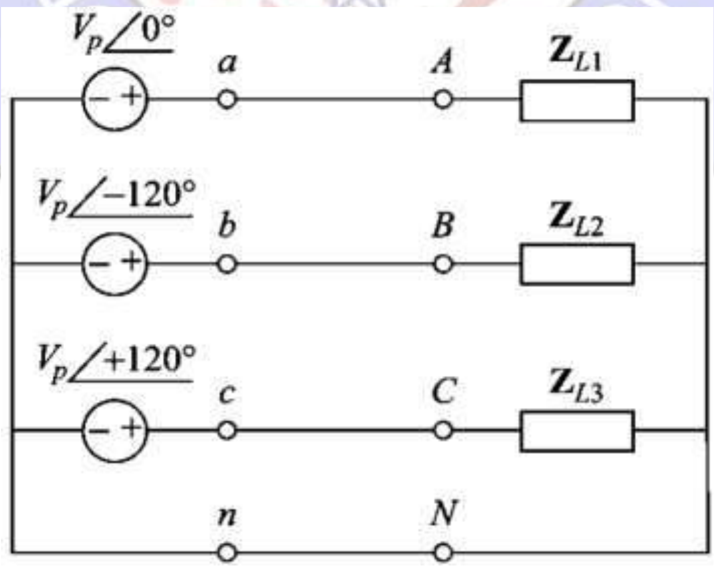


Fig. 5.2:

- Figure 5.3 shows the three generated voltages which are 120° apart from each other.

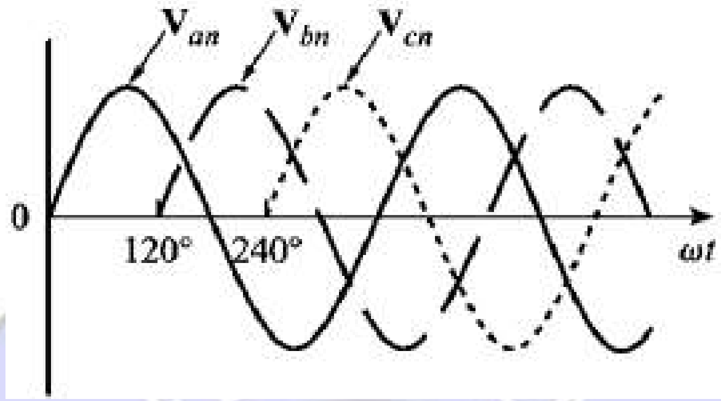


Fig. 5.3:

- A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines).
- A three-phase system is equivalent to three single-phase circuits. **The voltage sources** can be either wye-connected as shown in Figure 5.4 (a) or delta-connected as in Figure 5.4 (b).

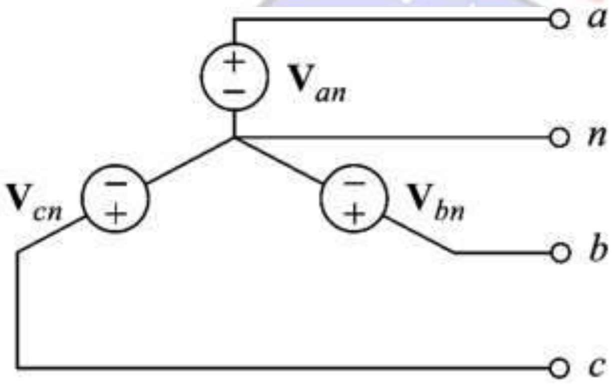


Fig. 5.4 (a)

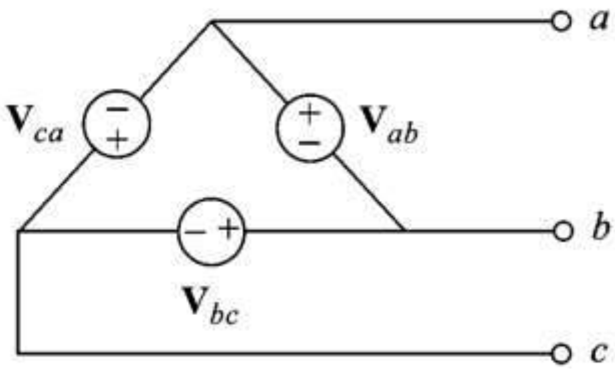


Fig. 5.4 (b)

- Let us consider the wye-connected voltages shown in figure 5.4 (a);

The voltages $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are respectively between lines *a*, *b*, and *c*, and the neutral line *n*. These voltages are called *phase voltages*. If the voltage sources have the same amplitude and frequency ω and are out of phase with each other by 120°, the voltages are said to be *balanced*. This implies that

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

- Balanced phase voltages are equal in magnitude and are out of phase with each other by 120°.

- Since the three-phase voltages are 120° out of phase with each other, there are two possible combinations. Firstly, the abc or positive sequence which shown in figure 5.5 can be expressed mathematically as;

$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$

$$\mathbf{V}_{bn} = V_p\underline{/-120^\circ}$$

$$\mathbf{V}_{cn} = V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ}$$

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = V_p\underline{/0^\circ} + V_p\underline{/-120^\circ} + V_p\underline{/+120^\circ}$$

$$= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866)$$

$$= 0$$

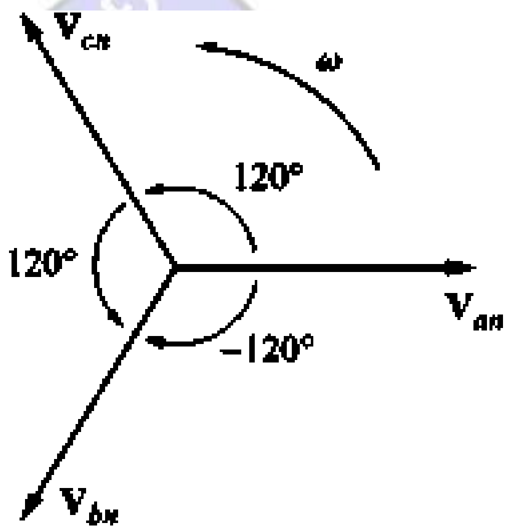


Fig. 5.5:

where V_p is the effective or rms value of the phase voltages. This is known as the *abc sequence* or *positive sequence*. In this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$, which in turn leads $\mathbf{V}_{cn}$.

- Secondly, the acb or negative sequence which shown in figure 5.6 can be expressed mathematically as;

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{cn} = V_p\angle -120^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -240^\circ = V_p\angle +120^\circ$$

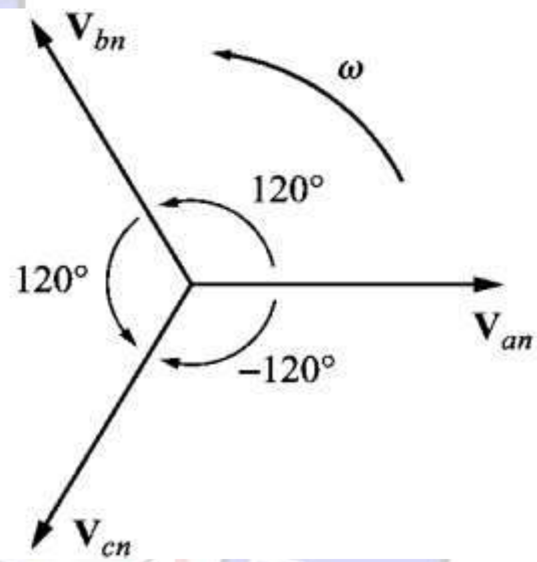


Fig. 5.6:

This is called the *acb sequence* or *negative sequence*. For this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$, which in turn leads $\mathbf{V}_{bn}$.

Example 5.1:

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$
$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

Solution:

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\angle 10^\circ \text{ V}, \quad \mathbf{V}_{bn} = 200\angle -230^\circ \text{ V}, \quad \mathbf{V}_{cn} = 200\angle -110^\circ \text{ V}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

- Regarding the load connections, a three-phase load can be either wye-connected or delta-connected as shown in figures 5.7(a) and 5.7 (b) respectively.

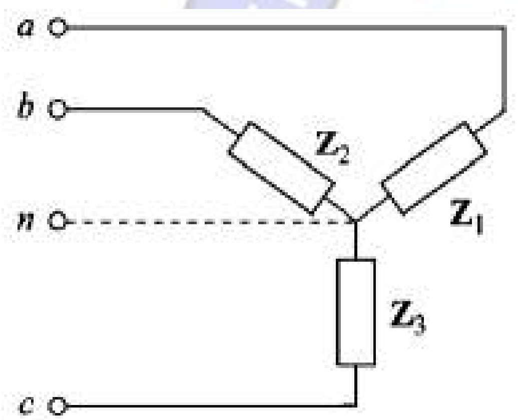


Fig. 5.7 (a):

For a *balanced* wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y$$

where $\mathbf{Z}_Y$ is the load impedance per phase.

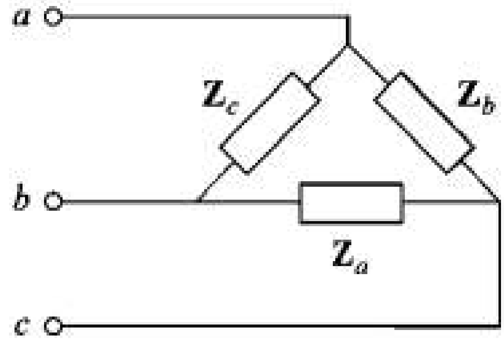


Fig. 5.7 (b):

For a *balanced* delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta$$

where $\mathbf{Z}_\Delta$ is the load impedance per phase in this case.

- A wye- or delta-connected load is said to be unbalanced if the phase impedances are not equal in magnitude or phase.

- We know that a wye-connected load can be transformed into a delta-connected load or vice versa as shown below;

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

- Since both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

  - Y-Y connection (i.e., Y-connected source with a Y-connected load).
  - Y-Δ connection.
  - Δ-Δ connection.
  - Δ-Y connection.

5.2- Balanced Wye-Wye connection:

- A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.
- Consider the balanced four-wire Y-Y system of Figure 5.8, where a Y-connected load is connected to a Y-connected source.

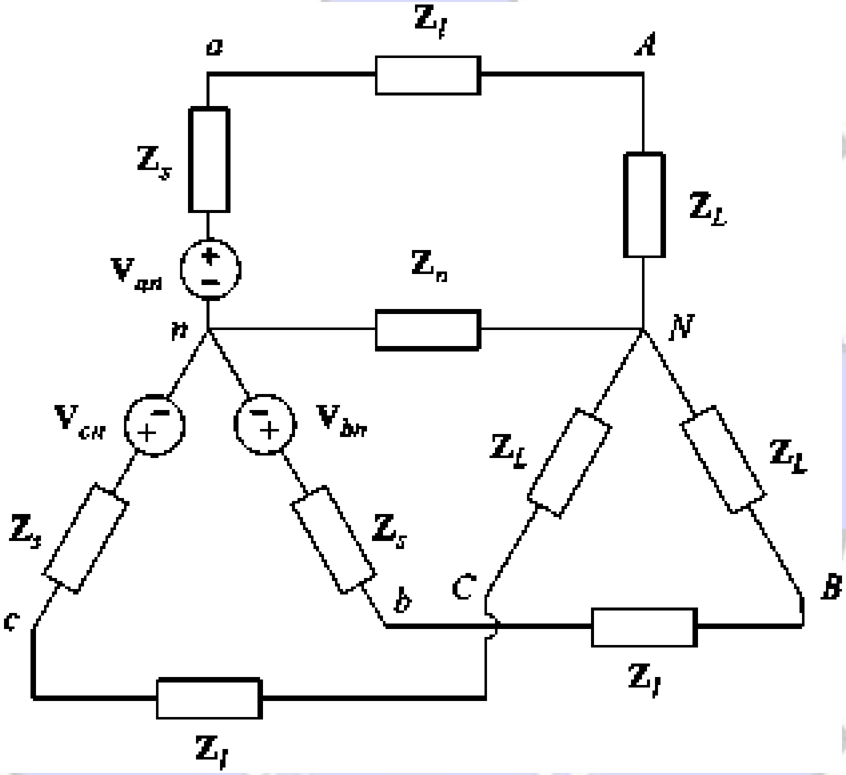


Fig. 5.8:

- We assume a balanced load so that load impedances are equal.

Although the impedance $\mathbf{Z}_Y$ is the total load impedance per phase, it may also be regarded as the sum of the source impedance $\mathbf{Z}_s$, line impedance $\mathbf{Z}_\ell$, and load impedance $\mathbf{Z}_L$ for each phase,

$\mathbf{Z}_s$ denotes the internal impedance of the phase winding of the generator;

$\mathbf{Z}_\ell$ is the impedance of the line joining a phase of the source with a phase of the load;

$\mathbf{Z}_L$ is the impedance of each phase of the load;

and $\mathbf{Z}_n$ is the impedance of the neutral line.

Thus, in general

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L$$

$\mathbf{Z}_s$ and $\mathbf{Z}_\ell$ are often very small compared with $\mathbf{Z}_L$, so one can assume that $\mathbf{Z}_Y = \mathbf{Z}_L$ if no source or line impedance is given.

- To simplify the calculations, Figure 5.9 will be considered instead of figure 5.8.

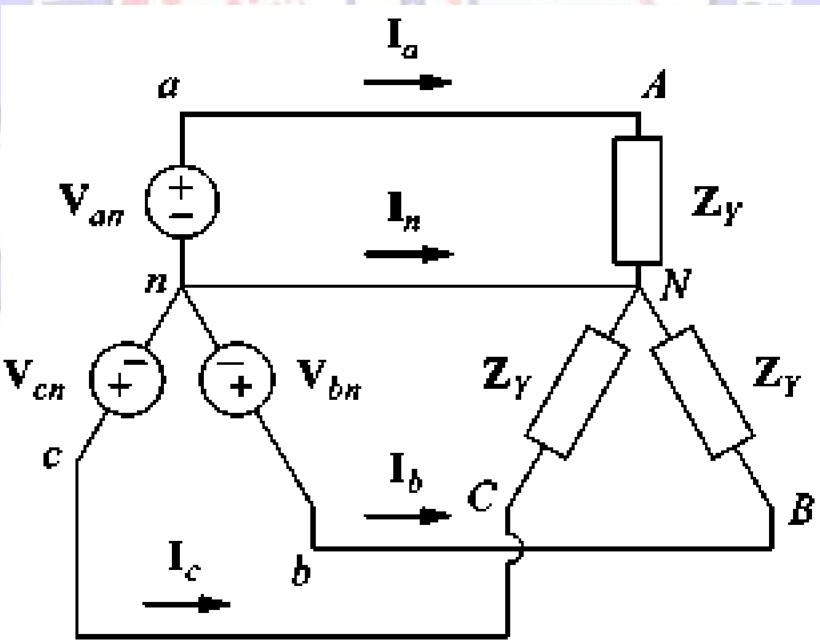


Fig. 5.9:

- Assuming the positive sequence, the phase voltages (or line-to-neutral voltages) are;

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ$$

The *line-to-line* voltages or simply *line* voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are related to the phase voltages. For example,

$$\mathbf{V}_{ab} = \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p\angle 0^\circ - V_p\angle -120^\circ$$
$$= V_p\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \sqrt{3}V_p\angle 30^\circ$$

Similarly, we can obtain

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p\angle -90^\circ$$
$$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p\angle -210^\circ$$

Thus, the magnitude of the line voltages V_L is $\sqrt{3}$ times the magnitude of the phase voltages V_p, or

$$\boxed{V_L = \sqrt{3}V_p}$$

where

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

and

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}|$$

- Figure 5.10 shows how to determine V_{ab} from the phase voltages.

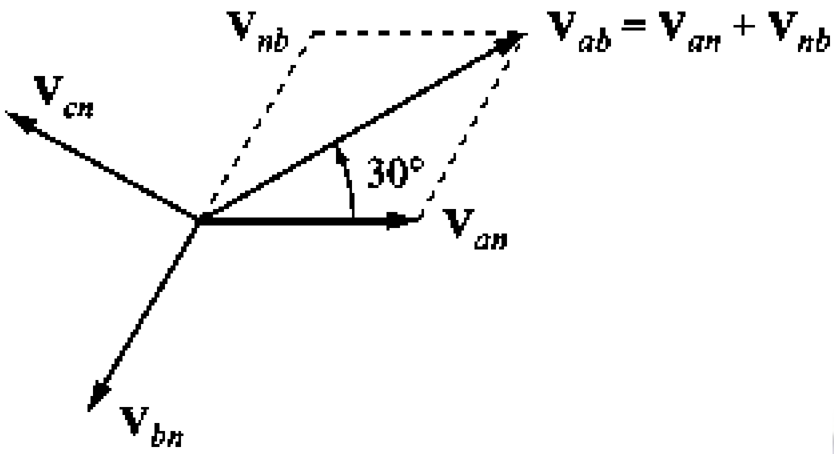


Fig. 5.10:

- Figure 5.11 shows the same for the three-line voltages.

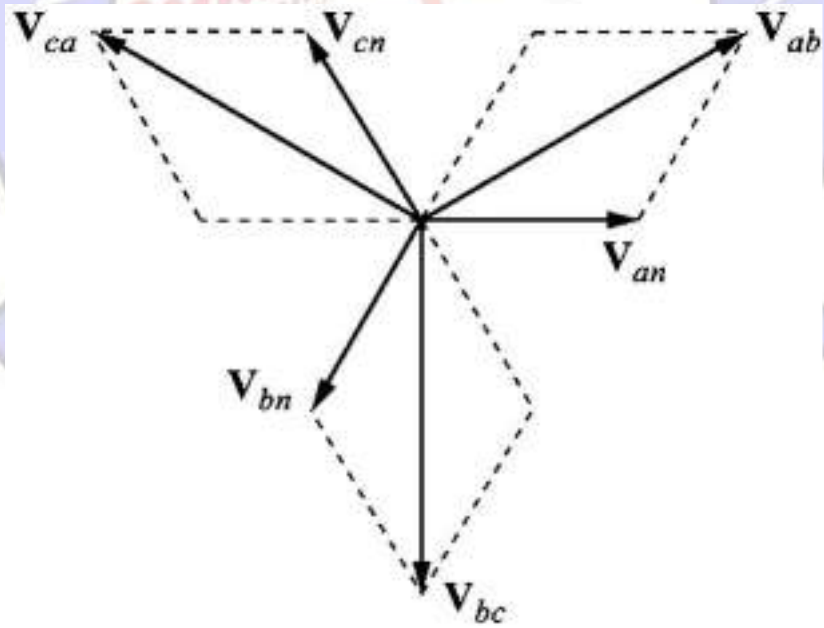


Fig. 5.11:

Notice that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{bc}$ by 120°, and $\mathbf{V}_{bc}$ leads $\mathbf{V}_{ca}$ by 120°, so that the line voltages sum up to zero as do the phase voltages.

- Applying KVL to each phase in Figure 5.9 we obtain the line currents as;

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle{-120^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\angle{-120^\circ}$$

$$\mathbf{I}_c = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle{-240^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\angle{-240^\circ}$$

We can readily infer that the line currents add up to zero,

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$$

so that

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0$$

or

$$\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0$$

The voltage across the neutral wire is zero. The neutral line can thus be removed without affecting the system.

While the *line* current is the current in each line, the *phase* current is the current in each phase of the source or load.

In the Y-Y system, the line current is the same as the phase current.

- The single-phase equivalent circuit in Figure 5.12. The single-phase analysis yields the line current I_a as;

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

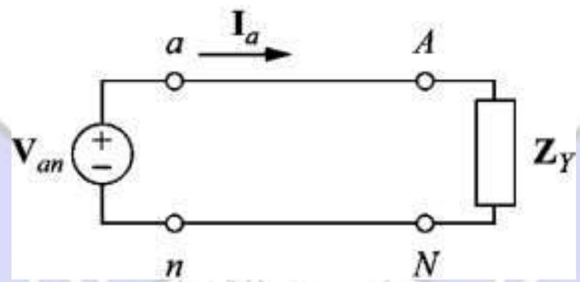
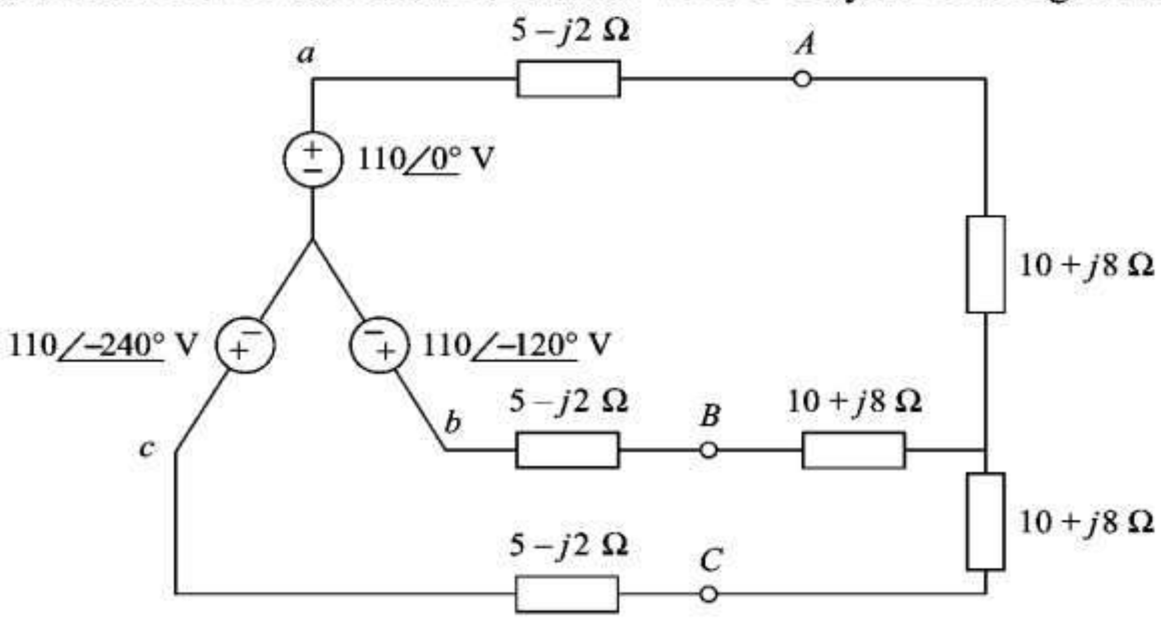


Fig. 5.12:

From $\mathbf{I}_a$, we use the phase sequence to obtain other line currents. Thus, as long as the system is balanced, we need only analyze one phase. We may do this even if the neutral line is absent, as in the three-wire system.

Example 12.2:

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13



Solution:

The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

where $\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\underline{/21.8^\circ}$. Hence,

$$\mathbf{I}_a = \frac{110\underline{/0^\circ}}{16.155\underline{/21.8^\circ}} = 6.81\underline{/-21.8^\circ}\text{ A}$$

Since the source voltages in Fig. 12.13 are in positive sequence, the line currents are also in positive sequence:

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 6.81\underline{/-141.8^\circ}\text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/-240^\circ} = 6.81\underline{/-261.8^\circ}\text{ A} = 6.81\underline{/98.2^\circ}\text{ A}$$

Example:

A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\mathbf{V}_{an} = 120\underline{/30^\circ}$ V, find: (a) the line voltages, (b) the line currents.

Answer: (a) $207.85\underline{/60^\circ}$ V, $207.85\underline{/-60^\circ}$ V, $207.85\underline{/-180^\circ}$ V,
(b) $3.75\underline{/-8.66^\circ}$ A, $3.75\underline{/-128.66^\circ}$ A, $3.75\underline{/-111.34^\circ}$ A.

5.3- Balanced Wye-Delta Connection.

- A balanced Y- Δ system consists of a balanced Y-connected source feeding a balanced Δ-connected load.
- The balanced Y-Δ system is shown in figure 5.13. There is no neutral connection from source to load for this case. Assuming the positive sequence, the phase voltages are again;

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ$$

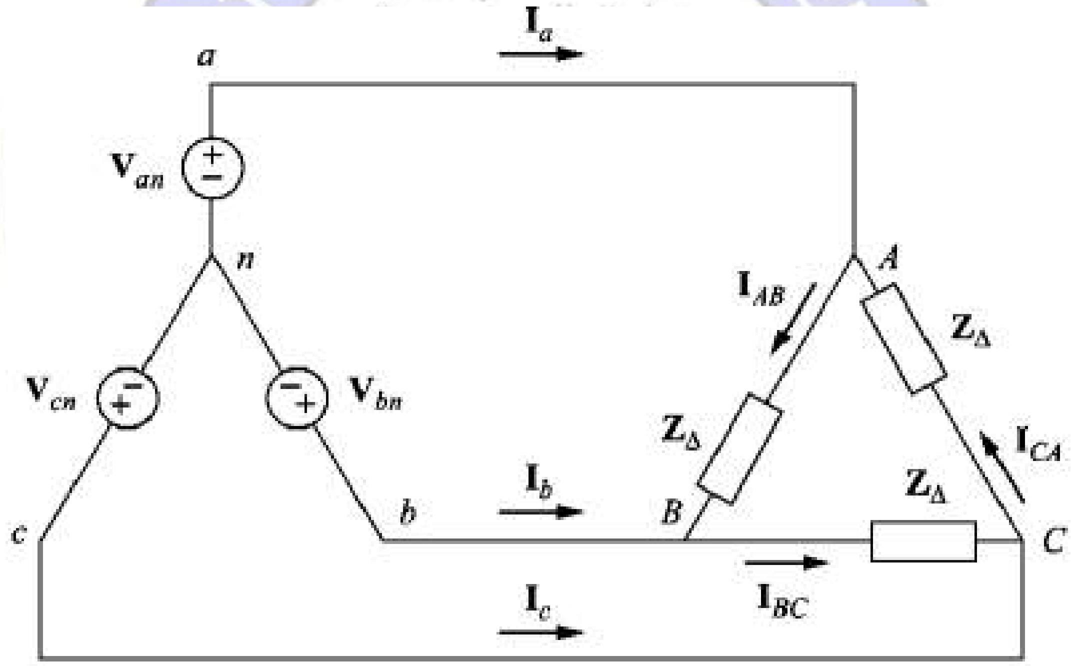


Fig. 5.13:

- The line voltages are given;

$$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \sqrt{3}V_p\angle -90^\circ = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \sqrt{3}V_p\angle -150^\circ = \mathbf{V}_{CA}$$

- The above equations are showing that the line voltages are equal to the voltages across the load impedances for this system configuration.
- From these voltages, we can obtain the phase currents as;

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta}$$

- These currents have the same magnitude but are out of phase with each other by 120°.

Another way to get these phase currents is to apply KVL. For example, applying KVL around loop *aABbna* gives

$$-\mathbf{V}_{an} + \mathbf{Z}_\Delta \mathbf{I}_{AB} + \mathbf{V}_{bn} = 0$$

or

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{an} - \mathbf{V}_{bn}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}$$

This is the more general way of finding the phase currents.

The line currents are obtained from the phase currents by applying KCL at nodes *A*, *B*, and *C*. Thus,

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$

Since $\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/-240^\circ}$,

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA} &= \mathbf{I}_{AB}(1 - 1\underline{/-240^\circ}) \\ &= \mathbf{I}_{AB}(1 + 0.5 - j0.866) = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ}\end{aligned}$$

Example:

A balanced *abc*-sequence Y-connected source with $\mathbf{V}_{an} = 100\angle 10^\circ$ V is connected to a Δ-connected balanced load $(8 + j4)\ \Omega$ per phase. Calculate the phase and line currents.

Solution:

This can be solved in two ways.

■ **METHOD 1** The load impedance is

$$\mathbf{Z}_\Delta = 8 + j4 = 8.944\angle 26.57^\circ\ \Omega$$

If the phase voltage $\mathbf{V}_{an} = 100\angle 10^\circ$, then the line voltage is

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\angle 30^\circ = 100\sqrt{3}\angle 10^\circ + 30^\circ = \mathbf{V}_{AB}$$

or

$$\mathbf{V}_{AB} = 173.2\angle 40^\circ \text{ V}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\angle 40^\circ}{8.944\angle 26.57^\circ} = 19.36\angle 13.43^\circ \text{ A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 19.36\angle -106.57^\circ \text{ A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 19.36\angle 133.43^\circ \text{ A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = \sqrt{3}(19.36)\angle 13.43^\circ - 30^\circ$$

$$= 33.53\angle -16.57^\circ \text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 33.53\angle -136.57^\circ \text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 33.53\angle 103.43^\circ \text{ A}$$

5.4- Balanced Delta-Delta Connection:

- A balanced Δ-Δ system is one in which both the balanced source and balanced load are Δ-connected.
- The source as well as the load may be delta-connected as shown in figure 5.14. Our goal is to obtain the phase and line currents as usual.

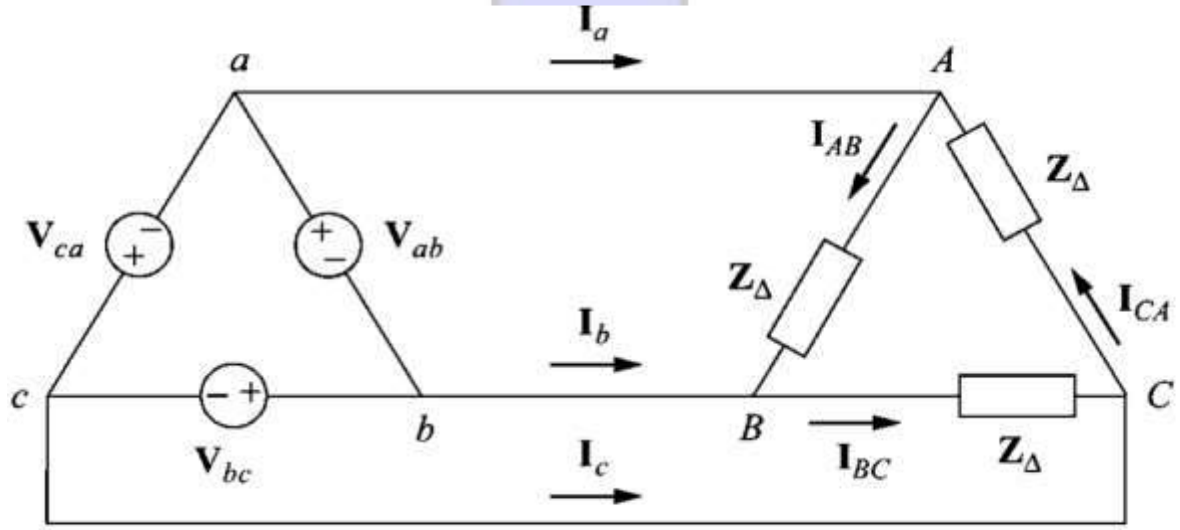


Fig. 5.14:

Assuming a positive sequence, the phase voltages for a delta-connected source are

$$\mathbf{V}_{ab} = V_p\underline{/0^\circ}$$
$$\mathbf{V}_{bc} = V_p\underline{/-120^\circ}, \qquad \mathbf{V}_{ca} = V_p\underline{/+120^\circ}$$

The line voltages are the same as the phase voltages.
assuming there is no line impedances, the phase voltages of the delta-connected source are equal to the voltages across the impedances; that is,

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA}$$

Hence, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_\Delta}$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_\Delta}$$

Since the load is delta-connected just as in the previous section, some of the formulas derived there apply here. The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C, as we did in the previous section:

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$

Example:

A balanced Δ-connected load having an impedance $20 - j15\ \Omega$ is connected to a Δ-connected, positive-sequence generator having $\mathbf{V}_{ab} = 330\underline{/0^\circ}$ V. Calculate the phase currents of the load and the line currents.

Solution:

The load impedance per phase is

$$\mathbf{Z}_\Delta = 20 - j15 = 25\underline{/-36.87^\circ}\ \Omega$$

Since $\mathbf{V}_{AB} = \mathbf{V}_{ab}$, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\underline{/0^\circ}}{25\underline{/-36.87}} = 13.2\underline{/36.87^\circ}\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 13.2\underline{/-83.13^\circ}\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 13.2\underline{/156.87^\circ}\ \text{A}$$

For a delta load, the line current always lags the corresponding phase current by 30° and has a magnitude $\sqrt{3}$ times that of the phase current. Hence, the line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = (13.2\underline{/36.87^\circ})(\sqrt{3}\underline{/-30^\circ})$$
$$= 22.86\underline{/6.87^\circ} \text{ A}$$
$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 22.86\underline{/-113.13^\circ} \text{ A}$$
$$\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ} = 22.86\underline{/126.87^\circ} \text{ A}$$

Example:

A positive-sequence, balanced Δ-connected source supplies a balanced Δ-connected load. If the impedance per phase of the load is $18 + j12\ \Omega$ and $\mathbf{I}_a = 19.202\underline{/35^\circ}$ A, find $\mathbf{I}_{AB}$ and $\mathbf{V}_{AB}$.

Answer: $11.094\underline{/65^\circ}$ A, $240\underline{/98.69^\circ}$ V.

5.5- Balanced Delta-Wye Connection:

- A balanced Δ-Y system consists of a balanced Δ-connected source feeding a balanced Y-connected load.
- Consider the circuit Δ-Y in Figure 5.15. Again, assuming the abc sequence, the phase voltages of a delta-connected source are;

$$\mathbf{V}_{ab} = V_p\angle 0^\circ, \qquad \mathbf{V}_{bc} = V_p\angle -120^\circ$$
$$\mathbf{V}_{ca} = V_p\angle +120^\circ$$

- These are also the line voltages as well as the phase voltages.

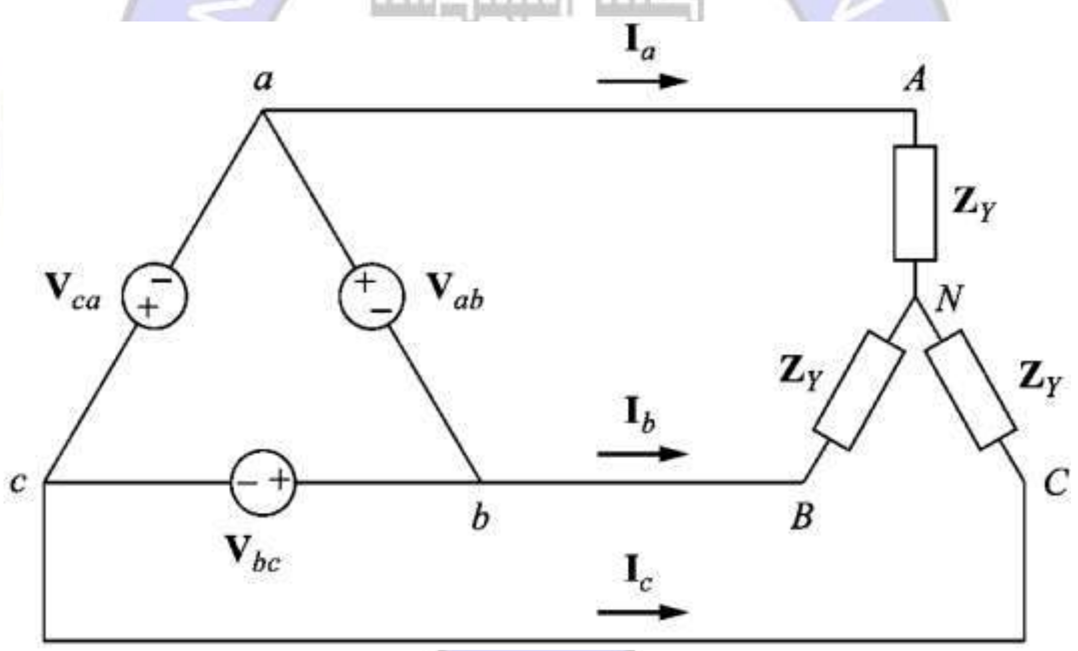


Fig. 5.15:

- We can obtain the line currents in many ways. One way is to apply KVL to loop aANBba in figure 5.15, writing

$$-\mathbf{V}_{ab} + \mathbf{Z}_Y\mathbf{I}_a - \mathbf{Z}_Y\mathbf{I}_b = 0$$

or

$$\mathbf{Z}_Y(\mathbf{I}_a - \mathbf{I}_b) = \mathbf{V}_{ab} = V_p\angle 0^\circ$$

Thus,

$$\mathbf{I}_a - \mathbf{I}_b = \frac{V_p\angle 0^\circ}{\mathbf{Z}_Y}$$

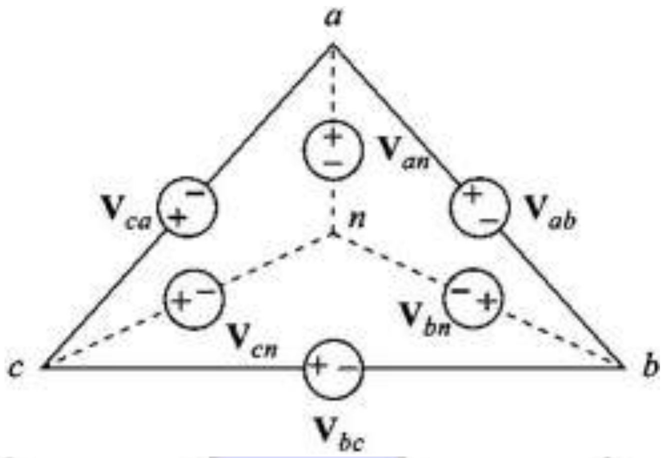
But $\mathbf{I}_b$ lags $\mathbf{I}_a$ by 120°, since we assumed the *abc* sequence; that is, $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$. Hence,

$$\begin{aligned}\mathbf{I}_a - \mathbf{I}_b &= \mathbf{I}_a(1 - 1\angle -120^\circ)\\ &= \mathbf{I}_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \mathbf{I}_a\sqrt{3}\angle 30^\circ\end{aligned}$$

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle -30^\circ}{\mathbf{Z}_Y}$$

From this, we obtain the other line currents $\mathbf{I}_b$ and $\mathbf{I}_c$ using the positive phase sequence, i.e., $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$, $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$. The phase currents are equal to the line currents.

Another way to obtain the line currents is to replace the delta-connected source with its equivalent wye-connected source, as shown in Fig.



Therefore, we obtain each phase voltage of the equivalent wye-connected source by dividing the corresponding line voltage of the delta-connected source by $\sqrt{3}$ and shifting its phase by $-30°$. Thus, the equivalent wye-connected source has the phase voltages

$$\mathbf{V}_{an} = \frac{V_p}{\sqrt{3}}\angle -30°$$

$$\mathbf{V}_{bn} = \frac{V_p}{\sqrt{3}}\angle -150°, \qquad \mathbf{V}_{cn} = \frac{V_p}{\sqrt{3}}\angle +90°$$

Once the source is transformed to wye, the circuit becomes a wye-wye system. The line current for phase a is

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle -30°}{\mathbf{Z}_Y}$$

- Table below presents a summary of the formulas for phase currents and voltages and line currents and voltages for the four connections.

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ Same as line currents	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Y-Δ	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ $\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$ $\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	Same as phase voltages $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Y	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$ Same as line currents	Same as phase voltages $\mathbf{I}_a = \dfrac{V_p\angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

Example:

A balanced Y-connected load with a phase impedance of $40 + j25\ \Omega$ is supplied by a balanced, positive sequence Δ-connected source with a line voltage of 210 V. Calculate the phase currents. Use $\mathbf{V}_{ab}$ as reference.

Solution:

The load impedance is

$$\mathbf{Z}_Y = 40 + j25 = 47.17\angle 32^\circ\ \Omega$$

and the source voltage is

$$\mathbf{V}_{ab} = 210\angle 0^\circ\ \text{V}$$

When the Δ-connected source is transformed to a Y-connected source,

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\angle -30^\circ = 121.2\angle -30^\circ\ \text{V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\angle -30^\circ}{47.12\angle 32^\circ} = 2.57\angle -62^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 2.57\angle -178^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle 120^\circ = 2.57\angle 58^\circ\ \text{A}$$

which are the same as the phase currents.

5.6- Power in Balanced System:

- The total instantaneous power in a balanced three-phase system if the load is Y or Δ-connected is;

$$p = 3V_p I_p \cos\theta$$

The average power per phase P_p for either the Δ-connected load or the Y-connected load is $p/3$, or

$$P_p = V_p I_p \cos\theta$$

and the reactive power per phase is

$$Q_p = V_p I_p \sin\theta$$

The apparent power per phase is

$$S_p = V_p I_p$$

The complex power per phase is

$$\mathbf{S}_p = P_p + jQ_p = \mathbf{V}_p \mathbf{I}_p^*$$

where $\mathbf{V}_p$ and $\mathbf{I}_p$ are the phase voltage and phase current with magnitudes V_p and I_p, respectively. The total average power is the sum of the average powers in the phases:

$$P = 3P_p = 3V_p I_p \cos\theta = \sqrt{3} V_L I_L \cos\theta$$

For a Y-connected load, $I_L = I_p$ but $V_L = \sqrt{3} V_p$, whereas for a Δ-connected load, $I_L = \sqrt{3} I_p$ but $V_L = V_p$. The total reactive power is

$$Q = 3V_p I_p \sin\theta = 3Q_p = \sqrt{3} V_L I_L \sin\theta$$

and the total complex power is

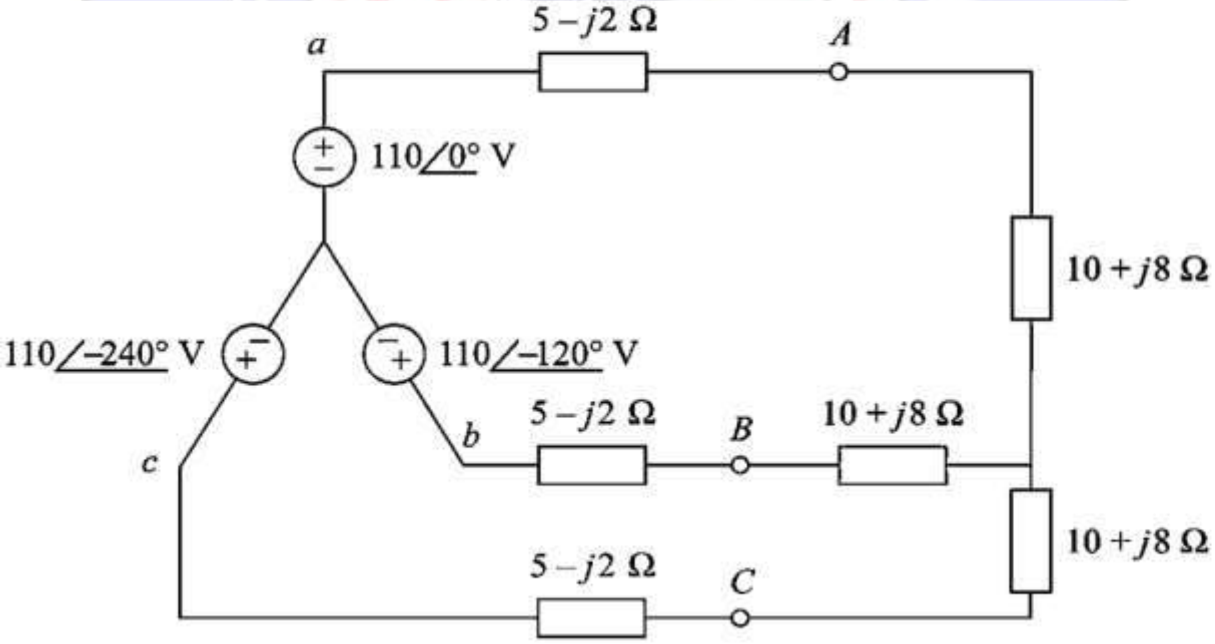
$$\mathbf{S} = 3\mathbf{S}_p = 3\mathbf{V}_p\mathbf{I}_p^* = 3I_p^2\mathbf{Z}_p = \frac{3V_p^2}{\mathbf{Z}_p^*}$$

where $\mathbf{Z}_p = Z_p\underline{/\theta}$ is the load impedance per phase. ($\mathbf{Z}_p$ could be $\mathbf{Z}_Y$ or $\mathbf{Z}_\Delta$.)

$$\mathbf{S} = P + jQ = \sqrt{3}V_L I_L\underline{/\theta}$$

Remember that V_p, I_p, V_L, and I_L are all rms values and that θ is the angle of the load impedance or the angle between the phase voltage and the phase current.

Example: For the circuit shown below, Determine the total average power, reactive power, and complex power at the source and at the load.



Solution:

It is sufficient to consider one phase, as the system is balanced. For phase a,

$$\mathbf{V}_p = 110\underline{/0^\circ}\text{ V} \qquad \text{and} \qquad \mathbf{I}_p = 6.81\underline{/-21.8^\circ}\text{ A}$$

Thus, at the source, the complex power absorbed is

$$\begin{aligned}\mathbf{S}_s = -3\mathbf{V}_p\mathbf{I}_p^* &= -3(110\underline{/0^\circ})(6.81\underline{/21.8^\circ}) \\ &= -2247\underline{/21.8^\circ} = -(2087 + j834.6)\text{ VA}\end{aligned}$$

The real or average power absorbed is -2087 W and the reactive power is -834.6 VAR.

At the load, the complex power absorbed is

$$\mathbf{S}_L = 3|\mathbf{I}_p|^2\mathbf{Z}_p$$

where $\mathbf{Z}_p = 10 + j8 = 12.81\underline{/38.66^\circ}$ and $\mathbf{I}_p = \mathbf{I}_a = 6.81\underline{/-21.8^\circ}$. Hence,

$$\begin{aligned}\mathbf{S}_L &= 3(6.81)^2 12.81\underline{/38.66^\circ} = 1782\underline{/38.66} \\ &= (1392 + j1113)\text{ VA}\end{aligned}$$

The real power absorbed is 1391.7 W and the reactive power absorbed is 1113.3 VAR. The difference between the two complex powers is absorbed by the line impedance $(5 - j2)\ \Omega$. To show that this is the case, we find the complex power absorbed by the line as

$$\mathbf{S}_\ell = 3|\mathbf{I}_p|^2\mathbf{Z}_\ell = 3(6.81)^2(5 - j2) = 695.6 - j278.3\text{ VA}$$

which is the difference between $\mathbf{S}_s$ and $\mathbf{S}_L$; that is, $\mathbf{S}_s + \mathbf{S}_\ell + \mathbf{S}_L = 0$, as expected.

Example 1:

What is the phase sequence of a balanced three-phase circuit for which $\mathbf{V}_{an} = 160\angle 30^\circ$ V and $\mathbf{V}_{cn} = 160\angle -90^\circ$ V? Find $\mathbf{V}_{bn}$.

Solution:

Since phase c lags phase a by 120°, this is an **acb sequence**.

$$\mathbf{V}_{bn} = 160\angle(30^\circ + 120^\circ) = \underline{\mathbf{160\angle 150^\circ\ V}}$$

Example 2:

A three-phase system with *abc* sequence and $\mathbf{V}_L = 200$ V feeds a Y-connected load with $Z_L = 40\angle 30^\circ\ \Omega$. Find the line currents.

Solution:

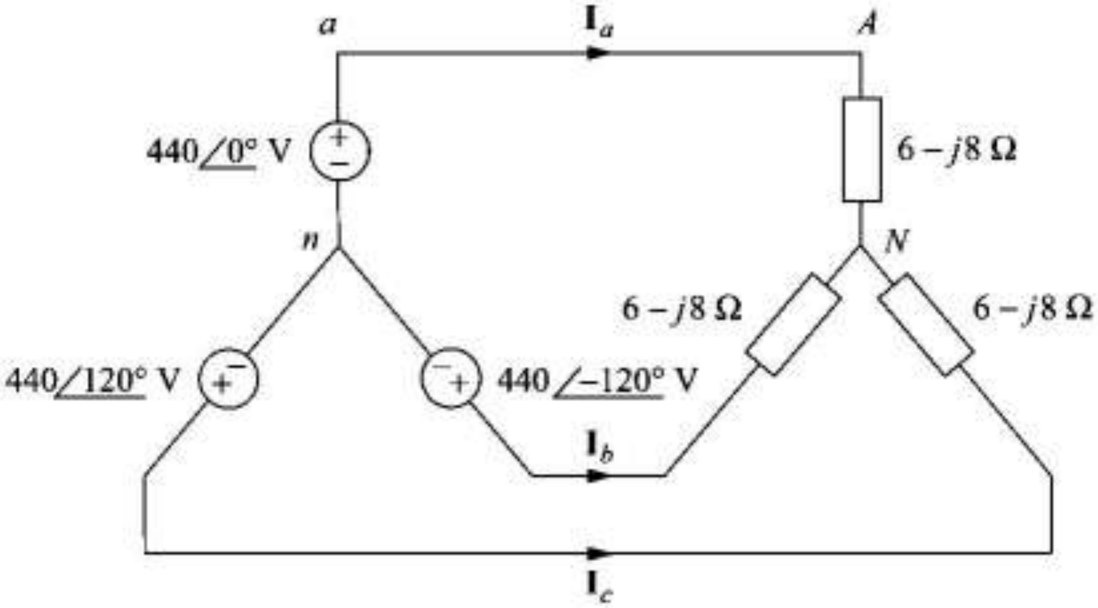
$$V_L = 200 = \sqrt{3}V_p \longrightarrow V_p = \frac{200}{\sqrt{3}}$$

$$I_a = \frac{V_{an}}{Z_Y} = \frac{200 < 0^o}{\sqrt{3}x40 < 30^o} = \underline{2.887 < -30^o\ \text{A}}$$

$$I_b = I_a < -120^o = \underline{2.887 < -150^o\ \text{A}}$$

$$I_c = I_a < +120^o = \underline{2.887 < 90^o\ \text{A}}$$

Example 3: Obtain the line currents in the three-phase circuit for the figure shown below:



Solution:

Using the per-phase circuit shown above,

$$I_a = \frac{440\angle 0°}{6 - j8} = \underline{44\angle 53.13°\ A}$$

$$I_b = I_a\angle -120° = \underline{44\angle -66.87°\ A}$$

$$I_c = I_a\angle 120° = \underline{44\angle 173.13°\ A}$$

Example 4:

In the Y-Δ system, the source is a positive sequence with $\mathbf{V}_{an} = 120\angle 0^\circ$ V and phase impedance $\mathbf{Z}_p = 2 - j3\ \Omega$. Calculate the line voltage $\mathbf{V}_L$ and the line current $\mathbf{I}_L$.

Solution:

$$V_{AB} = V_{ab} = \sqrt{3}V_p < 30^o = \sqrt{3}(120) < 30^o$$

$$V_L = |V_{ab}| = \sqrt{3}x120 = \underline{207.85\ \text{V}}$$

$$I_{AB} = \frac{V_{AB}}{Z_A} = \frac{\sqrt{3}V_p < 30^o}{2-j3}$$

$$I_a = I_{AB}\sqrt{3} < -30^o = \frac{3V_p < 0^o}{2-j3} = \frac{3x120}{2-j3} = 55.385 + j83.07$$

$$I_L = |I_a| = \underline{99.846\ \text{A}}$$

5.7- Unbalanced Three-Phase Systems:

- An unbalanced system is caused by two possible situations:
  - ✓ Source voltages are not equal in magnitude or differ in phase by angles that are unequal.
  - ✓ Load impedances are unequal.
- An unbalanced system is due to unbalanced voltage sources or an unbalanced load.
- To simplify analysis, source voltages are assumed to be balanced, but an unbalanced load.
- Figure 5.16 shows an example of an unbalanced three-phase system that consists of balanced source voltages (not shown in the figure) and an unbalanced Y-connected load (shown in the figure). Since the load is unbalanced, Z_A , Z_B , and Z_C are not equal.

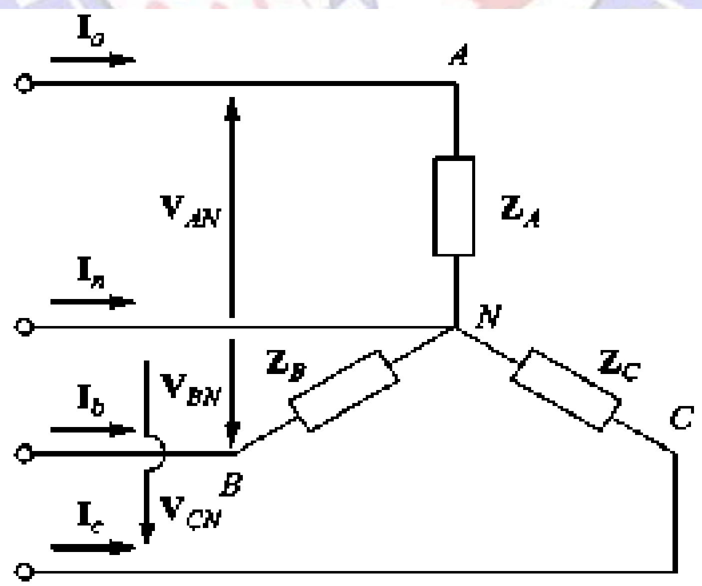


Fig. 5.16:

- The line currents are determined by Ohm's law as;

$$\mathbf{I}_a = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B}, \qquad \mathbf{I}_c = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C}$$

- This set of unbalanced line currents produces current in the neutral line, which is not zero as in a balanced system. Applying KCL at node N gives the neutral line current as;

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c)$$

In a three-wire system where the neutral line is absent, we can still find the line currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ using mesh analysis. At node N, KCL must be satisfied so that $\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$ in this case. The same could be done for an unbalanced Δ-Y, Y-Δ, or Δ-Δ three-wire system.

Example: The unbalanced Y-load of figure 5.16 has balanced voltages of 100 V and the acb sequence;

Calculate the line currents and the neutral current.
Take $\mathbf{Z}_A = 15\ \Omega$, $\mathbf{Z}_B = 10 + j5\ \Omega$, $\mathbf{Z}_C = 6 - j8\ \Omega$.

The line currents are

$$\mathbf{I}_a = \frac{100\underline{/0^\circ}}{15} = 6.67\underline{/0^\circ}\text{ A}$$

$$\mathbf{I}_b = \frac{100\underline{/120^\circ}}{10 + j5} = \frac{100\underline{/120^\circ}}{11.18\underline{/26.56^\circ}} = 8.94\underline{/93.44^\circ}\text{ A}$$

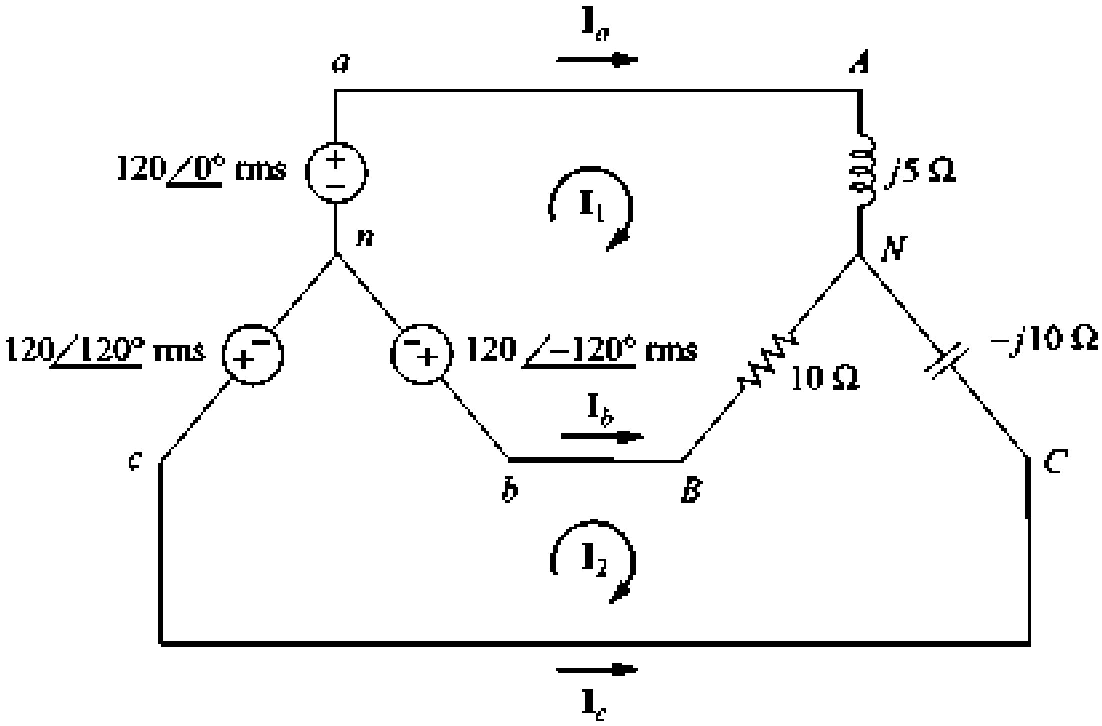
$$\mathbf{I}_c = \frac{100\underline{/-120^\circ}}{6 - j8} = \frac{100\underline{/-120^\circ}}{10\underline{/-53.13^\circ}} = 10\underline{/-66.87^\circ}\text{ A}$$

The current in the neutral line is

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = -(6.67 - 0.54 + j8.92 + 3.93 - j9.2)$$
$$= -10.06 + j0.28 = 10.06\underline{/178.4^\circ}\text{ A}$$

Example:

For the unbalanced circuit in Fig. 12.25, find: (a) the line currents, (b) the total complex power absorbed by the load, and (c) the total complex power absorbed by the source.



(a) We use mesh analysis to find the required currents. For mesh 1,

$$120\angle{-120^\circ} - 120\angle{0^\circ} + (10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 0$$

or

$$(10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 120\sqrt{3}\angle{30^\circ}$$

For mesh 2,

$$120\angle{120^\circ} - 120\angle{-120^\circ} + (10 - j10)\mathbf{I}_2 - 10\mathbf{I}_1 = 0$$

or

$$-10\mathbf{I}_1 + (10 - j10)\mathbf{I}_2 = 120\sqrt{3}\angle{-90^\circ}$$

Equations (12.10.1) and (12.10.2) form a matrix equation:

$$\begin{bmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\sqrt{3}\angle{30^\circ} \\ 120\sqrt{3}\angle{-90^\circ} \end{bmatrix}$$

The determinants are

$$\Delta = \begin{vmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{vmatrix} = 50 - j50 = 70.71\underline{/-45^\circ}$$

$$\Delta_1 = \begin{vmatrix} 120\sqrt{3}\underline{/30^\circ} & -10 \\ 120\sqrt{3}\underline{/-90^\circ} & 10 - j10 \end{vmatrix} = 207.85(13.66 - j13.66)$$

$$= 4015\underline{/-45^\circ}$$

$$\Delta_2 = \begin{vmatrix} 10 + j5 & 120\sqrt{3}\underline{/30^\circ} \\ -10 & 120\sqrt{3}\underline{/-90^\circ} \end{vmatrix} = 207.85(13.66 - j5)$$

$$= 3023.4\underline{/-20.1^\circ}$$

The mesh currents are

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{4015.23\underline{/-45^\circ}}{70.71\underline{/-45^\circ}} = 56.78 \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{3023.4\underline{/-20.1^\circ}}{70.71\underline{/-45^\circ}} = 42.75\underline{/24.9^\circ} \text{ A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_1 = 56.78 \text{ A}, \qquad \mathbf{I}_c = -\mathbf{I}_2 = 42.75\underline{/-155.1^\circ} \text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_2 - \mathbf{I}_1 = 38.78 + j18 - 56.78 = 25.46\underline{/135^\circ} \text{ A}$$

(b) We can now calculate the complex power absorbed by the load. For phase A,

$$\mathbf{S}_A = |\mathbf{I}_a|^2\mathbf{Z}_A = (56.78)^2(j5) = j16{,}120 \text{ VA}$$

For phase B,

$$\mathbf{S}_B = |\mathbf{I}_b|^2\mathbf{Z}_B = (25.46)^2(10) = 6480 \text{ VA}$$

For phase C,

$$\mathbf{S}_C = |\mathbf{I}_c|^2\mathbf{Z}_C = (42.75)^2(-j10) = -j18{,}276 \text{ VA}$$

The total complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 6480 - j2156 \text{ VA}$$

(c) We check the result above by finding the power absorbed by the source. For the voltage source in phase a,

$$\mathbf{S}_a = -\mathbf{V}_{an}\mathbf{I}_a^* = -(120\underline{/0^\circ})(56.78) = -6813.6 \text{ VA}$$

For the source in phase b,

$$\mathbf{S}_b = -\mathbf{V}_{bn}\mathbf{I}_b^* = -(120\underline{/-120^\circ})(25.46\underline{/-135^\circ})$$
$$= -3055.2\underline{/105^\circ} = 790 - j2951.1 \text{ VA}$$

For the source in phase c,

$$\mathbf{S}_c = -\mathbf{V}_{bn}\mathbf{I}_c^* = -(120\underline{/120^\circ})(42.75\underline{/155.1^\circ})$$
$$= -5130\underline{/275.1^\circ} = -456.03 + j5109.7 \text{ VA}$$

The total complex power absorbed by the three-phase source is

$$\mathbf{S}_s = \mathbf{S}_a + \mathbf{S}_b + \mathbf{S}_c = -6480 + j2156 \text{ VA}$$

showing that $\mathbf{S}_s + \mathbf{S}_L = 0$ and confirming the conservation principle of ac power.