Chapter Three

First-Order Circuits

3.1- Introduction

3.2- Source-free RC circuit

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3.1- Introduction:

- The three passive elements (resistors, capacitors, and inductors) and one active element (the op amp) have been considered.
- Two types of simple circuits have been considered such as a circuit which consists of resistor and capacitor called RC, another circuit called RL which consists of resistor and an inductor.
- RC and RL circuits can be analyzed by applying Kirchhoff's laws.
- Applying the Kirchhoff's laws to RC and RL circuits produces differential equations.
- The differential equations resulting from analyzing RC and RL circuits are called first order circuits.
- A first-order circuit is characterized by a first-order differential equation.
- There are two ways to excite RC and RL circuits.
- The first way is by initial conditions of the storage elements in the circuits which called source-free circuits. Assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors.
- The second way of exciting first-order circuits is by independent sources. the independent sources such as dc sources have been considered.

3.2- Source-free RC circuit:

- A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.
- Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 7.1.

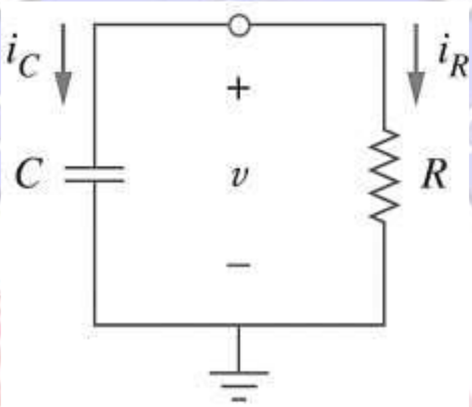


Figure 7.1
A source-free *RC* circuit.

- The objective is to determine the circuit response.
- Assume that at time t=0, initial voltage across the capacitor is

$$v(0) = V_0 \qquad (7.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad (7.2)$$

Applying KCL at the top node of the circuit in Fig. 7.1 yields

$$i_C + i_R = 0 \qquad (7.3)$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \tag{7.4a}$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \tag{7.4b}$$

This is a *first-order differential equation*, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \tag{7.5}$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC} \tag{7.6}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0e^{-t/RC} \tag{7.7}$$

- This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. The response is due to the initial energy stored which is called the natural response of the circuit.

- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- The natural response is illustrated graphically in Fig. 7.2. As t increases, the voltage decreases toward zero.

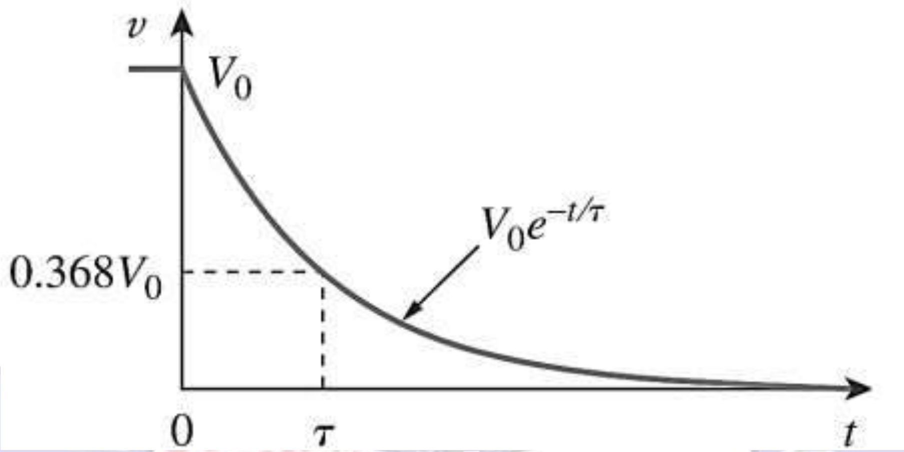


Figure 7.2
The voltage response of the *RC* circuit.

- The voltage decreasing is expressed in terms of the time constant, denoted by τ,
- The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

This implies that at $t = \tau$, Eq. (7.7) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$$

or

$$\boxed{\tau = RC} \tag{7.8}$$

In terms of the time constant, Eq. (7.7) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \tag{7.9}$$

- With a calculator it is easy to show that the value of is as shown in Table 7.1.

TABLE 7.1
Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

- Observe from Eq. (7.8) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.
- This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state)

quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state.

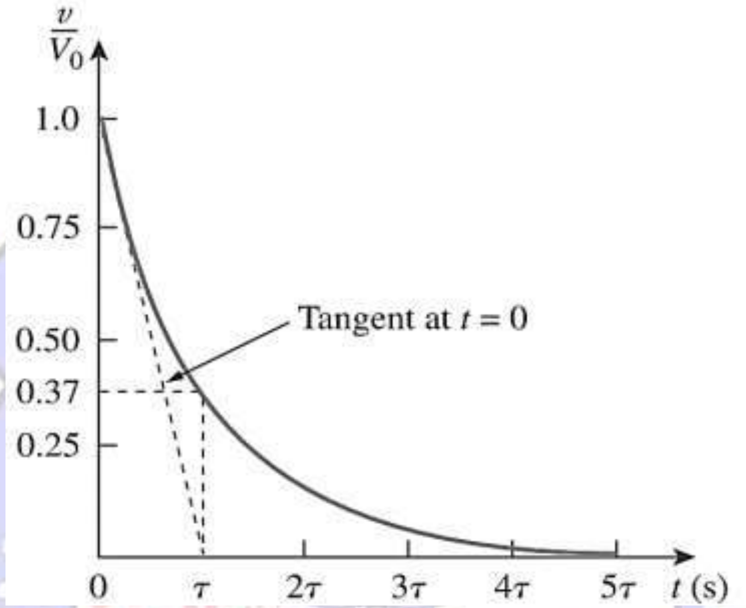


Figure 7.3
Graphical determination of the time constant τ from the response curve.

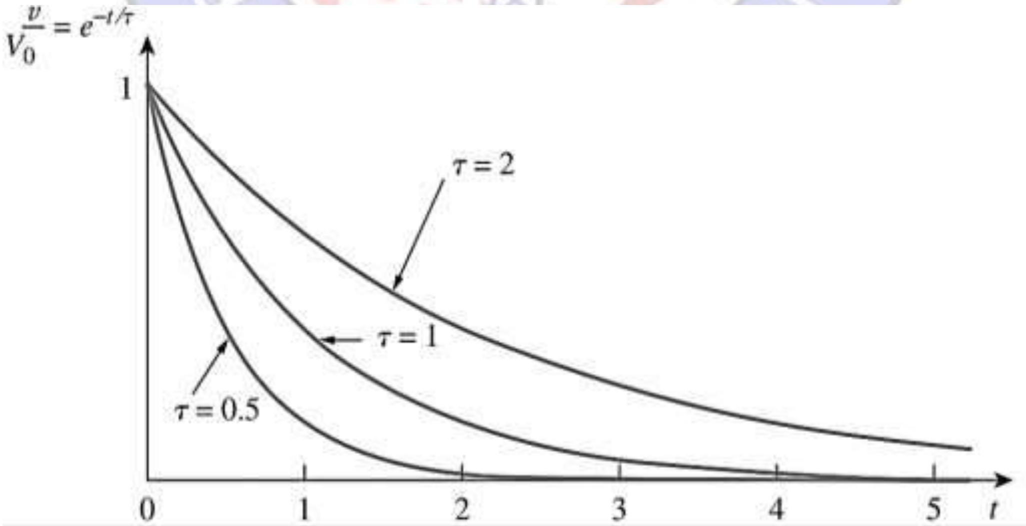


Figure 7.4
Plot of $v/V_0 = e^{-t/\tau}$ for various values of the time constant

With the voltage $v(t)$ in Eq. (7.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-t/\tau} \tag{7.10}$$

The Key to Working with a Source-Free *RC* Circuit Is Finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

Example 7.1:

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

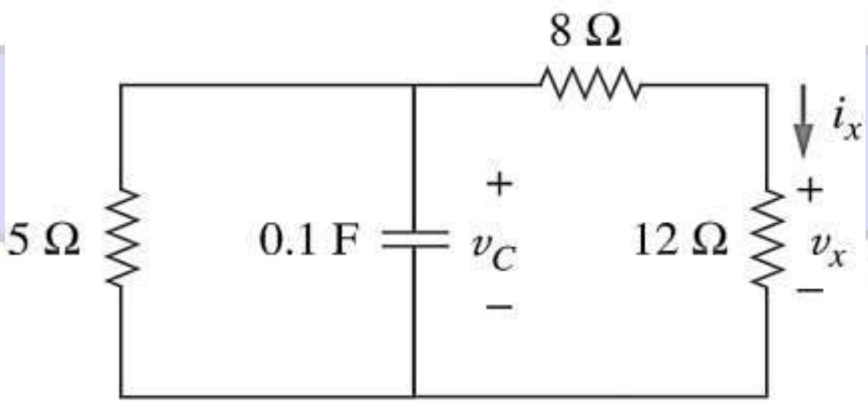


Figure 7.5
For Example 7.1.

- We first need to make the circuit in Fig. 7.5 conform with the standard RC circuit in Fig. 7.1.
- We find the equivalent resistance or the Thevenin resistance at the capacitor terminals.

- The objective is always to first obtain capacitor voltage v_c. From this, we can determine v_x and i_x.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12 + 8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

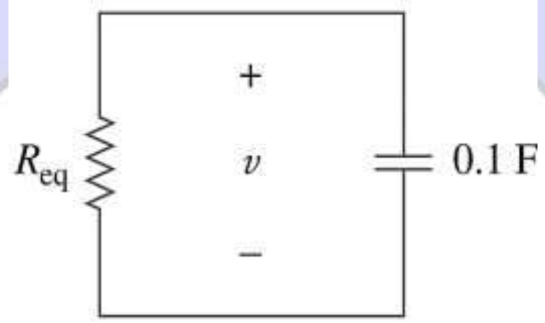


Figure 7.6
Equivalent circuit for the circuit in Fig. 7.5.

Example 7.2:

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

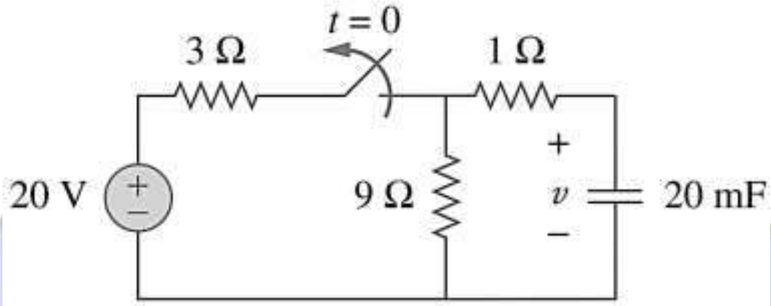


Figure 7.8
For Example 7.2

Solution:
For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15 \text{ V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15 \text{ V}$$

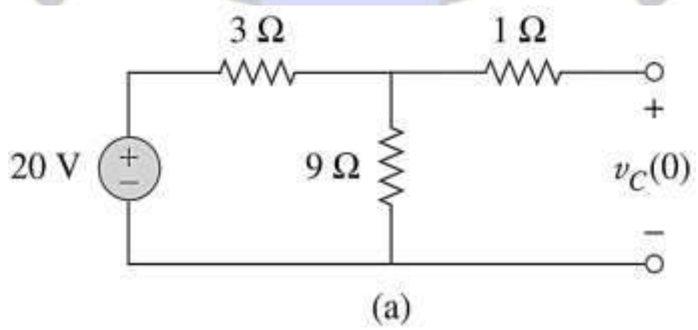


Figure 7.9
For Example 7.2: (a) $t < 0$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or

$$v(t) = 15e^{-5t}\ \text{V}$$

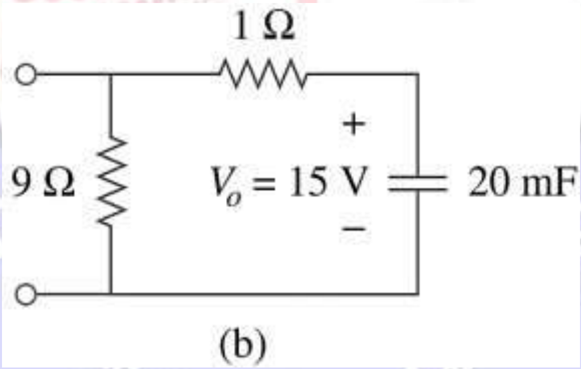


Figure 7.9
For Example 7.2: (b) $t > 0$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ \text{J}$$

3.3- Source-free RL circuit:

- Consider the series connection of a resistor and an inductor, as shown in Fig. 7.11.

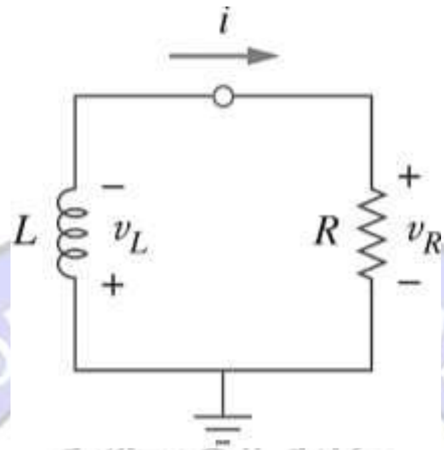


Figure 7.11
A source-free *RL* circuit

- The goal is to determine the circuit response, assume that the current $i(t)$ will through the inductor.
- The idea that the inductor current cannot change instantaneously. At $t = 0$, it is assumed that the inductor has an initial current I_o, or

$$i(0) = I_0 \qquad \textbf{(7.13)}$$

with the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2 \qquad \textbf{(7.14)}$$

Applying KVL around the loop in Fig. 7.11,

$$v_L + v_R = 0 \qquad \textbf{(7.15)}$$

But $v_L = L\,di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \qquad \textbf{(7.16)}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L}\,dt$$

$$\ln i\Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Rightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

or

$$\ln\frac{i(t)}{I_0} = -\frac{Rt}{L} \qquad \textbf{(7.17)}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \qquad \textbf{(7.18)}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig. 7.12. It is evident from Eq. (7.18) that the time constant for the RL circuit is

$$\boxed{\tau = \frac{L}{R}} \qquad \textbf{(7.19)}$$

with τ again having the unit of seconds. Thus, Eq. (7.18) may be written as

$$\boxed{i(t) = I_0 e^{-t/\tau}} \qquad \textbf{(7.20)}$$

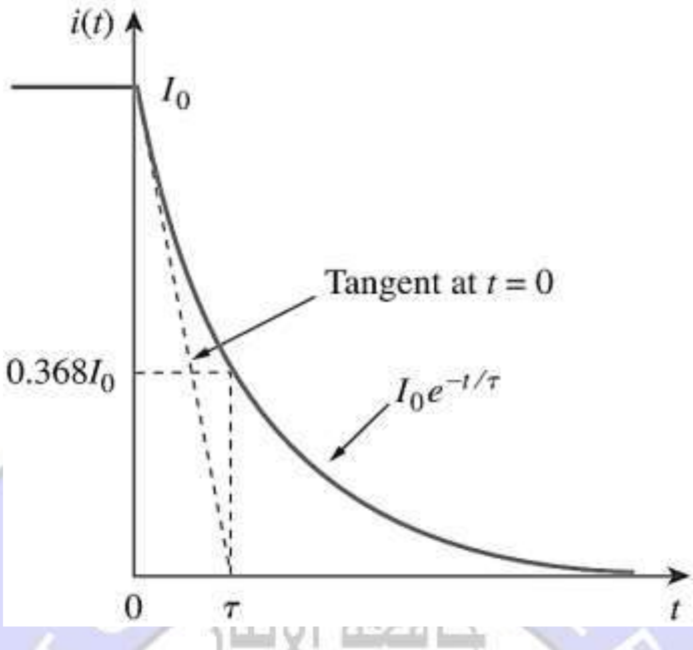


Figure 7.12
The current response of the *RL* circuit.

With the current in Eq. (7.20), we can find the voltage across the resistor as

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{7.21}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{7.22}$$

The Key to Working with a Source-Free *RL* Circuit Is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

Example 7.3:

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig. 7.13.

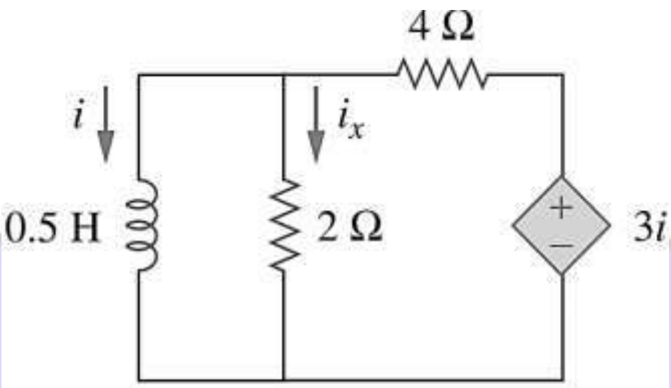


Figure 7.13
For Example 7.3

Solution:

There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (7.20). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

■ **METHOD 1** The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals *a-b*, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2} \qquad \textbf{(7.3.1)}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \qquad \textbf{(7.3.2)}$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

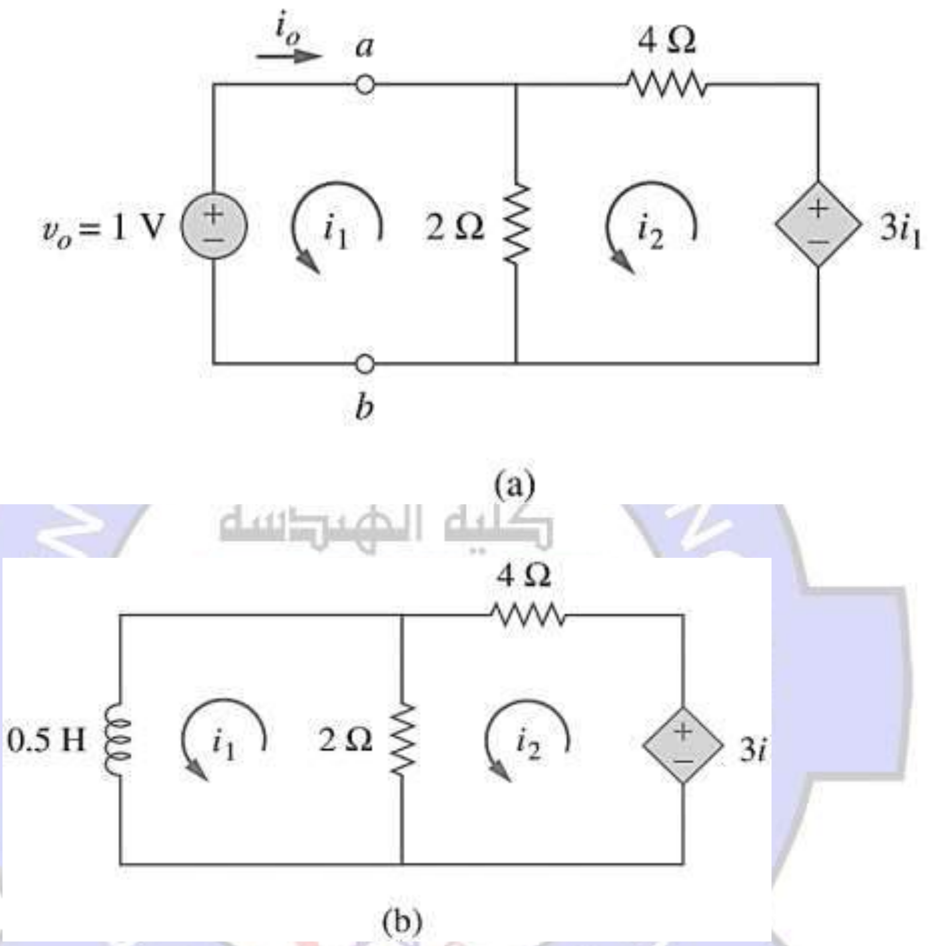


Figure 7.14
Solving the circuit in Fig. 7.13.

Hence,

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}\ \text{A}, \qquad t > 0$$

■ METHOD 2 We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \qquad \textbf{(7.3.3)}$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1 \qquad \textbf{(7.3.4)}$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t$$

or

$$\ln \frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Example 7.4:

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

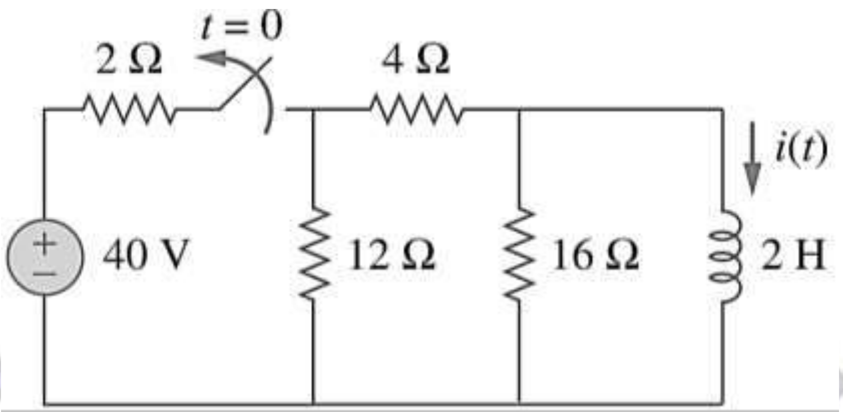


Figure 7.16
For Example 7.4.

Solution:

When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig. 7.17(a). To get i_1 in Fig. 7.17(a), we combine the 4-Ω and 12-Ω resistors in parallel to get

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

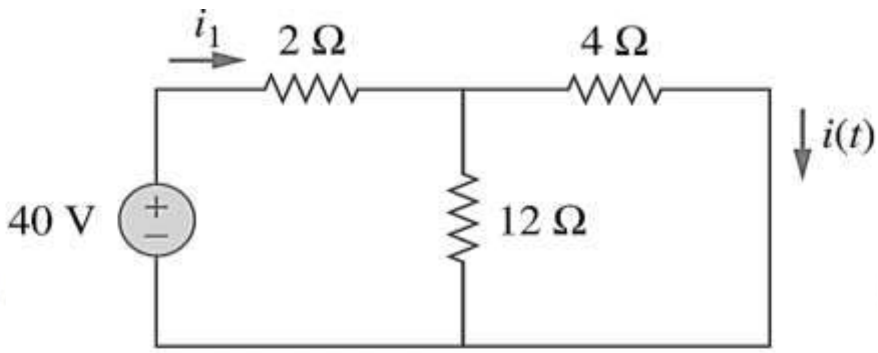
$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

We obtain $i(t)$ from i_1 in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12+4} i_1 = 6 \text{ A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6 \text{ A}$$



(a)

Figure 7.17
Fig. 7.16: (a) for $t < 0$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free *RL* circuit in Fig. 7.17(b). Combining the resistors, we have

$$R_{eq} = (12 + 4) \parallel 16 = 8 \ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} \text{ s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t} \text{ A}$$

Solution:

It is better to first find the inductor current i and then obtain other quantities from it.

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, so that we have the circuit shown in Fig. 7.20(a). Hence, $i_o = 0$, and

$$i(t) = \frac{10}{2+3} = 2\text{ A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6\text{ V}, \qquad t < 0$$

Thus, $i(0) = 2$.

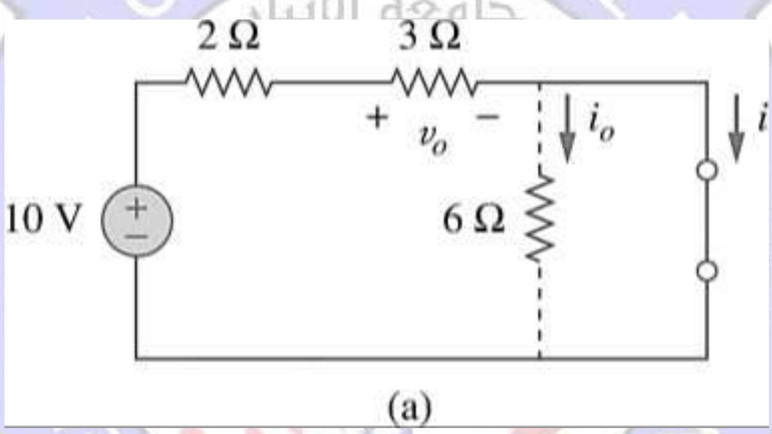


Figure 7.20
Fig. 7.19 for: (a) $t < 0$

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. 7.20(b). At the inductor terminals,

$$R_{Th} = 3 \parallel 6 = 2\,\Omega$$

so that the time constant is

$$\tau = \frac{L}{R_{Th}} = 1\text{ s}$$

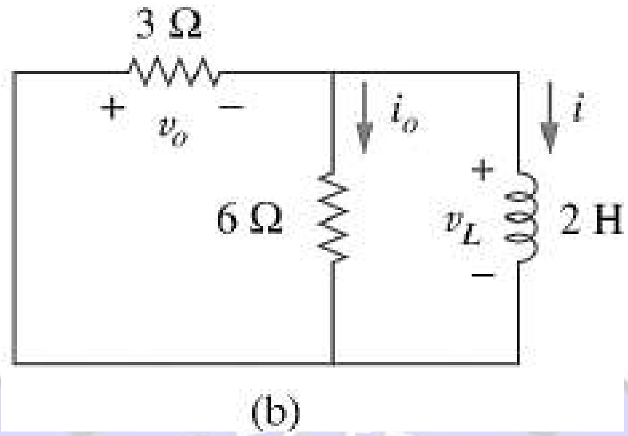


Figure 7.20
Fig. 7.19 (b) for $t > 0$

Hence,

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

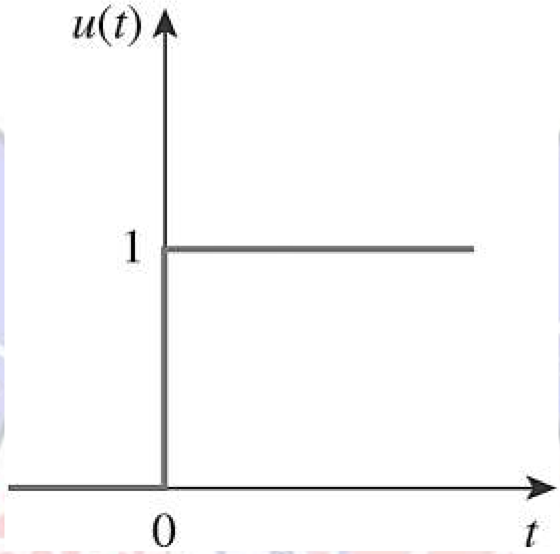
Since the inductor is in parallel with the 6-Ω and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

and

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$

- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.
- The unit step function u(t) is 0 for negative values of t and 1 for positive values of t as shown below



In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \quad (7.24)$$

If the change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \quad (7.25)$$

which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in Fig. 7.24(a).

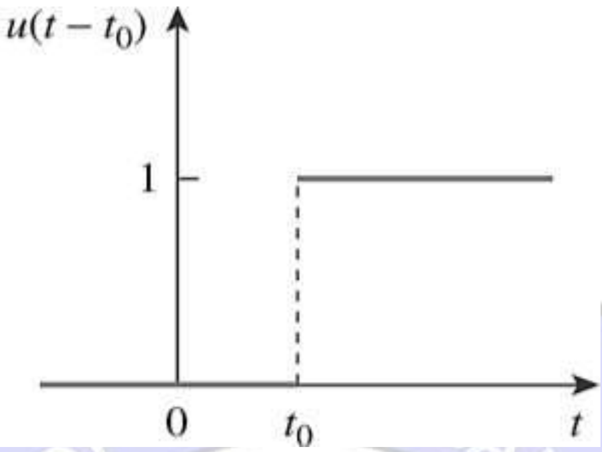


Figure 7.24
(a) The unit step function delayed by t_0,

If the change is at $t = -t_0$, the unit step function becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \quad (7.26)$$

meaning that $u(t)$ is advanced by t_0 seconds, as shown in Fig. 7.24(b).

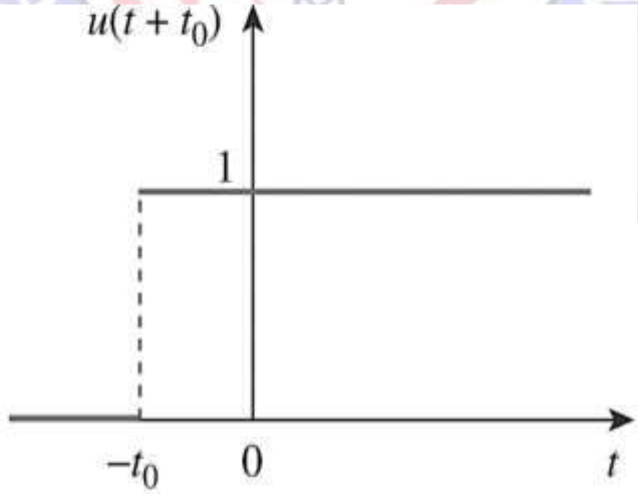


Figure 7.24
(b) the unit step advanced by t_0.

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \quad \textbf{(7.27)}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \quad \textbf{(7.28)}$$

If we let $t_0 = 0$, then $v(t)$ is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig. 7.25(a); its equivalent circuit is shown in Fig. 7.25(b). It is evident in Fig. 7.25(b) that terminals *a-b* are short-circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals

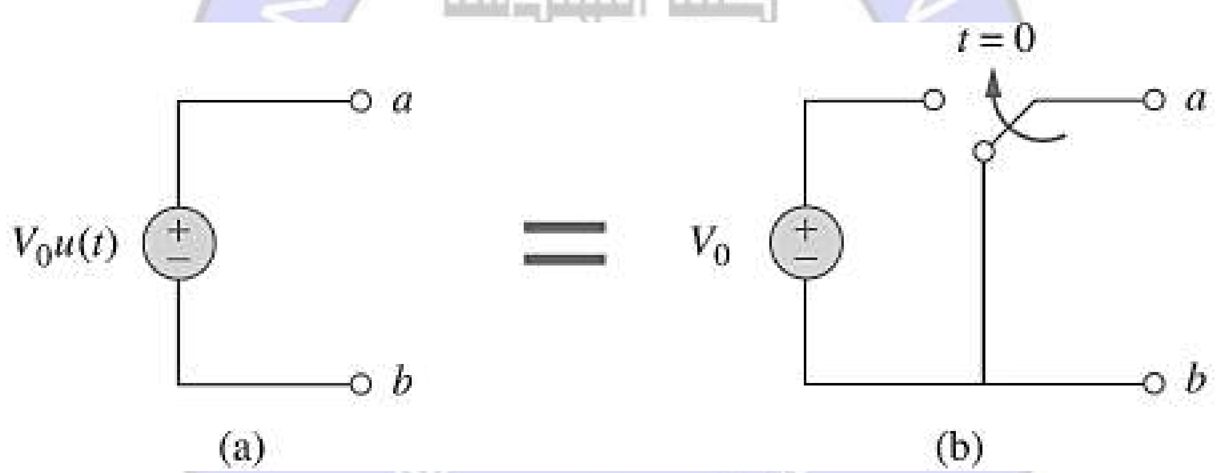


Figure 7.25
(a) Voltage source of $V_0u(t)$, (b) its equivalent circuit.

for $t > 0$. Similarly, a current source of $I_0 u(t)$ is shown in Fig. 7.26(a), while its equivalent circuit is in Fig. 7.26(b). Notice that for $t < 0$, there is an open circuit ($i = 0$), and that $i = I_0$ flows for $t > 0$.

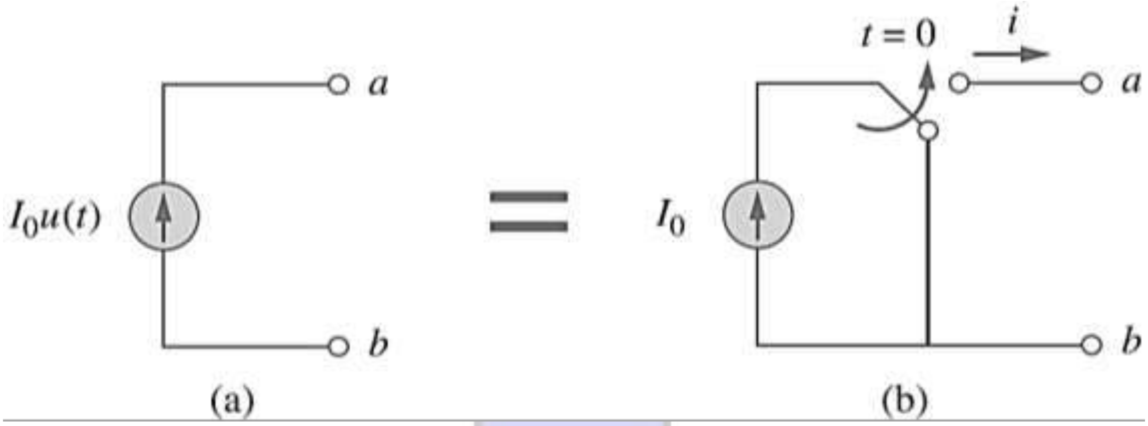


Figure 7.26
(a) Current source of $I_0u(t)$, (b) its equivalent circuit.