Course Description:
Functions, Limits and continuity, Differentiation, Applications of derivatives, Integration, Inverse functions. Applications of the Integral

Recommended Textbook(s):
Calculus, Early Transcendental By James Stewart, 6th Edition, 2008, Brooks/Cole

Prerequisites:
None

Course Topics:

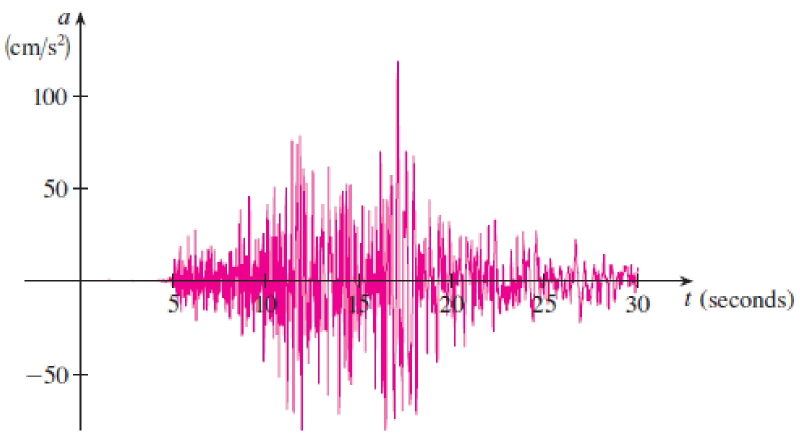
1. **Functions and models**: four ways to represent a function , mathematical models: a catalogue of essential functions , new functions from old functions , exponential functions, inverse functions and logarithms
2. **Limits**: the tangent and velocity problems. The limit of a function, calculating limits using the limit laws. Continuity, limits at infinity, horizontal asymptote. Infinite limits, vertical asymptotes. derivatives and rates of change
3. **Differentiation rules:** Differentiation of Polynomials. The Product and Quotient Rules. Derivatives of Trigonometric Functions. The Chain Rule, Implicit Differentiation. Related Rates, Indeterminate forms and l'hospital's rule.
7. **Applications of differentiation**: maximum and minimum values. The mean value theorem. How derivatives affect the shape of a graph. Summary of curve sketching. Optimization problems. Antiderivatives.
10.**Integrals**: the definite integral. The fundamental theorem of calculus. The indefinite integral and net change theorem. The substitution rule.
11. **Applications of integrals**: areas between curves. Volumes. Volumes by cylindrical shells. Average value of a function.
12.**Exponential and logarithmic functions**. Derivative and integrals involving logarithmic functions. Inverse functions. Derivative and integrals involving exponential functions. Derivative and integrals involving inverse trig functions. Hyperbolic functions.

1 Functions

Examples of functions

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula A= πr^2

B. The vertical acceleration of the ground as measured by a seismograph during an earthquake is a function of the elapsed time. Figure below shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of the graph provides a corresponding value of a.



So, function is y = f(x), expressing y as a dependent variable on f and x is an independent variable.

For example f(x) = 2x-1

If x = 1 then 2*1-1 = 1
X= -1 then 2 *-1-1 = -3

And so on

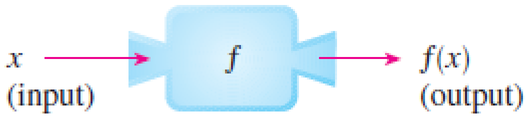
If an absolute value like f(x) = IxI then $x = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

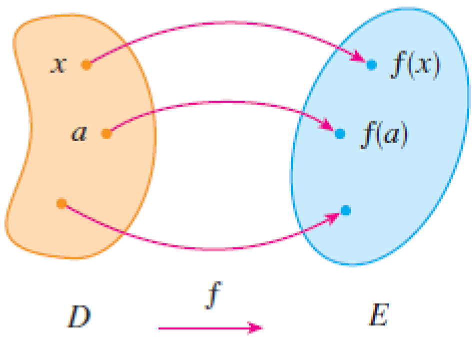
Note:

$|-a| = |a|$
$|ab| = |a|\,|b|$
$|a/b| = |a|/|b|$ but $b \neq 0$
$|a+b| = |a|+|b|$

2. Domain and Range

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the domain of the function. The number f(x) is the value of f at x and is read "f of x" The range of f is the set of all possible values of f(x) as varies throughout the domain.
Then domains and ranges of many functions are intervals of real numbers.





Example5: Find the domain and range of the following functions:

(a) $f(x) = 2x - 1$

(b) $f(x) = x^2$

(c) $f(x) = \tan x$

(d) $y = \sqrt{x}$

(e) $y = \dfrac{x-12}{x^2 - 5x + 6}$

Solution

(a) $f(x) = 2x - 1$

Domain: x = R $-\infty \le x \le \infty$

Range f(x) = R R: denotes as all real number

(b) $f(x) = x^2$

Domain: x = R $-\infty \le x \le \infty$

Range f(x) = R

(c) $f(x) = \tan x$

Domain: x = R excluding $\pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2}, ...$

(d) $f(x) = \sqrt{x}$

Domain $0 \le x$ and Range $0 \le y$

(e) $y = \dfrac{x-12}{x^2-5x+6}$

The denominator not equal to zero

$x^2-5x+6=0$

(x-3)(x-2) then domain all but $x \neq 3$ and $x \neq 2$

2. Sketch of functions

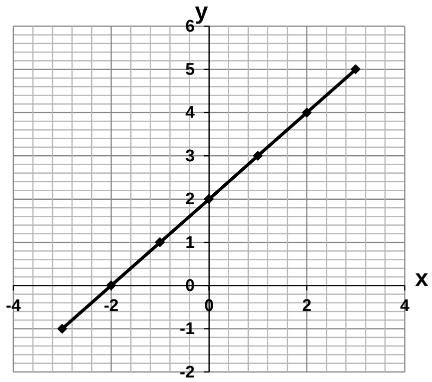
The points in the plane whose (x,y) are the input and output pairs of a function make up the graph of the function.

Definition:
Even function: if f(x) = f(-x)
Odd function: if f(x) = -f(-x)

Example6: sketch the function y= x+2

Solution:

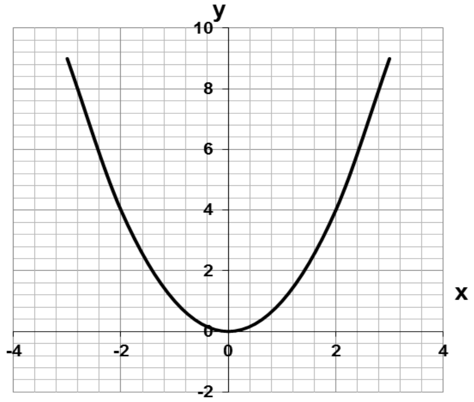
x	0	1	2	…..	-1	-2
y	2	3	4	….	1	0



Example7: sketch the power function y= x^2

Solution:

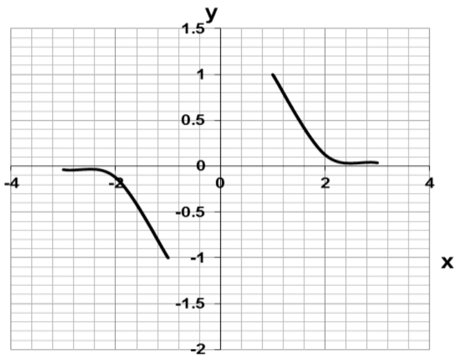
x	0	1	2	…..	-1	-2
y	0	1	4	….	1	4



Notes: if n is odd then symmetric about origin and pass through (1,1) and (-1,-1)

if n is even then symmetric about y axis and pass through (1,1) and (-1,1)
if the power is negative then $y=1/x^n$

like $y= 1/x^3$, $1/x$ …..



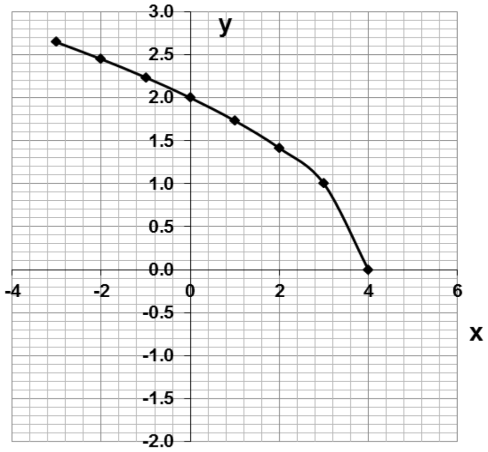
Graph of $y= 1/x^3$

Example8: Find the domain and range then sketch the function $y=\sqrt{4-x}$

Solution

Domain : $4-x \geq 0$ then $x \leq 4$

Range $y \geq 0$



2. Polynomials

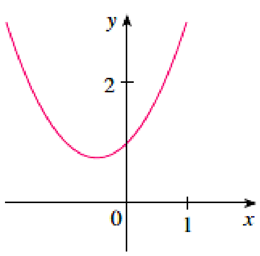
The general form is

$K_n x^n + K_{n-1} x^{n-1} + K_{n-2} x^{n-2} + \ldots\ldots + K_1 x + K_0$

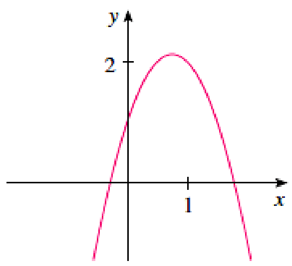
Ex: $x^3 + 5x^2 + 3$

$x^5 - x^3 + x^{0.5}$ ………….. etc.

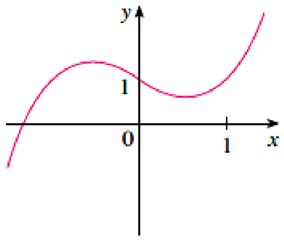
The graph is



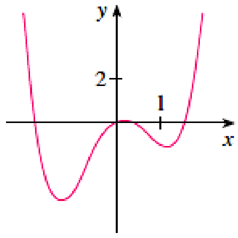
(a) $y = x^2 + x + 1$



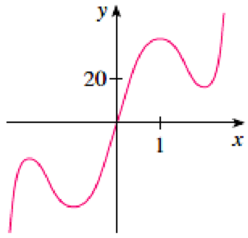
(b) $y = -2x^2 + 3x + 1$



(a) $y = x^3 - x + 1$



(b) $y = x^4 - 3x^2 + x$



(c) $y = 3x^5 - 25x^3 + 60x$

Example8: sketch the function $f_{(x)} = (x-2)(x+1)$

Solution

$f_{(x)} = y = x^2 + x - 2x - 2$

$y = x^2 - x - 2$ $\qquad$ $y = ax^2 + bx + c$

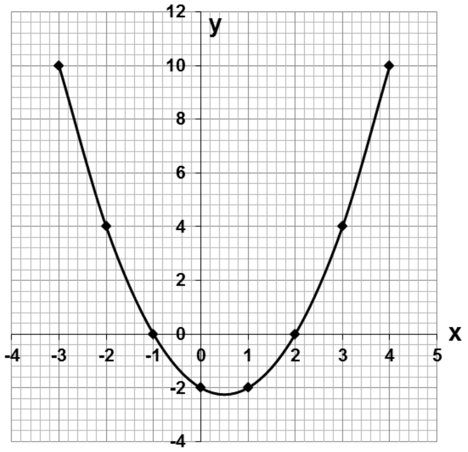
The vertex = x= -b/2a , $\quad$ x = -(-1)/2*1 = ½
y = $(1/2)^2$ -1/2 -2 = -9/4

vertex (1/2, -9/4)

points of intercept

at x= 0 y = -2
at y= 0 0 = (x-2)(x-1)

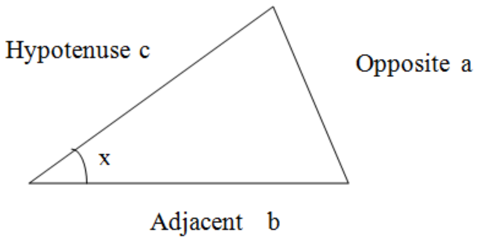
x= 2 (2,0)
x = -1 (-1,0)



2. Trigonometric functions

Sine $\quad$ sinx = a/c
Cosine $\quad$ cosx = b/c
Tangent $\quad$ tanx = a/b = sinx/cosx

Cotangent $\quad$ cotx = b/a = cosx/sinx
Secant $\quad$ secx = c/a = 1/cosx
Cosecant $\quad$ csecx = c/a = 1/sinx



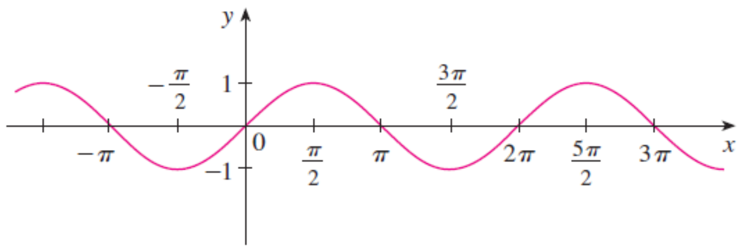
Identities

Trigonometric Identities – part 1 www.GIMathS.com

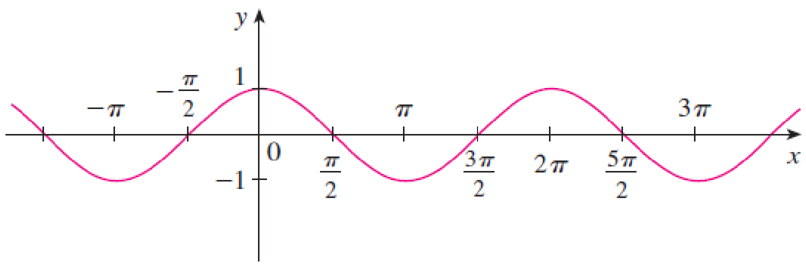
Reciprocal Identities		Half Angle Identities	Double Angle Identities	Pythagoras Identities
$\sin\theta = \dfrac{1}{\csc\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin^2\theta + \cos^2\theta = 1$
$\cos\theta = \dfrac{1}{\sec\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$	$1 + \tan^2\theta = \sec^2\theta$
				$1 + \cot^2\theta = \csc^2\theta$
$\tan\theta = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$	$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$	

Sum to Product Identities	Product to Sum Identities	Even/Odd Identities
		$\sin(-\theta) = -\sin\theta$
		$\cos(-\theta) = \cos\theta$
$\sin\alpha + \sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$	$\sin\alpha\sin\beta = \dfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$	$\tan(-\theta) = -\tan\theta$
$\sin\alpha - \sin\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$	$\cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$	$\csc(-\theta) = -\csc\theta$
$\cos\alpha + \cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$	$\sin\alpha\cos\beta = \dfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$	$\sec(-\theta) = \sec\theta$
$\cos\alpha - \cos\beta = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$	$\cos\alpha\sin\beta = \dfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$	$\cot(-\theta) = -\cot\theta$

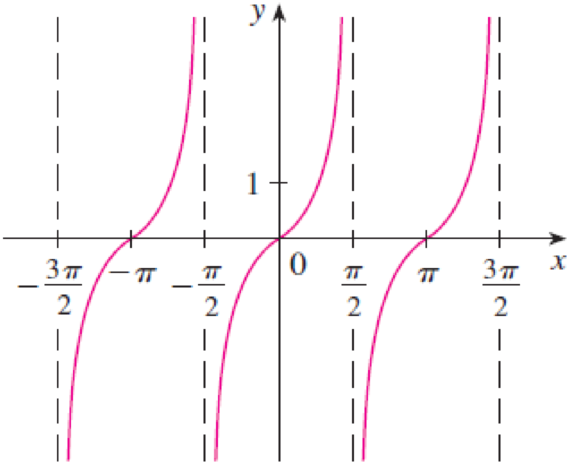
Graphs:



(a) $f(x) = \sin x$



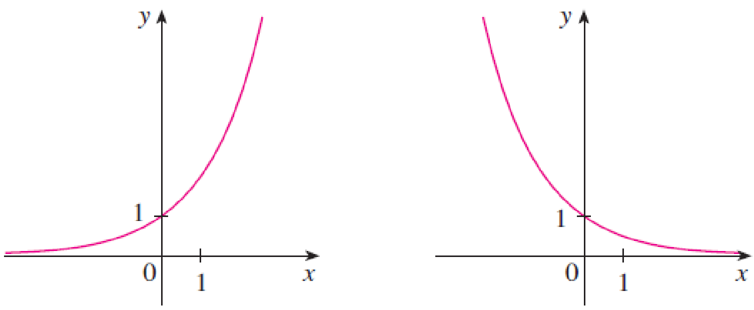
(b) $g(x) = \cos x$



$y = \tan x$

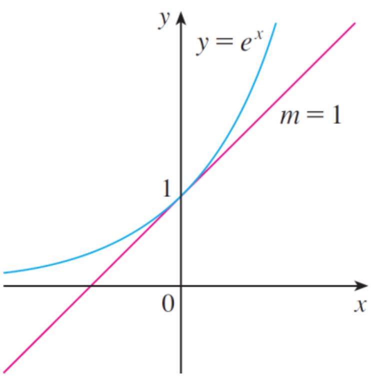
3. Exponential functions

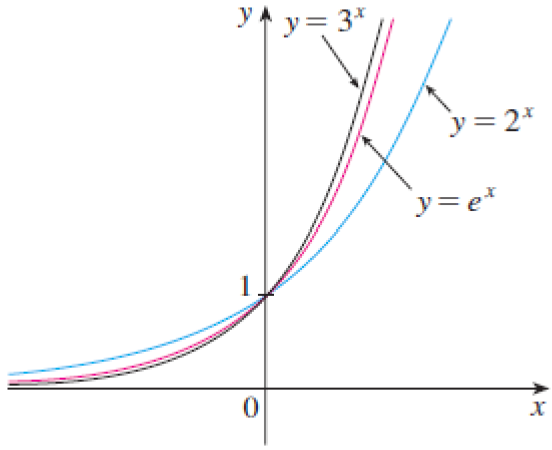
The exponential functions are the functions of the form f(x) = a^x , where the base a is a positive constant.



(a) $y = 2^x$ (b) $y = (0.5)^x$

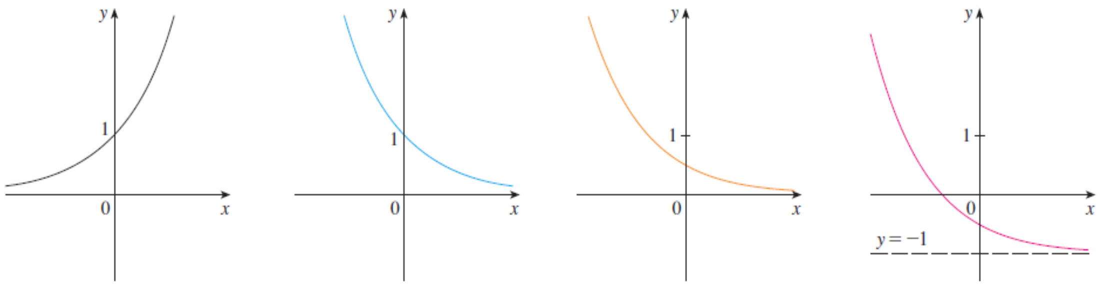
If we choose the base a so that the slope of the tangent line tangent line to the y = ax at (0,1) is exactly. In fact, there is such a number and it is denoted by the letter e. e =2.71828





Ilustrative example: Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

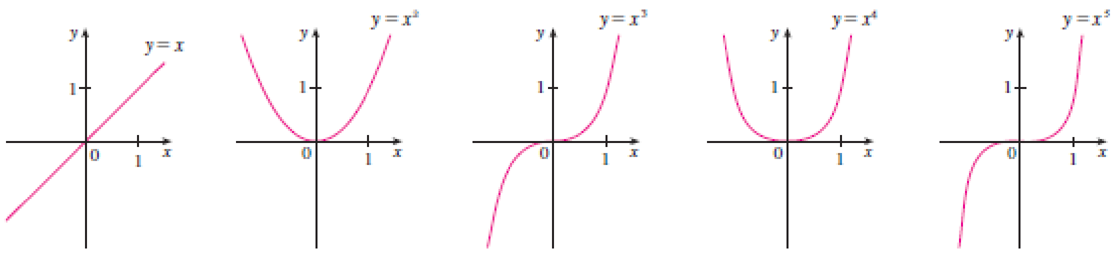
Solution We start with the graph of $y = e^x$ from Figure below, and reflect about the y-axis to get the graph of $y = e^{-x}$ in . Figure (b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure (c). . Finally, we shift the graph downward one unit to get the desired graph in Figure (d). The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.

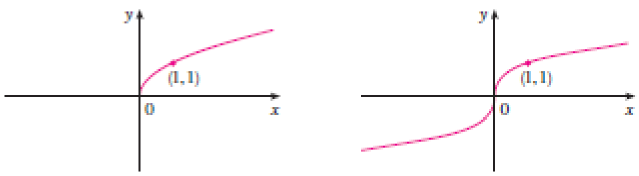


(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

4. Power functions

A function of the form $f(x) = x^a$, where is a constant, is called a power function. We consider several cases.

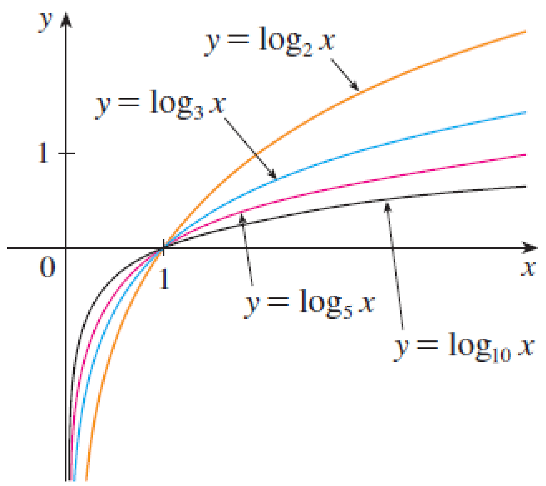




(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

5. Logarithmic functions

The logarithmic function f(x) = $\log_a$, where a is a positive constant, are the inverse function of the exponential functions. In each case the domain is(0, ∞) and the range is (-∞, ∞) and the function increases slowly when x>1.



Example9: Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Solution

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

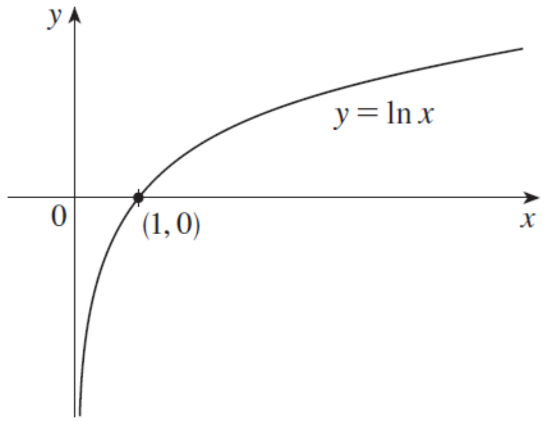
(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

NATURAL LOGARITHMS

The logarithm with base is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

$$\ln e = 1$$

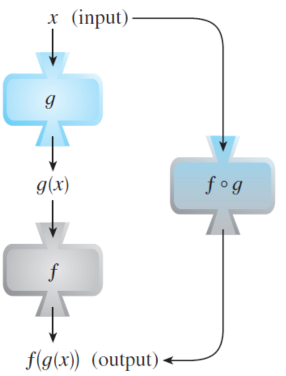


x	y= lnx
0	∞
1	0
2	0.693
3	1.098
4	1.386
5	1.609
-1	∞
0.9	-0.105
0.5	-0.693
0.2	-1.609
0.1	-2.302

6. Algebra of functions

Let f is a fynction of x then we get f(x) and g is a function of x also we get g(x)

Df is the domain of f(x)
Dg is the domain of g(x)



Then:

f+g = f(x) + g(x) and Df ∩ Dg
f – g = f(x) - g(x)
f.g = f(x) . g(x)

and the domain is as same before

if f/g then Df ∩ Dg but g(x) ≠ 0
if g/f then Dg ∩ Df but f(x) ≠ 0

and Dfog = {x: x ∈ Dg, g(x) ∈ Df}
where

fog(x) = f(g(x)) also called the composition of f and g

Example10: Find fog and gof if $f_{(x)} = \sqrt{1-x}$ and $g_{(x)} = \sqrt{5+x}$

Solution

(fog)x= f(g(x)) = f($\sqrt{5+x}$) = $\sqrt{1-\sqrt{5+x}}$
(1-x) ≥ 0 then x ≤ 1 Df: x≤1
5+x ≥ 0 then x ≥ -5 Dg: x≥-5
D fog = {x: x ≥ -5, $\sqrt{5+x}$ ≤1} = {x: -5 ≤ x ≤-4}

Example 11: Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9))$$
$$= [\cos(x + 9)]^2 = F(x)$$

Example 12: If $f_{(x)} = \sqrt{x}$ and $g_{(x)} = \sqrt{1-x}$

Find:

f+g, f-g, g-f, fog, f/g, g/f then graph fog and also f+g.

Solution

$f_{(x)} = \sqrt{x}$ domain $x \geq 0$

$g_{(x)} = \sqrt{1-x}$ domain $x \leq 1$

f+g = (f+g)x = $\sqrt{x} + \sqrt{1-x}$ domain $0 \leq x \leq 1$ or [0,1]

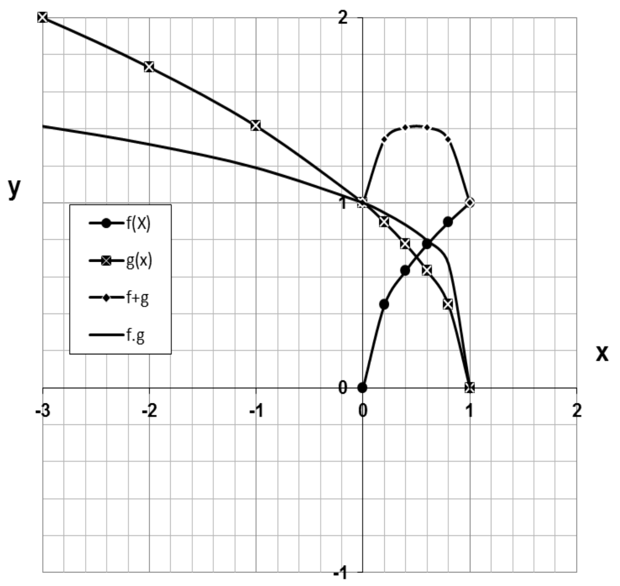
f-g = $\sqrt{x} - \sqrt{1-x}$ domain $0 \leq x \leq 1$

g-f = $\sqrt{1-x} - \sqrt{x}$ domain $0 \leq x \leq 1$

fog = f(x) g(x) = f(g(x)) =f($\sqrt{1-x}$) = $\sqrt{\sqrt{1-x}} = \sqrt[4]{1-x}$ domain (-∞, 1] (why?)

f/g = f(x)/g(x) = $\sqrt{\dfrac{x}{1-x}}$ domain (-∞, 1]

g/f = g(x)/f(x) = $\sqrt{\dfrac{1-x}{x}}$ domain (0, 1]


y
x
2
1
0
-1
-3
-2
-1
0
1
2
f(X)
g(x)
f+g
f.g

Inverse functions

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function y = ln x as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

> **DEFINITION** Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by
>
> $$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$
>
> The domain of f^{-1} is R and the range of f^{-1} is D.

Example 63:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

■

Note:

Only a one-to-one function can have an inverse

Q: What is the one to one function ?

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Note

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

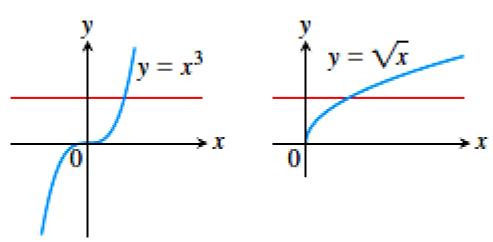
Example:

two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.

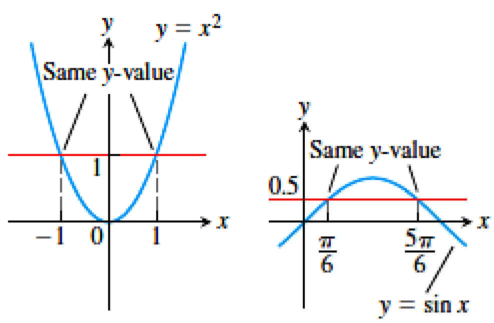
(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 64:

Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

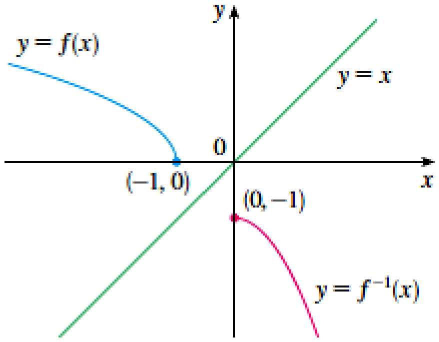
$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

Example 65:

Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

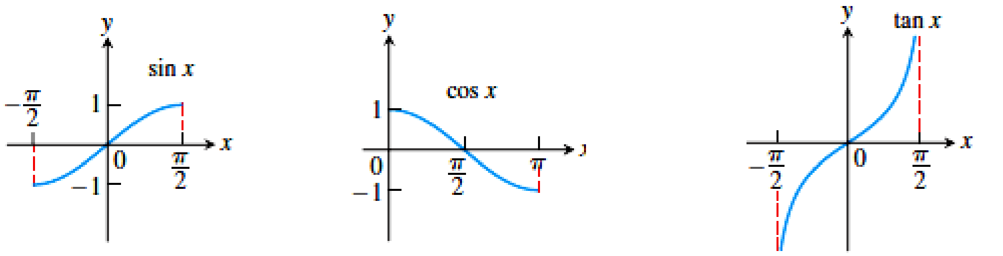
SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure



Inverse Trigonometric Functions

The six basic trigonometric functions of a general radian angle x were reviewed in Chapter 2. These functions are not one-to-one (their values repeat periodically).

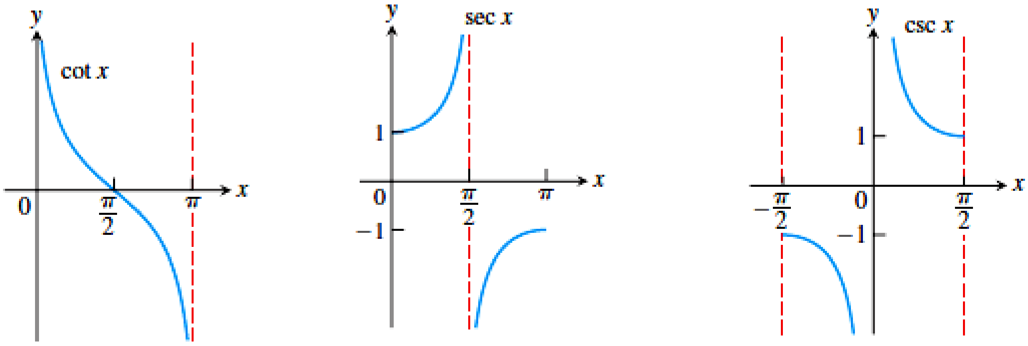
Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

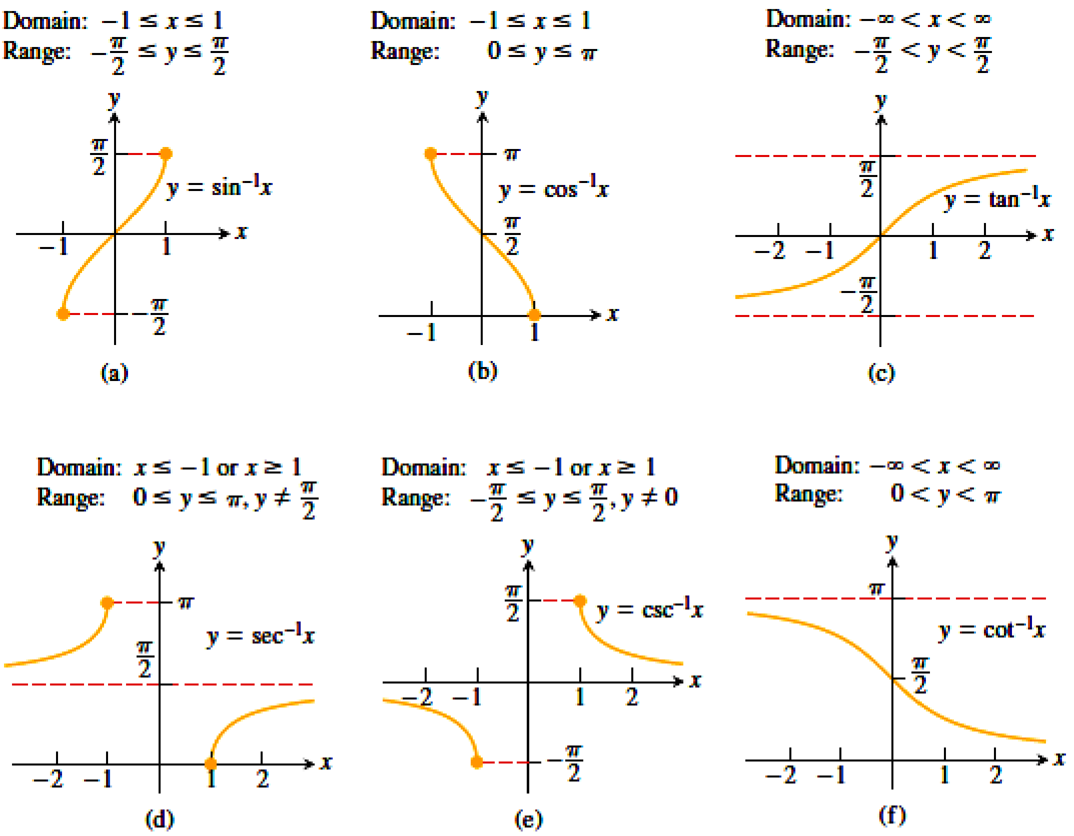
$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{array}{lcl}
y = \sin^{-1} x & \text{or} & y = \arcsin x \\
y = \cos^{-1} x & \text{or} & y = \arccos x \\
y = \tan^{-1} x & \text{or} & y = \arctan x \\
y = \cot^{-1} x & \text{or} & y = \operatorname{arccot} x \\
y = \sec^{-1} x & \text{or} & y = \operatorname{arcsec} x \\
y = \csc^{-1} x & \text{or} & y = \operatorname{arccsc} x
\end{array}
$$

Graph of inverse trig functions



Notes:

To convert from degree to radian

$$\pi \text{ radians} = 180°$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

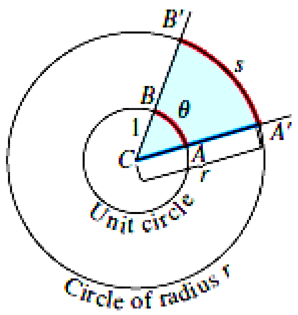


TABLE 1.1 Angles measured in degrees and radians

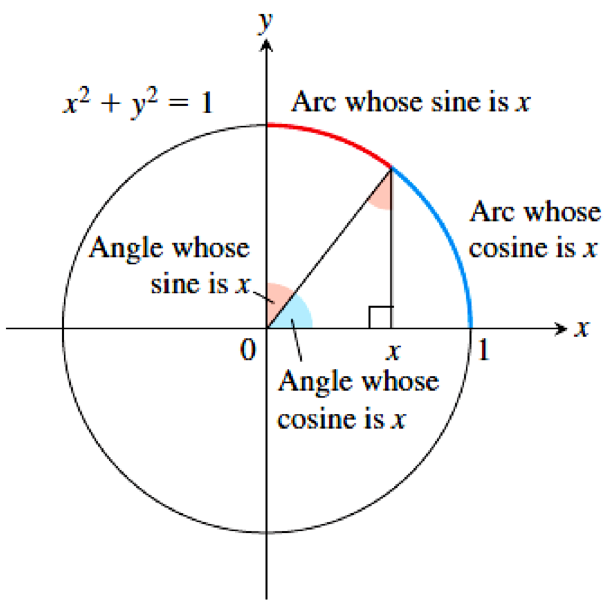
Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

TABLE 1.2 Values of sin θ, cos θ, and tan θ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

The "Arc" in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



Example 66:

Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.68a.

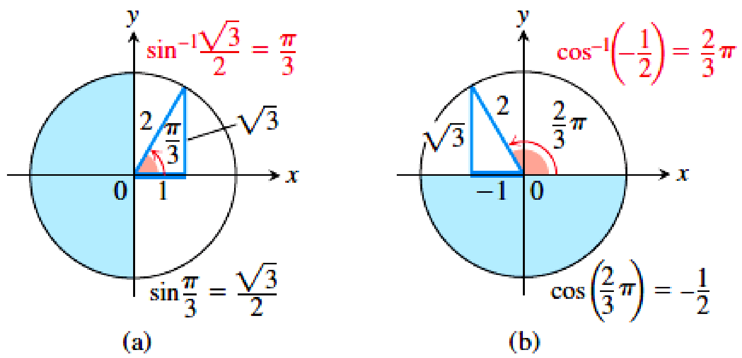
(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine

We can create the following table of common values for the arcsine and arccosine functions

x	$\sin^{-1}x$	$\cos^{-1}x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$



Example 67:

Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$

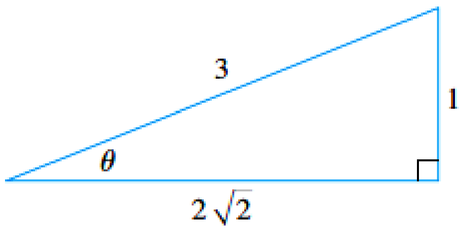
■

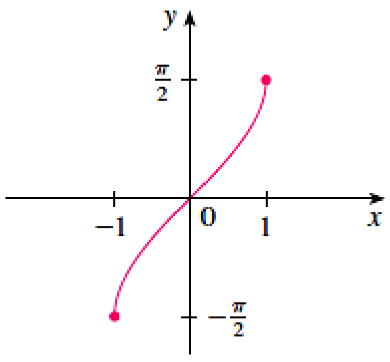
The cancellation equations for inverse functions become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function by reflection about the line $y = x$.





$y = \sin^{-1}x = \arcsin x$

Example 68:

Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

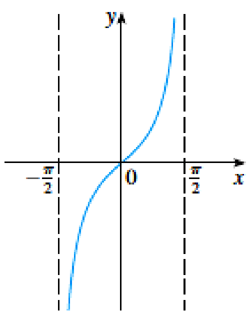
SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

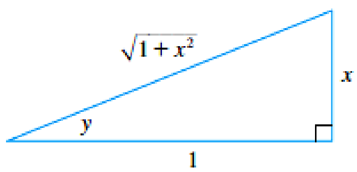
■

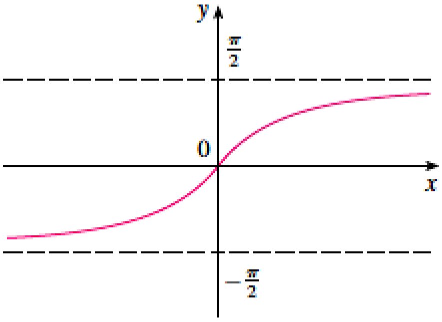
The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure .

$y = \cos^{-1}x = \arccos x$



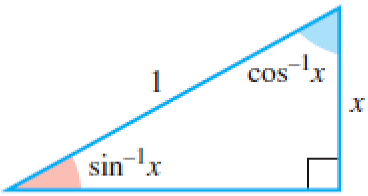
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



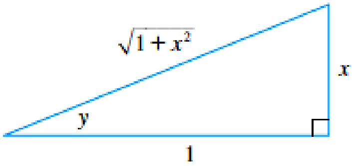


Identities of inverse trig functions

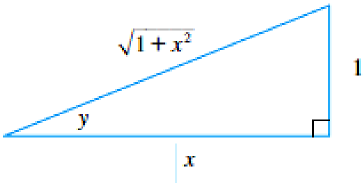
$$\sin^{-1}x + \cos^{-1}x = \pi/2.$$



sin^{-1}x



cos^{-1}x



and so on ..

2 Limits & Continuity

In this chapter, we'll define how limit of function values are defined and calculated.

Definition: the limit of f(x) as x tends to a is defined as the value of f(x) as x approaches closer and closer to a without actually reaching it and denoted by:

$\lim_{x \to a} f_{(x)} = L$ L is a single finite real number

It's important to know

1. We don't evaluate the limit by actually substituting x = a in f(x) in general, although in some cases its possible.

2. The value of the limit can depend on which side its approach

3. The limit may not exist at all.

Example 13: to explain the concept of limit, take the function f(x) = 2x – 4 if the

$\lim_{x \to 1} f_{(x)} = 2*1-4=-2$

But the following table express many values of x can be expressed close to 1.

x	0.5	0.8	0.9	0.99	0.999	1.001	1.01	1.1	1.2
f(x)	-3	-2.4	-2.2	-2.02	-2.002	-1.998	-1.98	-1.8	-1.6

Question: Why we take values approaches to 2 in example 13 instead we take x = 1 directly?

Solution: the answer about this question can be expressed in the following example:

$$f_{(x)} = 3^{-\frac{1}{x^2}} + 1$$

If x = 0 then 1/0 = ∞

So..

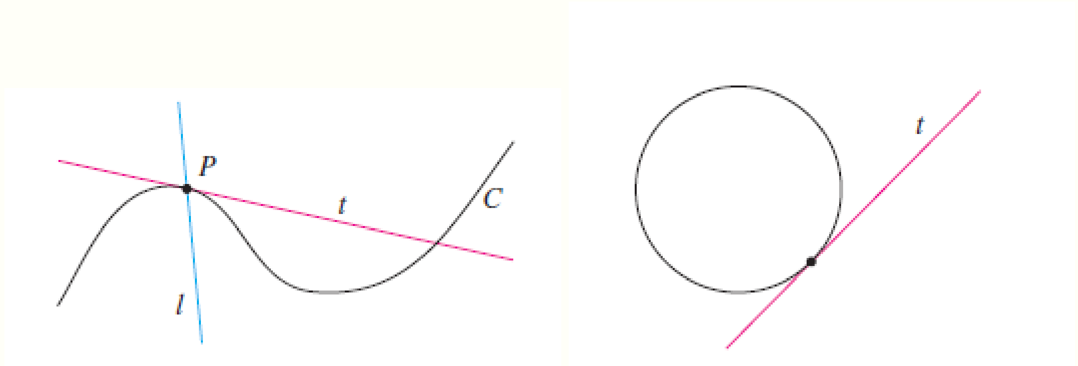
x	±0.2	±0.5	etc..
f(x)	1.00000	1.012345679	

In limits we avoid ∞

THE TANGENT PROBLEM

The word tangent is derived from the Latin word tangens, which means "touching." Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure (a). For more complicated curves this definition is inadequate as shown in Figure (b)



Example 14: Find an eq. of the tangent line to the parabola y = x^2 at point (1,1)?

Solution

We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

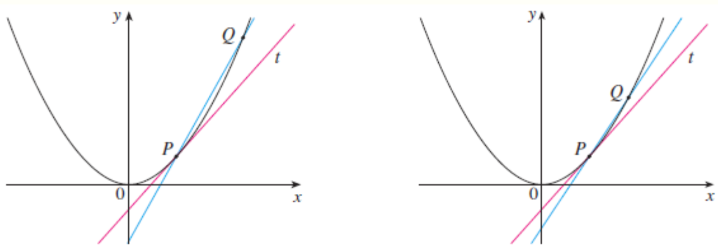
The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

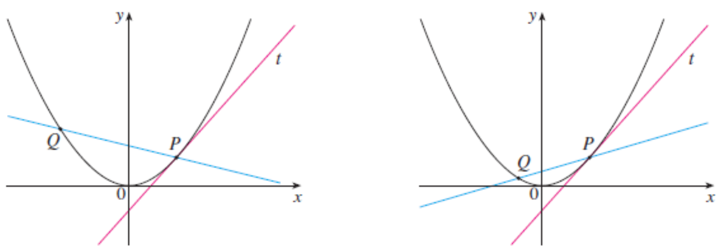
$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$



Q approaches *P* from the right



Q approaches *P* from the left

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

THE VELOCITY PROBLEM

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

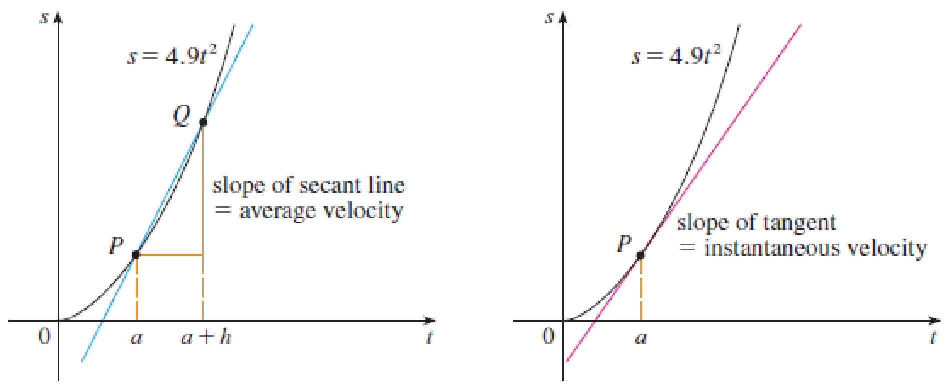
It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$ □

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).



Example 15: Discuss the function $f_{(x)} = \dfrac{x^2 - 9}{x - 3}$

If (1) x = 1, x = 2

(2) x = 3

(3) x→ 1, x→ 2

(4) x→ 3

Solution:

$f_{(x)} = \frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{(x-3)} = x + 3$ and x ≠ 3

Its equivalent to g(x) = x+3 and x ≠ 3, then:

f(1) = g(1) = 4

f(2) = g(2) = 5

if x → 1 then f(x) = 4 and $\lim_{x \to 1} f_{(x)} = 4$

if x = 3 then f(3) = 0/0 = ∞

if x → 3 then $\lim_{x \to 3} f_{(x)} = 6$

note: if f(x) is defined by two different forms before and after x = a then we must discuss the left limit and the right limit.

Properties of limits:

If $\lim_{x \to a} f_{(x)} = b$ $\lim_{x \to a} f_{(x)} = c$

Then:

1. $\lim_{x \to a} k\, f_{(x)} = kb$ for any constant k
2. $\lim_{x \to a} [\, f_{(x)} \pm g_{(x)}\,] = \lim_{x \to a} f_{(x)} \pm \lim_{x \to a} g_{(x)} = b + c$
3. $\lim_{x \to a} [\, f_{(x)} . g_{(x)}\,] = \lim_{x \to a} f_{(x)} . \lim_{x \to a} g_{(x)} = b.c$
4. $\lim_{x \to a} [\, f_{(x)} / g_{(x)}\,] = \lim_{x \to a} f_{(x)} / \lim_{x \to a} g_{(x)} = b / c \;\; if \;\; c \neq 0$
5. $\lim_{x \to a} [\, f_{(x)}\,]^{1/n} = b^{1/n} \quad real\ values\ only\ for\ n$

The limit must exist before applying the above results.

Example 16: find the limits of the following functions:

1. $\lim_{x\to 3} = \frac{x^2-9}{x-3}$ = $\lim_{x\to 3} = \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3}(x+3) = 3+3 = 6$

2. $\lim_{x\to 2} = \frac{\sqrt{x+2}-2}{x-2} = \lim_{x\to 2} = \frac{\sqrt{x+2}-2}{x-2} * \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x\to 2} = \frac{(x+2)-4}{x-2\sqrt{x+2}+2}$

$= \frac{1}{\sqrt{2+2}+2} = \frac{1}{2+2} = \frac{1}{4}$

3. $\lim_{x\to\infty} = \frac{3x^2+2x+1}{5x^2+7x+1}$ ÷ x^2

$\lim_{x\to\infty} = \frac{3+\frac{2}{x}+\frac{1}{x^2}}{5+\frac{7}{x}+\frac{4}{x^2}} = \frac{3}{5}$

Note: $\lim_{x\to\infty} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_o}{b_m x^m + a_{m-1}x^{m-1} + \ldots\ldots + b_o} = \begin{cases} 0 & n<m \\ \frac{a}{b} & n=m \\ \infty & n>m \end{cases}$

Example 17: find

$\lim_{x\to 1} = \frac{x^2-1}{(x-1)\sqrt{x^2+2x+3}}$

Solution:

$= \lim_{x\to 1} = \frac{(x-1)(x+1)}{(x-1)\sqrt{x^2+2x+3}} = \lim_{x\to 1}(x+1) - \lim_{x\to 1}\sqrt{x^2+2x+3} = 2 \div \sqrt{\lim_{x\to 1}(x^2+2x+3)}$

$= 2 \div \sqrt{6} = \frac{2}{\sqrt{6}}$

Theorem I   If g(x) ≤ f(x) ≤ h(x) and $\lim_{x\to a} g_{(x)} = \lim_{x\to a} h_{(x)} = L$

Theorem II $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ or $\lim_{x\to a} \frac{\sin(x-a)}{(x-a)} = 1$

Example 18:

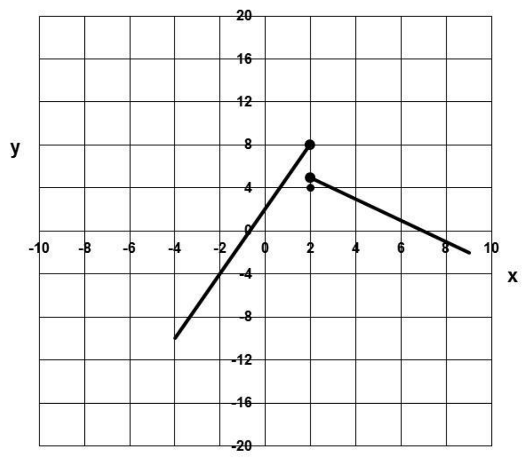
1. $\lim_{x\to 0} \frac{\sin 5x}{\sin 7x} = \frac{\frac{\sin 5x}{5x}.5x}{\frac{\sin 7x}{7x}.7x} = 5/7$

2. $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x+2)} = \frac{1}{(x+2)} = 1/4$

Left and right – side limits

Example 19: Discuss the $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 3x+2 & x<2 \\ 4 & x=2 \\ 8-x & x>2 \end{cases}$

Solution:



If x > 2

Then f (2^+) = $\lim_{x\to 2} f(x) = \lim_{x\to 2} (8-x) = 8-2 = 6$

If x < 2 then

f(2^-) $\lim_{x\to 2} f(x) = \lim_{x\to 2} (3x+2) = 8$

Then right limit ≠ left limit at x = 2

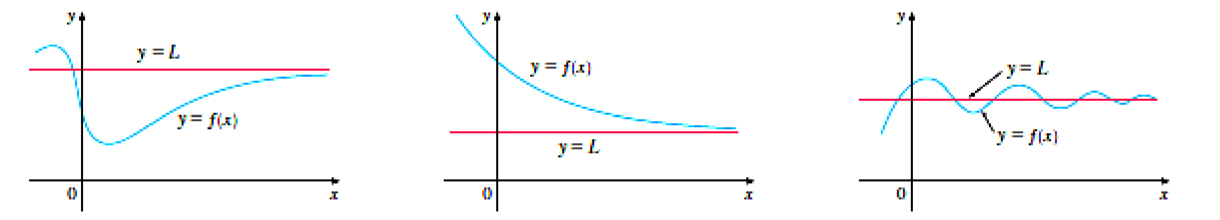
Then, we say that $\lim_{x\to 2} f(x)$ doesn't exists

Limits at Infinity: Horizontal Asymptote

1 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



The line L is called horizontal asymptote of the graph of the function (f). If the value of f(x) increases without bound as x $\to +\infty$ or x $\to -\infty$, then we write:

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{Or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

If the value of f(x) decreases without bound as x + ∞ or x - ∞, then we write:

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{Or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

2 DEFINITION Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

3 DEFINITION The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example 20: Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$

Solution:

Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.) □

5 THEOREM If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

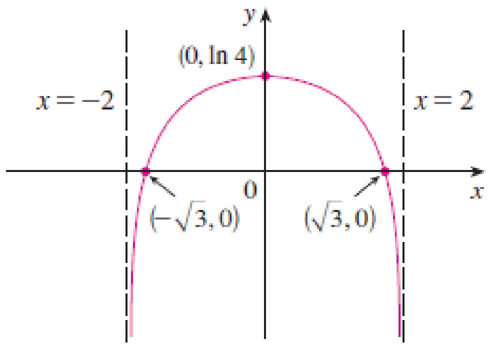
Infinite limits and Vertical Asymptotes

As the line x = a is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Example 21: $\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$



Continuity

If the limit of a function as approaches can often be found simply by calculating the value of the function at . Functions with this property are called continuous at a.

1 DEFINITION A function f is continuous at a number a if

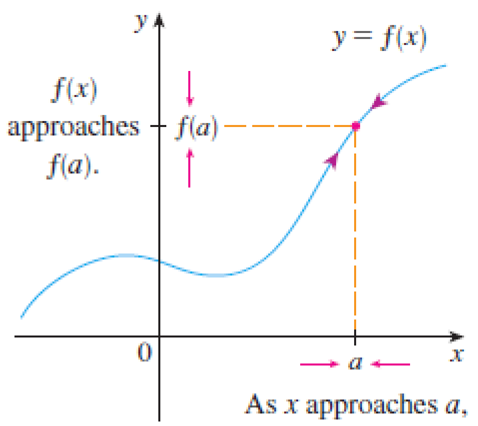
$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

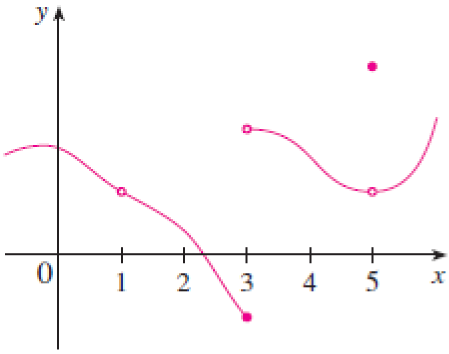
Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

■ As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.



Example 22: In figure below, at which numbers the function f is discontinuous? Why?

Solution:



It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x\to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. □

Example 23: Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2} (x + 1) = 3$$

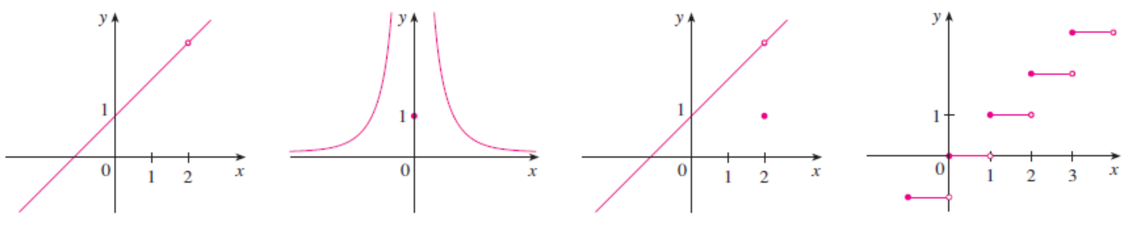
exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer.

Figure shows the graphs of the functions in Example 23. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x) = \frac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

5 THEOREM
(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 THEOREM The following types of functions are continuous at every number in their domains:

polynomials	rational functions	root functions
trigonometric functions	inverse trigonometric functions	
exponential functions	logarithmic functions	

Example 24: $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$

Solution:

The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x\to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

□

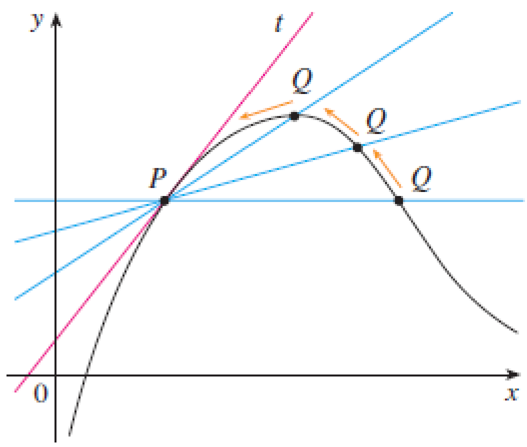
Tangent line, Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in previous section. This special type of limit is called a derivative and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

1 DEFINITION The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Example 25: Find an equation of the tangent line to the parabola $y = x^2$ at point P(1,1).

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1,1) is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Note:

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 26: Find an equation of the tangent line to the hyperbola y = 3/x at point (3,1).

Solution:

Let $f(x) = 3/x$. Then the slope of the tangent at $(3, 1)$ is

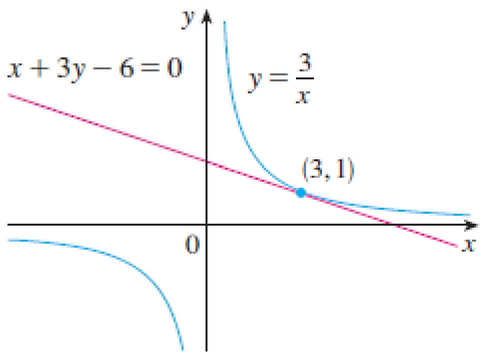
$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $$x + 3y - 6 = 0$$



4 DEFINITION **The derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 27: Find the derivative of the function f(x) = $x^2 - 8x + 9$ at the number a.

Solution: From Definition 4 we have

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

RATES OF CHANGE

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment of** x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

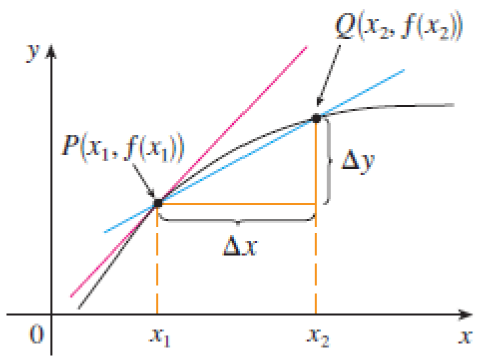
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over an interval

6 $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



average rate of change $= m_{PQ}$
instantaneous rate of change = slope of tangent at P

Example 28: A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is C= f(x) dollars,

(a) What is the meaning of the derivative f'(x), what are its units?

(b) In practical terms, what does it mean to say that f'(1000) = 9?

(c) Which do you think is greater f'(50) or f'(500), what about f'(5000)?

Solution:

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.*

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ □

3 Differentiation

Introduction

Derivative: it's a function we use to measure the rates at which things change, like slope and velocity and accelerations.

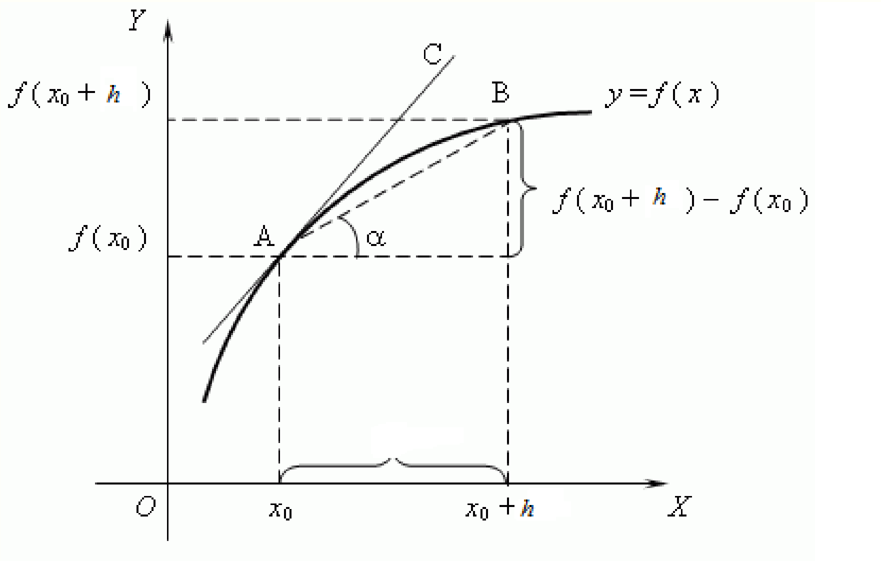
The derivative of a function is a function f′ where value at x is defined in the equation:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The function $\frac{f(x+h)-f(x)}{h}$ is the difference quotient for f at x.

h is the difference increment.

f′(x) is the first derivate of the function f at x. See figure below.



The most common notation for the differentiation of a function y = f(x) besides f′(x) or dy/dx and df/dx Dx(f) (Dx of f) .. etc..

Application of differentiation:

- The velocity and acceleration at time t and
- Problems of cost , maxima and minima
- Electrical circuits' problem
- Any other problems related to rate of change.

Example 29: Find the derivative of f(x) = $x^2 - 2x$ using the definition.

Solution

f(x) = $x^2 - 2x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 2(x+h) = x^2 + 2xh + h^2 - 2x - 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 - 2x - 2h - (x^2 - 2x)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} = \frac{h^2 + 2hx - 2h}{h} = h + 2x - 2$$

We can take the limit as h $\longrightarrow$ 0: -

$$f'(x) = \lim_{h \to 0} (h + 2x - 2) = 2x - 2$$

Example 30: Show that the derivative of $y = \sqrt{x}$ is $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Solution:

$f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$$ ÷ 0 (Not Ok)

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{(x+h)-x}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{h\sqrt{x+h}+\sqrt{x}}$$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{1}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$ Ok

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

FIRST PROOF The formula

$$x^n-a^n=(x-a)(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x)=x^n$, we can use Equation 2.7.5 for $f'(a)$ and the equation above to write

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$=\lim_{x\to a}(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1})$$

$$=a^{n-1}+a^{n-2}a+\cdots+aa^{n-2}+a^{n-1}$$

$$=na^{n-1}$$

k

Example 31: Differentiate (a) $f(x)=\frac{1}{x^2}$ (b) $y=\sqrt[3]{x^2}$

Solution:

In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$

Slope and tangent lines:

Example 32: Find an eq. for the tangent to the curve y = 2/x at x = 3

Solution:

$$m = f'(x) = \lim_{h \to -0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{2}{x+h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \frac{\frac{2x-2x-2h}{(x+h)x}}{h} = \frac{-2h}{(x+h)x} = \frac{-2}{x^2}$$

$m = f'(x) = -2/x^2$ at x = 3 m = $-2/(3)^2$

Then y = -2/9

y + 2/3 = -2/9 (x-3)

Rules for differentiation

If f and g are differentiable functions, the following differentiation rules are valid

1. $\frac{d}{dx}\{f(x)+g(x)\}=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)=f'(x)+g'(x)$ **(Addition Rule)**

2. $\frac{d}{dx}\{f(x)-g(x)\}=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)=f'(x)-g'(x)$

3. $\frac{d}{dx}\{Cf(x)\}=C\frac{d}{dx}f(x)=Cf'(x)$ **where C is any constant**

4. $\frac{d}{dx}\{f(x)g(x)\}=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)=f(x)g'(x)+g(x)f'(x)$ **(Product Rule)**

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$ **if $g(x)\neq 0$ (Quotient Rule)**

6. $\frac{d}{dx}(C)=0$

7. $\frac{d}{dx}(x^n)=nx^{n-1}$

8. $\frac{d}{dx}(\ln x)=\frac{dx}{x}$ or $\frac{1}{x}dx$

9. $\frac{d}{dx}(e^x)=e^x\ dx$

10. $\frac{d}{dx}(a^x)=a^x\ \ln a\ dx$

11. $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a .e=\frac{1}{x}.\frac{1}{\ln a}dx$

Example 33: If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

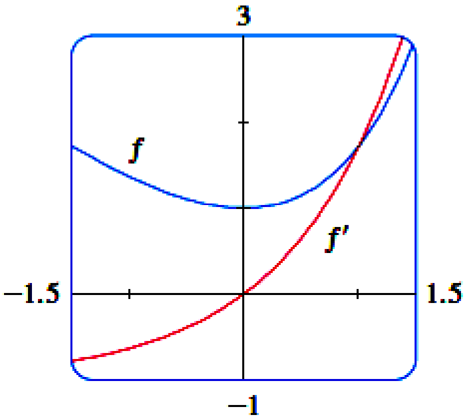
Solution: Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure . Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■



Example 34: Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Example 35:

Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $(1, \frac{1}{2}e)$.

According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

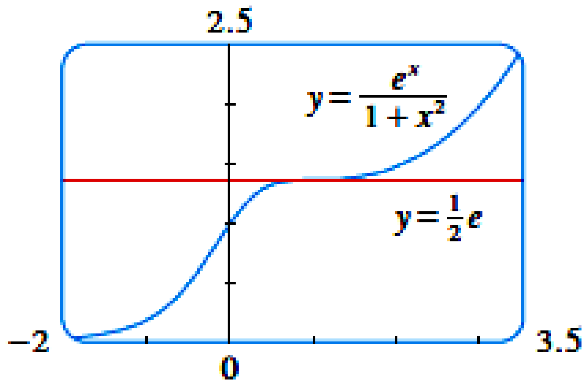
$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-2x+x^2)}{(1+x^2)^2}$$

$$= \frac{e^x(1-x)^2}{(1+x^2)^2}$$

So the slope of the tangent line at $(1, \frac{1}{2}e)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

5 This means that the tangent line at $(1, \frac{1}{2}e)$ is horizontal and its equation is $y = \frac{1}{2}e$.



Derivatives of trigonometric functions

- X is measured in radians

- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 36: Find d/dx {sinx} ?

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

1 $$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h \to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h \to 0} \cos x = \cos x$$

But :

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1} \qquad \boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0}$$

Then

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

4 $$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 37: Differentiate $y = x^2 \sin x$.

Solution: Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$
$$= x^2 \cos x + 2x \sin x$$

Example 38:

Differentiate $f(x) = \frac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

Solution:

$$f'(x) = \frac{(1 + \tan x)\frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

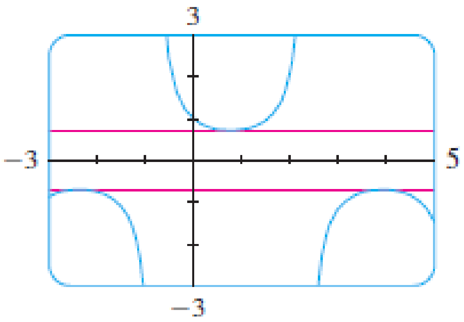
$$= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure). ■



Example 39:

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$
$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$
$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$
$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Example 40:

The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14 \text{ cm/s}^2$. ■

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

Higher order derivatives

Example 43: Find y′, y″ , and y‴ for the following functions:

1. $y = 2x^3 + x - 5$
2. $y = \dfrac{2x}{1-2x}$

Solution

$y = 2x^3 + x - 5$

$y' = 6x^2 + 1$

$y'' = 12x$

$y''' = 12$

$y = \dfrac{2x}{1-2x}$

$$y' = \frac{(1-2x)*2 - 2x*(-2)}{(1-2x)^2} = \frac{2-4x+4x}{(1-2x)^2} = \frac{2}{(1-2x)^2}$$

$$y'' = \frac{-2*2(1-2x)*(-2)}{(1-2x)^4} = \frac{8}{(1-2x)^3}$$

$$y'' = \frac{-8 * 3(1-2x)^2 * (-2)}{(1-2x)^6} = \frac{48}{(1-2x)^4}$$

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

1 $$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$ (Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous.)

$$= \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

2 $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 44: Find F(x)′ if $F'_{(x)} = \sqrt{x^2 + 1}$

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have
$$F'(x) = f'(g(x)) \cdot g'(x)$$
$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$ ■

NOTE In using the Chain Rule we work from the outside to the inside. Formula 2 says that *we differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Example 45 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\text{outer function}}\ \underbrace{(x^2)}_{\text{evaluated at inner function}} = \underbrace{\cos}_{\text{derivative of outer function}}\ \underbrace{(x^2)}_{\text{evaluated at inner function}} \cdot \underbrace{2x}_{\text{derivative of inner function}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)^2}_{\text{inner function}} = \underbrace{2}_{\text{derivative of outer function}} \cdot \underbrace{(\sin x)}_{\text{evaluated at inner function}} \cdot \underbrace{\cos x}_{\text{derivative of inner function}}$$

Example 46: Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sqrt[3]{1 + 4x}$
2. $y = (2x^3 + 5)^4$
3. $y = \tan \pi x$
4. $y = \sin(\cot x)$
5. $y = e^{\sqrt{x}}$
6. $y = \sqrt{2 - e^x}$

Solution

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \frac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\pi) = \pi \sec^2 \pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\frac{e^x}{2\sqrt{2-e^x}}$.

Implicit Differentiation

To find dy/dx for any equation involving x and y differentiation each of term in the equation with respect to x instead of finding y in terms of x.

Example 47 Find y'' if $x^4 + y^4 = 16$.

Solution Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} \tag{3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

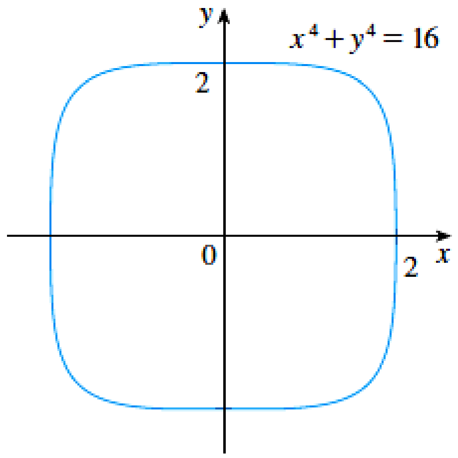
If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$



Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem Solving Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 72:

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

SOLUTION We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown:*

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation:*

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

Given: $\dfrac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\dfrac{dr}{dt}$ when $r = 25$ cm

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

Example 73:

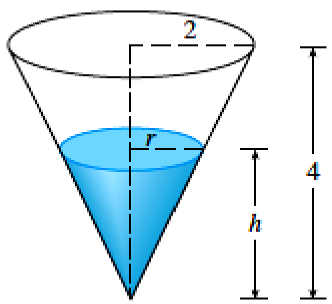
A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use



the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2\ \text{m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min.

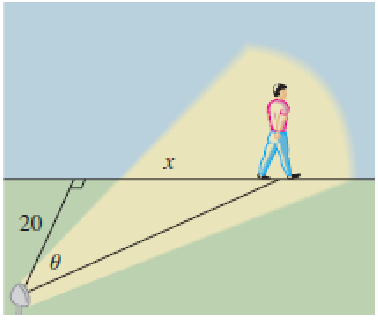
Example 74:

A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$



Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

so
$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt}$$

$$= \frac{1}{20}\cos^2\theta\,(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

Indeterminate Forms and L'Hospital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or +∞. The rule is known today as l'Hôpital's Rule, after Guillaume de l'Hôpital.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 75:

The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $$\lim_{x\to0}\frac{3x-\sin x}{x}=\lim_{x\to0}\frac{3-\cos x}{1}=\left.\frac{3-\cos x}{1}\right|_{x=0}=2$$

(b) $$\lim_{x\to0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$$

(c) $$\lim_{x\to0}\frac{\sqrt{1+x}-1-x/2}{x^2}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$=\lim_{x\to0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(d) $$\lim_{x\to0}\frac{x-\sin x}{x^3}$$ $\frac{0}{0}$; apply l'Hôpital's Rule.

$$=\lim_{x\to0}\frac{1-\cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to0}\frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to0}\frac{\cos x}{6}=\frac{1}{6}$$ Not $\frac{0}{0}$; limit is found.

Example 76: Calculate $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

SOLUTION Since $\ln x\to\infty$ and $\sqrt{x}\to\infty$ as $x\to\infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

Example 77:

Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$ (b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x} \quad \frac{\infty}{\infty} \text{ from the left so we apply l'Hôpital's Rule.}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

Example 78:

Evaluate $\lim_{x\to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$ ■

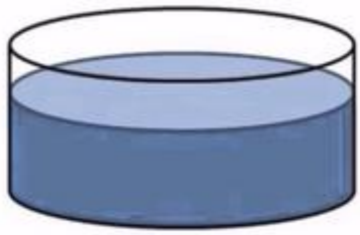
4 Applications of Differentiation

Introduction

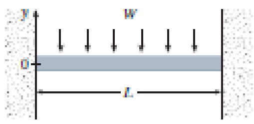
We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).

Change of velocity with time

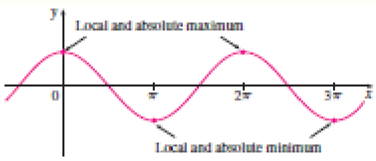
flow of tank



Displacement

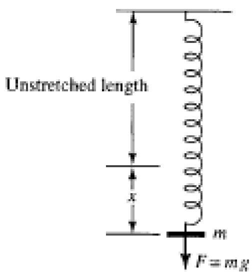


Maximum and Minimum Values



Simple circuit with light

Engineering mechanics



Summary

Mechanics

$v = \frac{dx}{dt}$, where v = velocity, x = distance, t = time.

$a = \frac{dv}{dt}$, where a = acceleration, v = velocity, t = time.

$F = \frac{dW}{dx}$, where F = force, W = work done (or energy used), x = distance moved in the direction of the force.

$F = \frac{dp}{dt}$, where F = force, p = momentum, t = time.

$P = \frac{dW}{dt}$, where P = power, W = work done (or energy used), t = time.

$\frac{dE}{dv} = p$, where E = kinetic energy, v = velocity, p = momentum.

Gases

$\frac{dW}{dV} = p$, where p = pressure, W = work done under isothermal expansion, V = volume.

Circuits

$I = \frac{dQ}{dt}$, where I = current, Q = charge, t = time.

$V = \left(L\frac{dI}{dt}\right)$, where V is the voltage drop across an inductor, L = inductance, I = current, t = time.

Electrostatics

$E = -\frac{dV}{dx}$, where V = potential, E = electric field, x = distance.

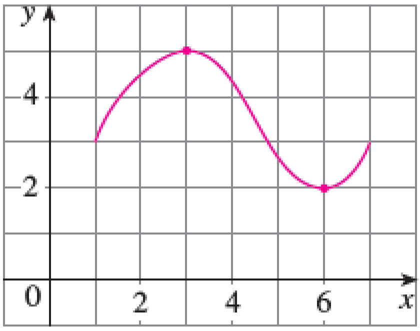
Maximum and Minimum Values

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something.

These problems can be reduced to finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function f shown in Figure is the point (3,5). In other words, the largest value of f is f(3)= 5. Likewise, the smallest value is f(6)= 2. We say that f (3)= 5 is the ***absolute maximum*** of f and f (6)= 2 is the ***absolute minimum***.

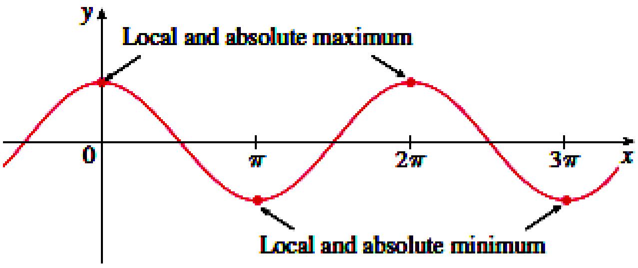


In general, we use the following definition

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Example 48 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leqslant \cos x \leqslant 1$ for all x. (See Figure) Likewise, $\cos(2n+1)\pi = -1$ is its minimum value, where n is any integer.



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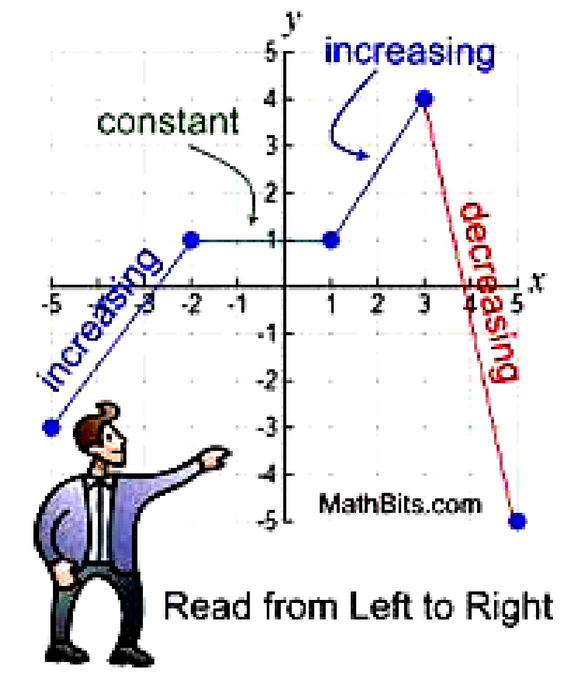
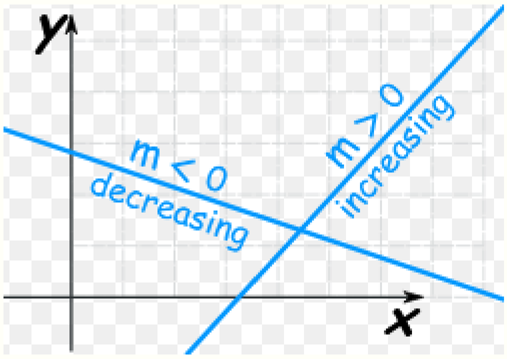
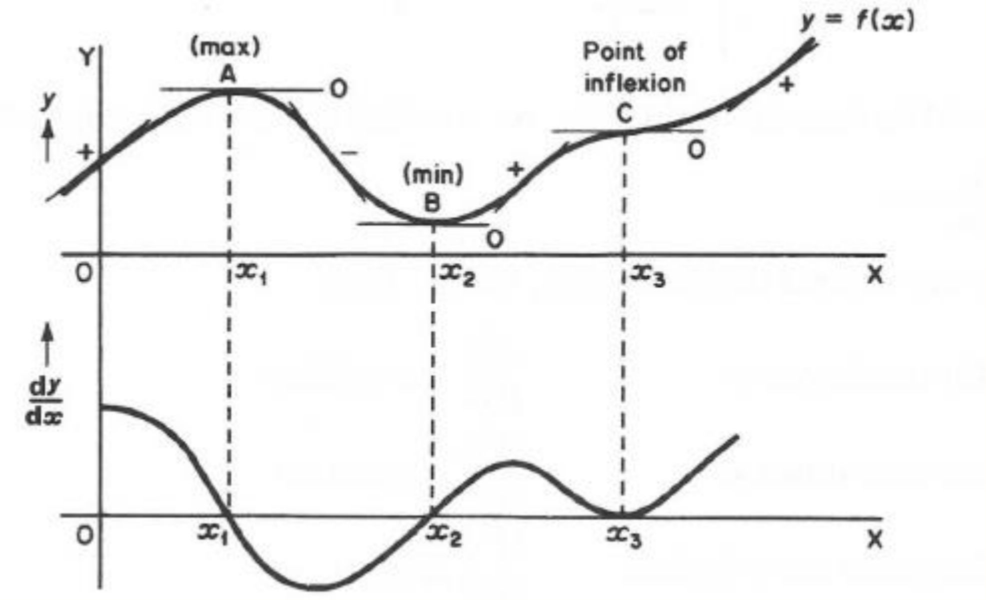




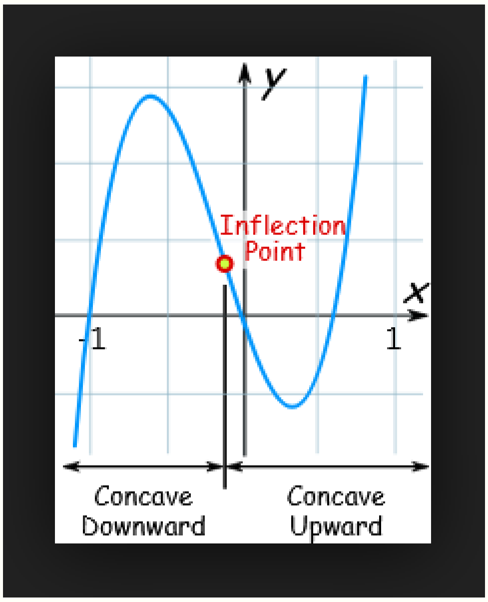
If f(x2) > f(x1) then the function is called **increasing** on its interval

If f(x2) < f(x1) then the function is called **decreasing** on its interval

If f(x2) = f(x1) then the function is called **constant** on its interval



Concavity



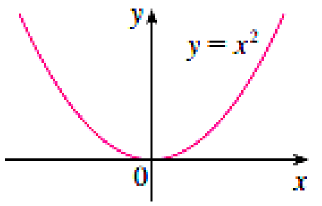
Remember:

The graph of y= f(x) is

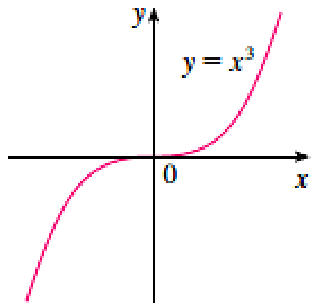
Concve up when y" > 0

Concave down when y" < 0

Example 49:



Mimimum value 0, no maximum

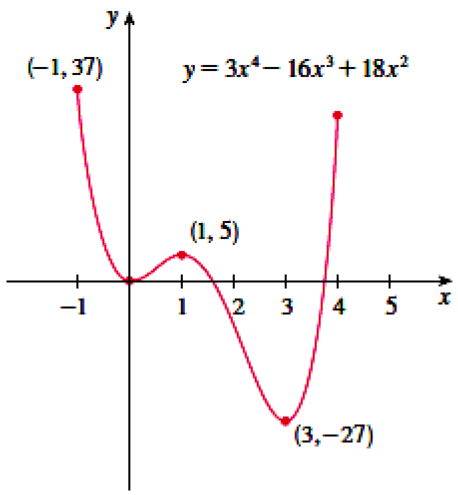


No mimimum, no maximum

Example 50: The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

is shown in Figure . You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$.



We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

Extrema of a function (maxima and minima)

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Second Derivative Test Suppose f'' is continuous near c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 51: Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. (Note that f' is a polynomial and hence defined everywhere.) To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

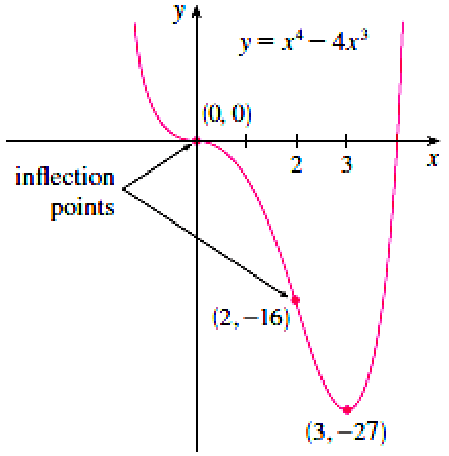
Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.] Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

The point $(0, 0)$ is an inflection point since the curve changes from concave up to concave downward there. Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points,



6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 52: Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

n

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. ■

Procedures for finding and distinguishing between stationary points:

1. Given y = f(x), determine dy/dx (i.e. f′(x)).
2. Let dy/dx = 0 and solve for the values of x.
3. Substitute the values of x into the original function y= f(x) to find the corresponding y ordinate values. This would establish the nature of stationary points.
4. Find d^2y/dx^2 and sub into the values found in 2 above. If the result is:

i. Positive then min. point

ii. Negative then max. point

iii. Zero then its point of inflexion (inflection)

5. Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign changes for the gradient of the curve is:

a) Positive to negative then point is max.

b) Negative to positive then point is min

c) Positive to positive or negative to negative then it's a point of inflection.

Example 53: Find the local minimum and maximum values of the function *f*

$$f_{(x)} = x^3 - 3x^2 + 4$$

Solution

$f'_{(x)} = 3x^2 - 6x$, $f''_{(x)} = 6x - 6$

$f'_{(x)} = 0$, $0 = 3x^2 - 6x$

X = 0 or 3x – 6 = 0 then x = 2

Finding values of f″(x) at x = 0, 2

$f''_{(0)} = -6$ Relative maximum point

$f''_{(2)} = 6$ Relative minimum point

Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f (x) by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function

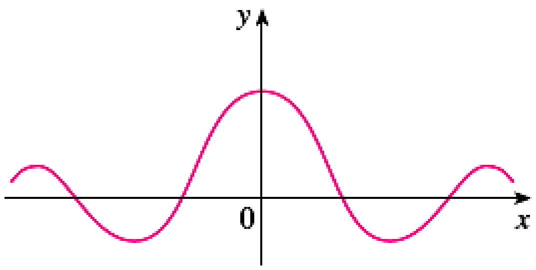
A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

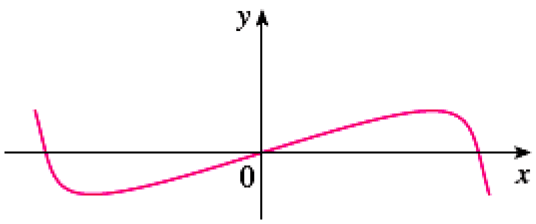
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure (b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

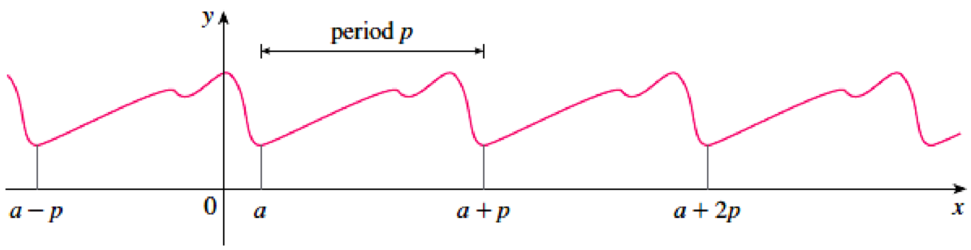


(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure).



D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from chapter 2 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from chapter 2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

1

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

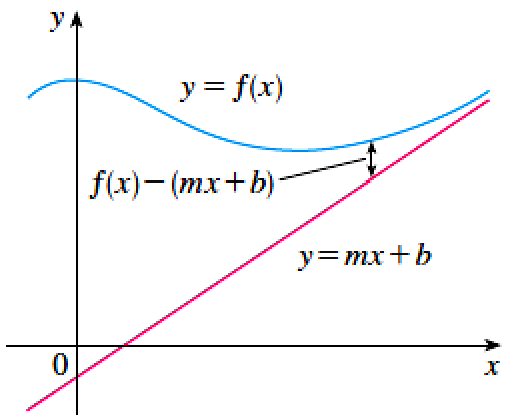
(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.*

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure . (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.



E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

Example 54:

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E.
$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

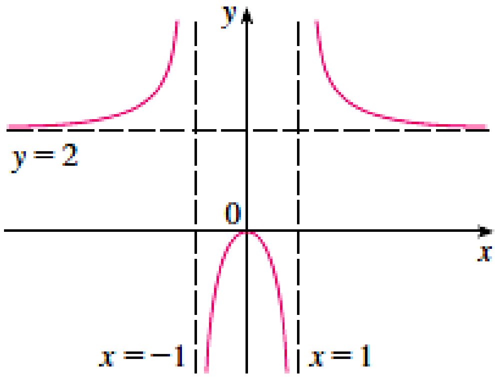
F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(x^2 - 1)^2(-4) + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.



Example 55:

Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

A. Domain $= \{x \mid x+1>0\} = \{x \mid x>-1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

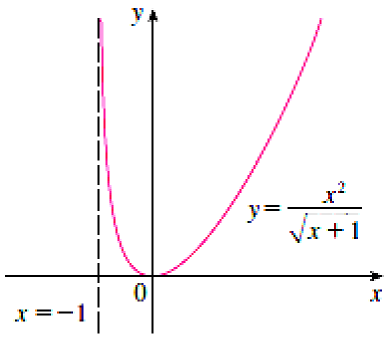
E. $$f'(x) = \frac{\sqrt{x+1}\,(2x) - x^2 \cdot 1/\left(2\sqrt{x+1}\right)}{x+1} = \frac{3x^2+4x}{2(x+1)^{3/2}} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. $$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.



Example 56:

Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (\pi/2) + n\pi$, where n is an integer.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.

D. Asymptotes: None

E. $$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

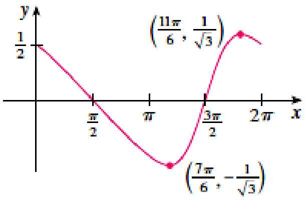
The denominator is always positive, so $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.

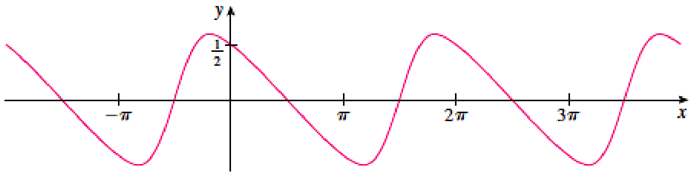
F. From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.

G. If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.





Example 57:

Sketch the graph of $y = \ln(4 - x^2)$.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

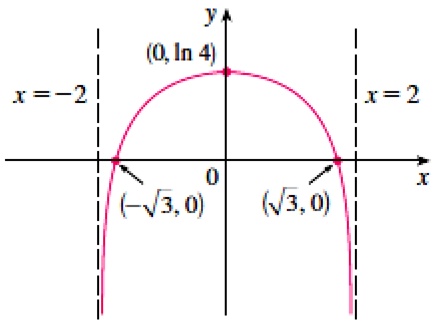
E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.



Optimization Problems

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized

Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 58:
A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can?.

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

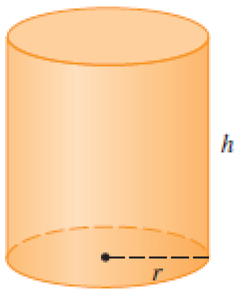


FIGURE 3

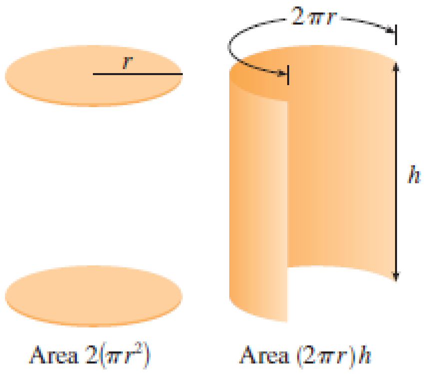


FIGURE 4

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

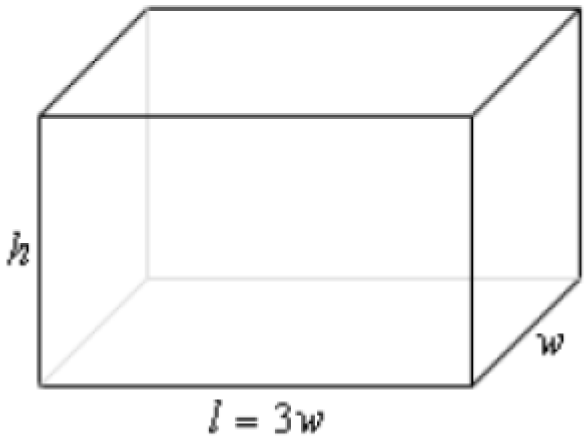
Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

Example 59

We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10/ft² and the material used to build the sides cost $6/ft². If the box must have a volume of 50ft³ determine the dimensions that will minimize the cost to build the box.

Solution:

First, we sketch a figure as below:



We want to minimize the cost of the materials subject to the constraint that the volume must be $50 ft^3$. Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\text{Minimize : } C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh$$
$$\text{Constraint : } 50 = lwh = 3w^2h$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for h so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later…) derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad C''(w) = 120 + 1600w^{-3}$$

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

First, we know that whatever the value of w that we get it will have to be positive and we can see second derivative above that provided $w > 0$ we will have $C''(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

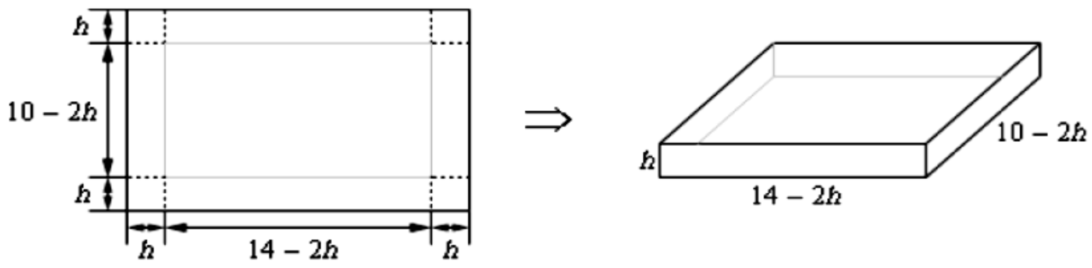
All we need to do now is to find the remaining dimensions.

$$w = 1.8821$$
$$l = 3w = 3(1.8821) = 5.6463$$
$$h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050$$

Also, even though it was not asked for, the minimum cost is : $C(1.8821) = \$637.60$.

Example 60:

We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



In this case we want to maximize the volume. Here is the volume, in terms of h and its first derivative.

$$V(h) = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3 \qquad V'(h) = 140 - 96h + 12h^2$$

Setting the first derivative equal to zero and solving gives the following two critical points,

$$h = \frac{12 \pm \sqrt{39}}{3} = 1.9183, \quad 6.0817$$

So, knowing that whatever h is it must be in the range $0 \le h \le 5$ we can see that the second critical point is outside this range and so the only critical point that we need to worry about is 1.9183.

Finally, since the volume is defined and continuous on $0 \le h \le 5$ all we need to do is plug in the critical points and endpoints into the volume to determine which gives the largest volume. Here are those function evaluations.

$$V(0) = 0 \qquad V(1.9183) = 120.1644 \qquad V(5) = 0$$

So, if we take $h = 1.9183$ we get a maximum volume.

Example 61: a rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $\left(x, \sqrt{4-x^2}\right)$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.40). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

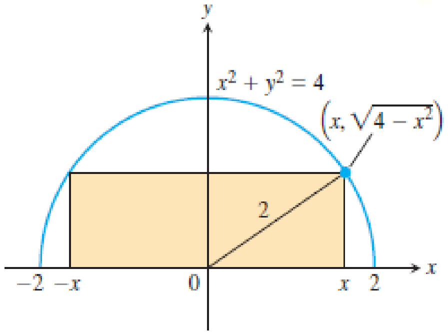
$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4-x^2}, \qquad \text{Area: } 2x\sqrt{4-x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4-x^2}$$

on the domain $[0, 2]$.



The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

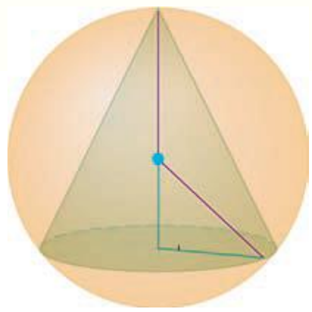
is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0$$
$$-2x^2 + 2(4-x^2) = 0$$
$$8 - 4x^2 = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\text{Critical point value: } \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$$
$$\text{Endpoint values: } \quad A(0) = 0, \qquad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4-x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ■



Example 62 **<u>(Homework)</u>** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3?

V = 32π / 3

5 Integrals

The Indefinite Integral

An integral can be considered to be an **antiderivative**. Thus, if we know that the derivative of $F(x)$ is $f(x)$ $[=F'(x)]$, an integral of $f(x)$ is $F(x)$. For example, the derivative of $\frac{1}{3}x^3$ is x^2, and an integral of x^2 is $\frac{1}{3}x^3$. Note that we have used the article an. Since the derivative of a constant is zero, $F(x)$ is arbitrary to the extent of an arbitrary constant. The integral we have defined is known as an **indefinite integral** which is usually denoted by the symbol $\int$. Thus, we write

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

Example 79: Evaluate the following indifinite integral $\int x^4 + 3x - 9\ dx$

Solution

The indefinite integral is

$$\int x^4 + 3x - 9\ dx = \frac{1}{x}x^5 + \frac{3}{2}x^2 - 9x + c$$

PROPERTIES OF INDEFINITE INTEGRALS

1. *A constant factor can be taken outside the integral sign*:

$$\int af(x)\,dx = a\int f(x)\,dx \qquad (a = \text{const}).$$

2. *Integral of the sum or difference of functions (additivity):*

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Computing Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Example 80: Evaluate each of the following integrals

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\int x^8 + x^{-8}\,dx$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution:

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

$$\int 5t^3 - 10t^{-6} + 4\,dt = 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c$$

$$= \frac{5}{4}t^4 + 2t^{-5} + 4t + c$$

(b) $\int x^8 + x^{-8}\,dx$

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

$$\int 3\sqrt[4]{x^3}+\frac{7}{x^5}+\frac{1}{6\sqrt{x}}\,dx=\int 3x^{\frac{3}{4}}+7x^{-5}+\frac{1}{6}x^{-\frac{1}{2}}\,dx$$

$$=3\frac{1}{7/4}x^{\frac{7}{4}}-\frac{7}{4}x^{-4}+\frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}}+c$$

$$=\frac{12}{7}x^{\frac{7}{4}}-\frac{7}{4}x^{-4}+\frac{1}{3}x^{\frac{1}{2}}+c$$

(d) $\int dy$

$$\int dy=\int 1\,dy=y+c$$

(e) $\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw$

$$\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw=\int 4w-w^3+4w^{\frac{1}{3}}-w^{\frac{7}{3}}dw$$

$$=2w^2-\frac{1}{4}w^4+3w^{\frac{4}{3}}-\frac{3}{10}w^{\frac{10}{3}}+c$$

(f) $\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx$

$$\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx=\int \frac{4x^{10}}{x^3}-\frac{2x^4}{x^3}+\frac{15x^2}{x^3}dx$$

$$=\int 4x^7-2x+\frac{15}{x}dx$$

$$=\frac{1}{2}x^8-x^2+15\ln|x|+c$$

Furtehr examples:

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

You should check this answer by differentiating it. ■

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

SOLUTION This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta\,\cot\theta\,d\theta = -\csc\theta + C$$

■

Substitution Rule for Indefinite Integrals

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1
$$\int 2x\sqrt{1+x^2}\,dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.*

4 **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 1 Find $\int x^3\cos(x^4+2)\,dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using

$x^3\,dx = \frac{1}{4}\,du$ and the Substitution Rule, we have

$$\int x^3\cos(x^4+2)\,dx = \int \cos u\cdot\tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$

$$= \tfrac{1}{4}\sin u + C$$

$$= \tfrac{1}{4}\sin(x^4+2) + C$$

Notice that at the final stage we had to return to the original variable x.

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$

$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x+1}$. Then

$$du = \frac{dx}{\sqrt{2x+1}} \qquad \text{so} \qquad dx = \sqrt{2x+1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\int \sqrt{2x+1}\,dx = \int u\cdot u\,du = \int u^2\,du$$

$$= \frac{u^3}{3} + C = \tfrac{1}{3}(2x+1)^{3/2} + C$$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$

$$= -\tfrac{1}{8}(2\sqrt{u}) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C$$

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^u\,du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3 \cos(x^4+2)\,dx = \int \cos(x^4+2)\cdot x^3\,dx = \tfrac{1}{4}\int \cos(x^4+2)\cdot(4x^3)\,dx$$

$$= \tfrac{1}{4}\int \cos(x^4+2)\cdot\frac{d}{dx}(x^4+2)\,dx = \tfrac{1}{4}\sin(x^4+2)+C$$

Similarly, the solution to Example 4 could be written like this:

$$\int e^{5x}\,dx = \tfrac{1}{5}\int 5e^{5x}\,dx = \tfrac{1}{5}\int \frac{d}{dx}(e^{5x})\,dx = \tfrac{1}{5}e^{5x}+C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\,x^5\,dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4\cdot x$. Let $u=1+x^2$. Then $du=2x\,dx$, so $x\,dx=\frac{1}{2}du$. Also $x^2=u-1$, so $x^4=(u-1)^2$:

$$\int \sqrt{1+x^2}\,x^5\,dx = \int \sqrt{1+x^2}\,x^4\cdot x\,dx$$

$$= \int \sqrt{u}\,(u-1)^2\cdot\tfrac{1}{2}du = \tfrac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du$$

$$= \tfrac{1}{2}\int (u^{5/2}-2u^{3/2}+u^{1/2})\,du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2}-2\cdot\tfrac{2}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}\right)+C$$

$$= \tfrac{1}{7}(1+x^2)^{7/2}-\tfrac{2}{5}(1+x^2)^{5/2}+\tfrac{1}{3}(1+x^2)^{3/2}+C \qquad \blacksquare$$

EXAMPLE 6 Calculate $\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 6 can also be written as

5

$$\int \tan x\, dx = \ln|\sec x| + C$$

6 Inverse Trig Functions & Hyperbolic Functions

Derivatves of inverse trig functions

Recall the definition of the arcsine function:

$$y = \sin^{-1}x \qquad \text{means} \qquad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$, since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}}$$

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1}x$, then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\boxed{\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}}$$

Summary

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Example 60:

Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

Further examples:

Example 1

Find $\dfrac{dy}{dx}$, given that $y = (1-x^2)\sin^{-1}x$

Here we have a product

$$\therefore \frac{dy}{dx} = (1-x^2)\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x.(-2x)$$
$$= \sqrt{1-x^2} - 2x.\sin^{-1}x$$

Example 2

If $y = \tan^{-1}(2x - 1)$, find $\frac{dy}{dx}$

This time, it is a function of a function

$$\frac{dy}{dx} = \frac{1}{1 + (2x - 1)^2} . 2 = \frac{2}{1 + 4x^2 - 4x + 1}$$
$$= \frac{2}{2 + 4x^2 - 4x} = \frac{1}{2x^2 - 2x + 1}$$

and so on.

Additional Exercise

For each of the following problems differentiate the given function.

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$
2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$
3. $y = 5x^6 - \sec^{-1}(x)$
4. $f(w) = \sin(w) + w^2 \tan^{-1}(w)$
5. $h(x) = \dfrac{\sin^{-1}(x)}{1+x}$

Hyperbolic functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

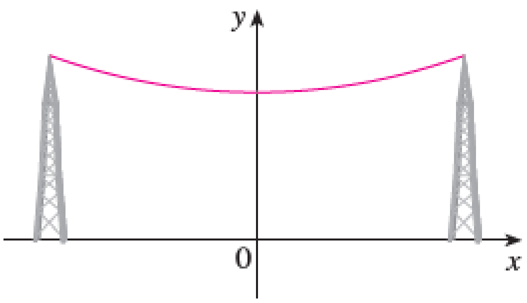
Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

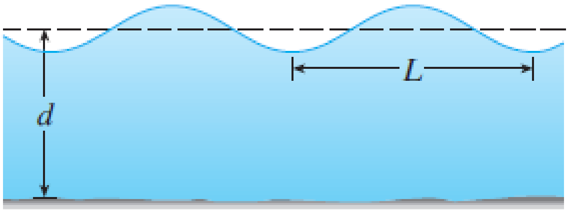
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire.

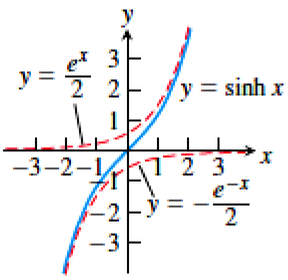


A catenary $y = c + a\cosh(x/a)$



Idealized ocean wave

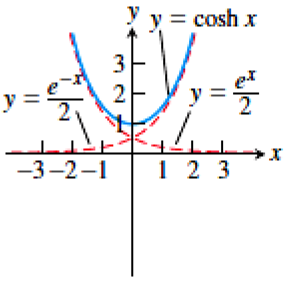
The six basic hyperbolic functions



(a)

Hyperbolic sine:

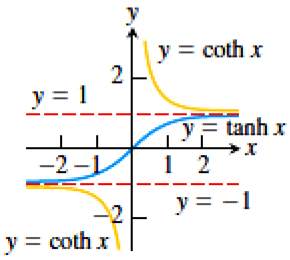
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



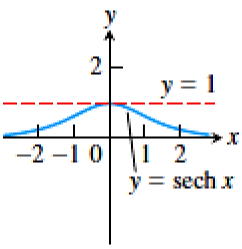
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

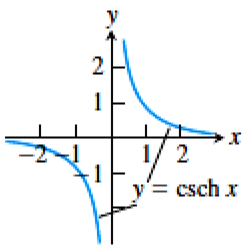
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Evaluation of hyperbolic functions

To evaluate sinh 1·275

Now $\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \therefore \quad \sinh 1{\cdot}275 = \frac{1}{2}(e^{1\cdot275} - e^{-1\cdot275})$.

We now have to evaluate $e^{1\cdot275}$ and $e^{-1\cdot275}$.
Using your calculator, you will find that:

$$e^{1\cdot275} = 3{\cdot}579 \text{ and } e^{-1\cdot275} = \frac{1}{3{\cdot}579} = 0{\cdot}2794$$

$$\therefore \quad \sinh 1{\cdot}275 = \frac{1}{2}(3{\cdot}579 - 0{\cdot}279)$$

$$= \frac{1}{2}(3{\cdot}300) = 1{\cdot}65$$

$$\therefore \quad \sinh 1{\cdot}275 = 1{\cdot}65$$

Example 70:

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 71: Differentiate each of the following functions

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$