CHAPTER-3
Equilibrium of a Particle

CHAPTER OBJECTIVES:

- ***To introduce the concept of the free-body diagram for a particle.***
- ***To show how to solve particle equilibrium problems using the equations of equilibrium.***

3.1 Condition for the equilibrium of a particle.

A particle is said to be in ***equilibrium*** if it ***remains at rest*** *if* ***originally at rest, or has a constant velocity*** *if* ***originally in motion***. To maintain equilibrium, it is necessary to satisfy Newton's first law of motion which requires the resultant force acting on a particle to be equal to zero. This condition may be stated mathematically as:

$$\Sigma \mathbf{F} = 0 \qquad \ldots\ldots\ldots\ldots\ (3.1)$$

Where ΣF is the vector sum of all the forces acting on the particle.

3.2 The free body diagram

A **drawing** that **shows** the particle with **all the forces** that act on it is called a **free body diagram** (**FBD**).

We will consider ***a springs connections*** often encountered in particle equilibrium problems.

Springs: If a linearly elastic spring of undeformed length l0 is used to support a particle, **the length of the spring will change in direct proportion to the force F acting on it**, Fig 3.1. A **characteristic** that defines the **elasticity of a spring** is the **spring constant** or **stiffness** *k*. The magnitude of force exerted on a linearly elastic spring is stated as:

$$F = k\,s$$

Where:

$s = l - l_0$, measured from its *unloaded* position.

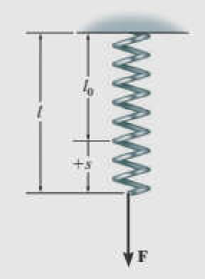
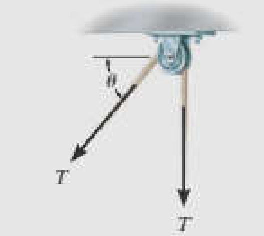


Fig. 3–1

Cables and Pulleys: All cables (or cords), unless otherwise mentioned, will be assumed to have negligible weight and they cannot stretch. Also, a cable can support ***only*** a tension or "pulling" force, and this force always acts in the direction of the cable. It will be shown that the tension force developed in a continuous cable which passes over a frictionless pulley must have ***a constant magnitude*** to keep the cable in equilibrium. Hence, for any angle u, shown in Fig. 3–2 , the cable is subjected to a constant tension ***T*** throughout its length.



Cable is in tension

Fig. 3–2

Procedure for Drawing a Free-Body Diagram

Since we must account for *all the forces acting on the particle* when applying the equations of equilibrium, the importance of first drawing a free-body diagram cannot be overemphasized. To construct a free-body diagram, the following three steps are necessary.

Draw Outlined Shape.

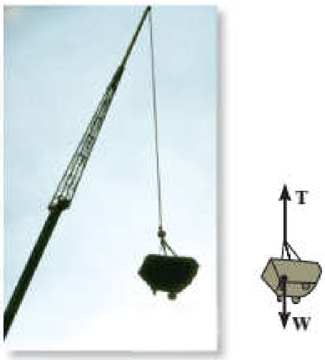
Imagine the particle to be *isolated* or cut "free" from its surroundings by drawing its outlined shape.

Show All Forces.

Indicate on this sketch *all* the forces that act *on the particle*. These forces can be *active forces*, which tend to set the particle in motion, or they can be *reactive forces* which are the result of the constraints or supports that tend to prevent motion. To account for all these forces, it may be helpful to trace around the particle's boundary, carefully noting each force acting on it.

Identify Each Force.

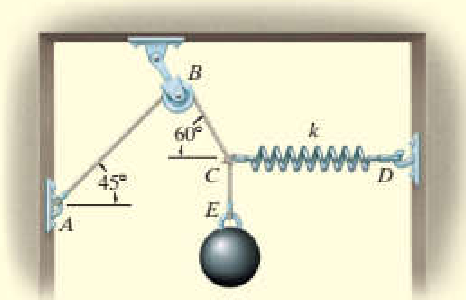
The forces that are *known* should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are unknown.



The bucket is held in equilibrium by the cable, and instinctively we know that the force in the cable must equal the weight of the bucket. By drawing a free-body diagram of the bucket we can understand why this is so. This diagram shows that there are only two forces *acting on the bucket*, namely, its weight **W** and the force **T** of the cable. For equilibrium, the resultant of these forces must be equal to zero, and so $T = W$.

EXAMPLE 3.1

The sphere in Fig. 3–3*a* has a mass of 6 kg and is supported as shown. Draw a free-body diagram of the sphere, the cord *CE*, and the knot at *C*.



(a)

(b)

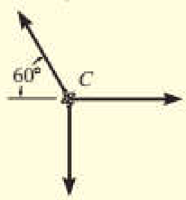
(c)

SOLUTION

Sphere. By inspection, there are only two forces acting on the sphere, namely, its weight, 6 kg $(9.81 \text{ m/s}^2) = 58.9$ N, and the force of cord *CE*. The free-body diagram is shown in Fig. 3–3*b*.

Cord *CE*. When the cord *CE* is isolated from its surroundings, its free-body diagram shows only two forces acting on it, namely, the force of the sphere and the force of the knot, Fig. 3–3*c*. Notice that $\mathbf{F}_{CE}$ shown here is equal but opposite to that shown in Fig. 3–3*b*, a consequence of Newton's third law of action–reaction. Also, $\mathbf{F}_{CE}$ and $\mathbf{F}_{EC}$ pull on the cord and keep it in tension so that it doesn't collapse. For equilibrium, $F_{CE} = F_{EC}$.

Knot. The knot at *C* is subjected to three forces, Fig. 3–3*d*. They are caused by the cords *CBA* and *CE* and the spring *CD*. As required, the free-body diagram shows all these forces labeled with their magnitudes and directions. It is important to recognize that the weight of the sphere does not directly act on the knot. Instead, the cord *CE* subjects the knot to this force.



(d)

Fig. 3–3

3.3 Coplanar Force Systems

If a particle is subjected to a system of coplanar forces that lie in the *x*–*y* plane, as in Fig. 3–4 , then each force can be resolved into its **i** and **j** components. For equilibrium, these forces must sum to produce a zero force resultant, i.e.,

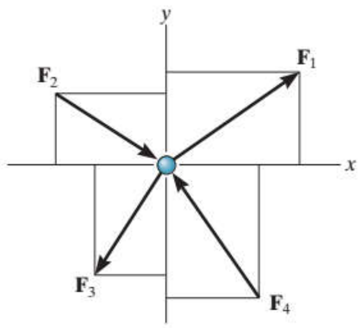


Fig. 3–4

$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma F_x\,\mathbf{i} + \Sigma F_y\,\mathbf{j} = \mathbf{0}$$

For this vector equation to be satisfied, the resultant force's *x* and *y* components must both be equal to zero. Hence,

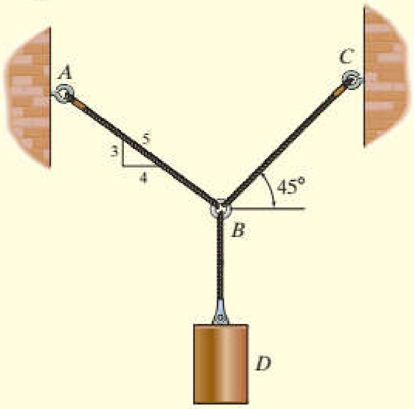
$$\Sigma Fx = 0 \quad \text{and} \quad \Sigma Fy = 0 \qquad \text{.......(3–3)}$$

These two equations can be solved for at most two unknowns, generally represented as angles and magnitudes of forces shown on the particle's free-body diagram.

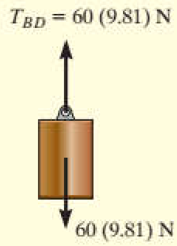
Note: When applying each of the two equations of equilibrium, we must account for the sense of direction of any component by using an ***algebraic sign*** which corresponds to the arrowhead direction of the component along the *x* or *y* axis. It is important to note that if a force has an ***unknown magnitude***, then the arrowhead sense of the force on the free-body diagram can be ***assumed.*** Then if the ***solution*** yields **a *negative scalar***, this indicates that the sense of the force is opposite to that which was assumed.

EXAMPLE 3.2

Determine the tension in cables BA and BC necessary to support the 60-kg cylinder in Fig. 3–6a.



(a)



(b)

SOLUTION

Free-Body Diagram. Due to equilibrium, the weight of the cylinder causes the tension in cable BD to be $T_{BD} = 60(9.81)$ N, Fig. 3–6b. The forces in cables BA and BC can be determined by investigating the equilibrium of ring B. Its free-body diagram is shown in Fig. 3–6c. The magnitudes of $\mathbf{T}_A$ and $\mathbf{T}_C$ are unknown, but their directions are known.

Equations of Equilibrium. Applying the equations of equilibrium along the x and y axes, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad T_C \cos 45° - \left(\tfrac{4}{5}\right)T_A = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0; \qquad T_C \sin 45° + \left(\tfrac{3}{5}\right)T_A - 60(9.81)\text{ N} = 0 \qquad (2)$$

Equation (1) can be written as $T_A = 0.8839T_C$. Substituting this into Eq. (2) yields

$$T_C \sin 45° + \left(\tfrac{3}{5}\right)(0.8839T_C) - 60(9.81)\text{ N} = 0$$

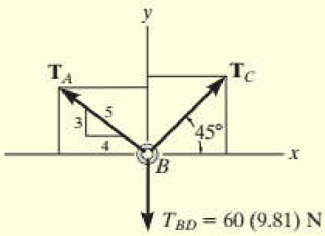
so that

$$T_C = 475.66\text{ N} = 476\text{ N} \qquad \textit{Ans.}$$

Substituting this result into either Eq. (1) or Eq. (2), we get

$$T_A = 420\text{ N} \qquad \textit{Ans.}$$

NOTE: The accuracy of these results, of course, depends on the accuracy of the data, i.e., measurements of geometry and loads. For most engineering work involving a problem such as this, the data as measured to three significant figures would be sufficient.

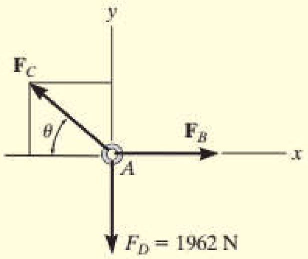


(c)

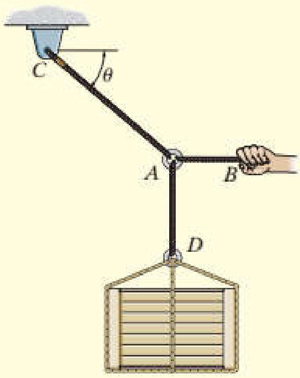
Fig. 3–6

EXAMPLE 3.3

The 200-kg crate in Fig. 3–7*a* is suspended using the ropes *AB* and *AC*. Each rope can withstand a maximum force of 10 kN before it breaks. If *AB* always remains horizontal, determine the smallest angle θ to which the crate can be suspended before one of the ropes breaks.



(b)



(a)

Fig. 3–7

SOLUTION

Free-Body Diagram. We will study the equilibrium of ring *A*. There are three forces acting on it, Fig. 3–7*b*. The magnitude of $\mathbf{F}_D$ is equal to the weight of the crate, i.e., $F_D = 200\,(9.81)\text{ N} = 1962\text{ N} < 10\text{ kN}$.

Equations of Equilibrium. Applying the equations of equilibrium along the *x* and *y* axes,

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad -F_C\cos\theta + F_B = 0; \quad F_C = \frac{F_B}{\cos\theta} \tag{1}$$

$$+\uparrow\Sigma F_y = 0; \qquad F_C\sin\theta - 1962\text{ N} = 0 \tag{2}$$

From Eq. (1), F_C is always greater than F_B since $\cos\theta \le 1$. Therefore, rope *AC* will reach the maximum tensile force of 10 kN *before* rope *AB*. Substituting $F_C = 10\text{ kN}$ into Eq. (2), we get

$$[10(10^3)\text{ N}]\sin\theta - 1962\text{ N} = 0$$

$$\theta = \sin^{-1}(0.1962) = 11.31^\circ = 11.3^\circ \qquad \textit{Ans.}$$

The force developed in rope *AB* can be obtained by substituting the values for θ and F_C into Eq. (1).

$$10(10^3)\text{ N} = \frac{F_B}{\cos 11.31^\circ}$$

$$F_B = 9.81\text{ kN}$$

EXAMPLE 3.4

Determine the required length of cord AC in Fig. 3–8*a* so that the 8-kg lamp can be suspended in the position shown. The *undeformed* length of spring AB is $l'_{AB} = 0.4$ m, and the spring has a stiffness of $k_{AB} = 300$ N/m.

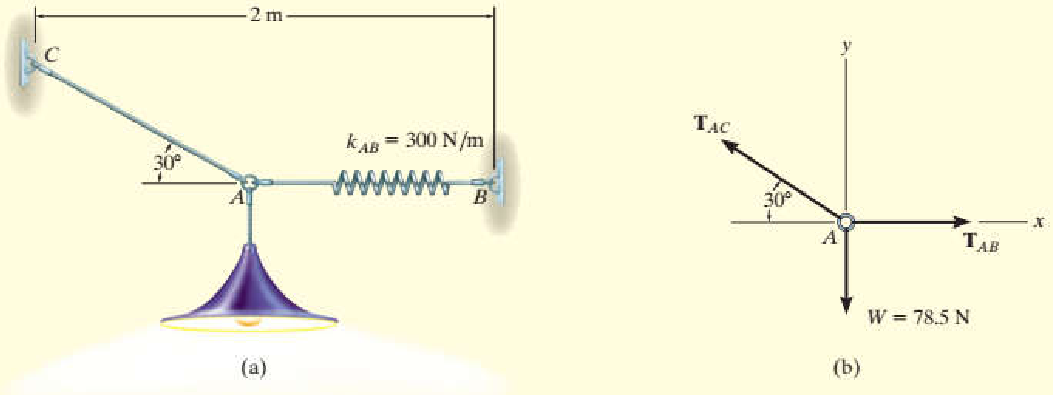


Fig. 3–8

SOLUTION

If the force in spring AB is known, the stretch of the spring can be found using $F = ks$. From the problem geometry, it is then possible to calculate the required length of AC.

Free-Body Diagram. The lamp has a weight $W = 8(9.81) = 78.5$ N and so the free-body diagram of the ring at A is shown in Fig. 3–8*b*.

Equations of Equilibrium. Using the x, y axes,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad T_{AB} - T_{AC} \cos 30^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad T_{AC} \sin 30^\circ - 78.5 \text{ N} = 0$$

Solving, we obtain

$$T_{AC} = 157.0 \text{ N}$$

$$T_{AB} = 135.9 \text{ N}$$

The stretch of spring AB is therefore

$$T_{AB} = k_{AB} s_{AB}; \qquad 135.9 \text{ N} = 300 \text{ N/m}(s_{AB})$$

$$s_{AB} = 0.453 \text{ m}$$

so the stretched length is

$$l_{AB} = l'_{AB} + s_{AB}$$

$$l_{AB} = 0.4 \text{ m} + 0.453 \text{ m} = 0.853 \text{ m}$$

The horizontal distance from C to B, Fig. 3–8*a*, requires

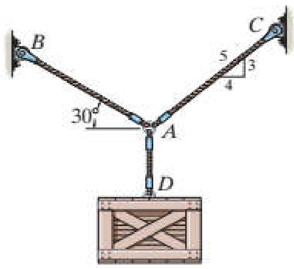
$$2 \text{ m} = l_{AC} \cos 30^\circ + 0.853 \text{ m}$$

$$l_{AC} = 1.32 \text{ m} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

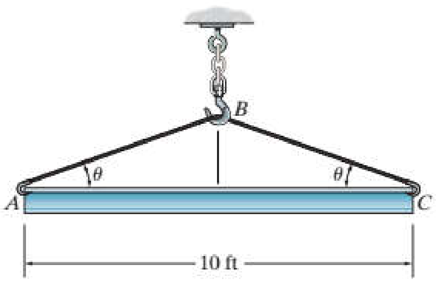
All problem solutions must include an FBD.

F3–1. The crate has a weight of 550 lb. Determine the force in each supporting cable.



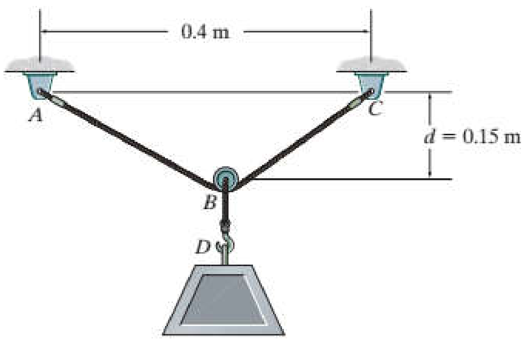
F3–1

F3–2. The beam has a weight of 700 lb. Determine the shortest cable ABC that can be used to lift it if the maximum force the cable can sustain is 1500 lb.



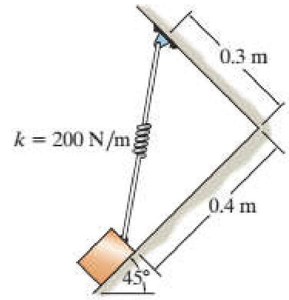
F3–2

F3–3. If the 5-kg block is suspended from the pulley B and the sag of the cord is $d = 0.15$ m, determine the force in cord ABC. Neglect the size of the pulley.



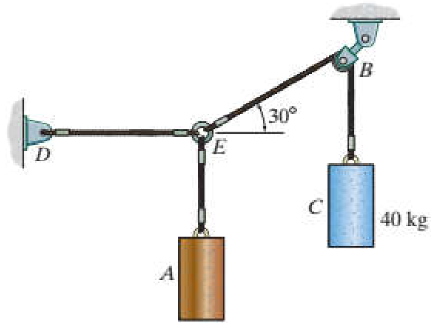
F3–3

F3–4. The block has a mass of 5 kg and rests on the smooth plane. Determine the unstretched length of the spring.



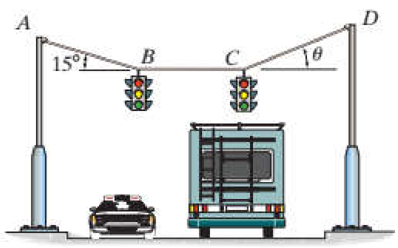
F3–4

F3–5. If the mass of cylinder C is 40 kg, determine the mass of cylinder A in order to hold the assembly in the position shown.



F3–5

F3–6. Determine the tension in cables AB, BC, and CD, necessary to support the 10-kg and 15-kg traffic lights at B and C, respectively. Also, find the angle θ.



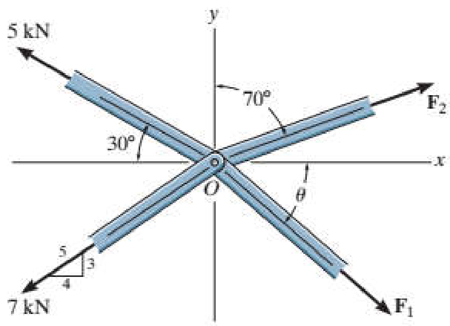
F3–6

PROBLEMS

All problem solutions must include an FBD.

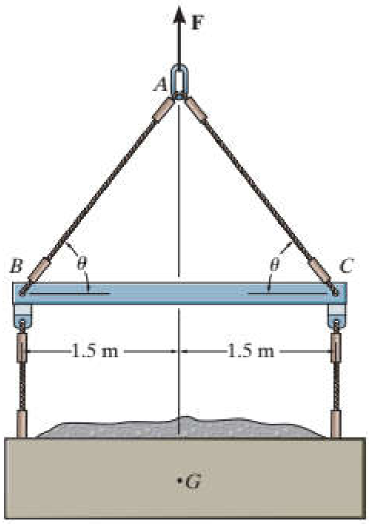
3–1. The members of a truss are pin connected at joint O. Determine the magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ for equilibrium. Set $\theta = 60°$.

3–2. The members of a truss are pin connected at joint O. Determine the magnitude of $\mathbf{F}_1$ and its angle θ for equilibrium. Set $F_2 = 6$ kN.



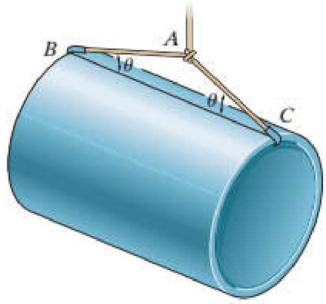
Probs. 3–1/2

3–3. The lift sling is used to hoist a container having a mass of 500 kg. Determine the force in each of the cables AB and AC as a function of θ. If the maximum tension allowed in each cable is 5 kN, determine the shortest lengths of cables AB and AC that can be used for the lift. The center of gravity of the container is located at G.



Prob. 3–3

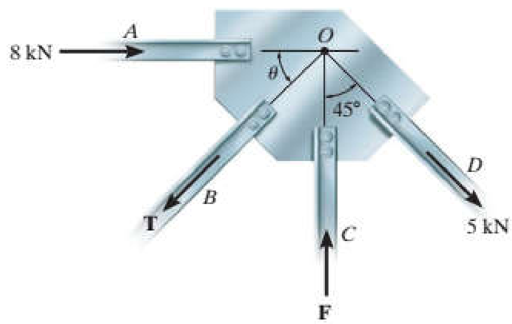
***3–4.** Cords AB and AC can each sustain a maximum tension of 800 lb. If the drum has a weight of 900 lb, determine the smallest angle θ at which they can be attached to the drum.



Prob. 3–4

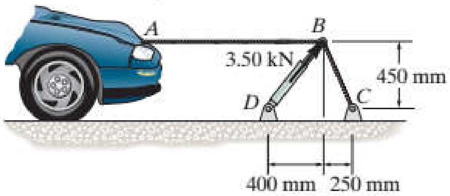
3–5. The members of a truss are connected to the gusset plate. If the forces are concurrent at point O, determine the magnitudes of $\mathbf{F}$ and $\mathbf{T}$ for equilibrium. Take $\theta = 30°$.

3–6. The gusset plate is subjected to the forces of four members. Determine the force in member B and its proper orientation θ for equilibrium. The forces are concurrent at point O. Take $F = 12$ kN.



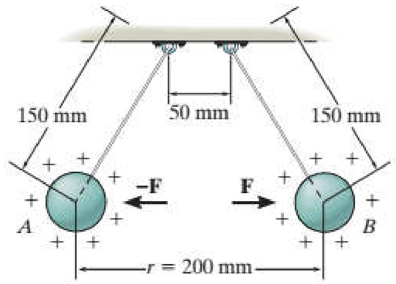
Probs. 3–5/6

3–7. The device shown is used to straighten the frames of wrecked autos. Determine the tension of each segment of the chain, i.e., AB and BC, if the force which the hydraulic cylinder DB exerts on point B is 3.50 kN, as shown.



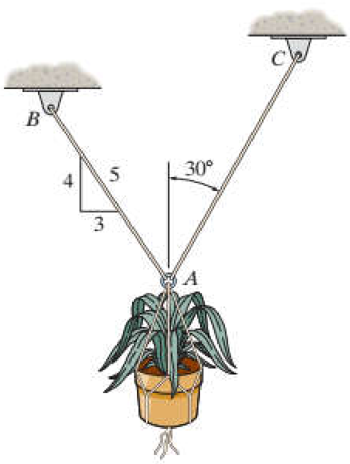
Prob. 3–7

***3–8.** Two electrically charged pith balls, each having a mass of 0.2 g, are suspended from light threads of equal length. Determine the resultant horizontal force of repulsion, F, acting on each ball if the measured distance between them is $r = 200$ mm.



Prob. 3–8

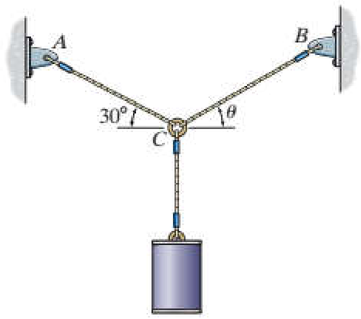
3–9. Determine the maximum weight of the flowerpot that can be supported without exceeding a cable tension of 50 lb in either cable AB or AC.



Prob. 3–9

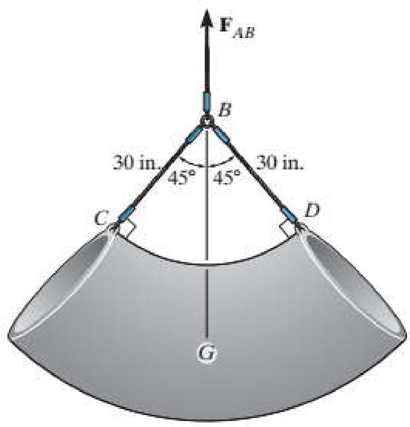
3–10. Determine the tension developed in wires CA and CB required for equilibrium of the 10-kg cylinder. Take $\theta = 40°$.

3–11. If cable CB is subjected to a tension that is twice that of cable CA, determine the angle θ for equilibrium of the 10-kg cylinder. Also, what are the tensions in wires CA and CB?



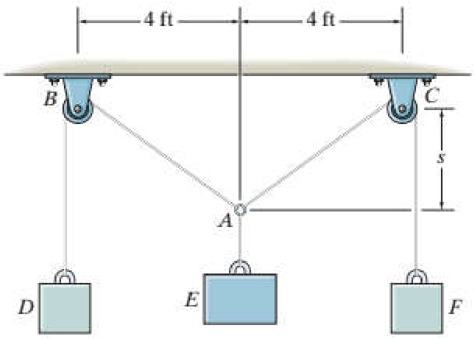
Probs. 3–10/11

***3–12.** The concrete pipe elbow has a weight of 400 lb and the center of gravity is located at point G. Determine the force $\mathbf{F}_{AB}$ and the tension in cables BC and BD needed to support it.



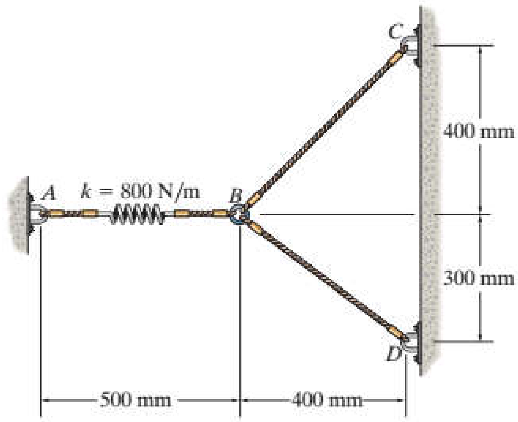
Prob. 3–12

3–14. If blocks D and F weigh 5 lb each, determine the weight of block E if the sag $s = 3$ ft. Neglect the size of the pulleys.



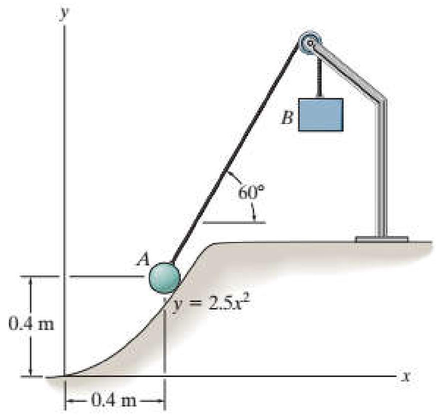
Probs. 3–13/14

3–15. The spring has a stiffness of $k = 800$ N/m and an unstretched length of 200 mm. Determine the force in cables BC and BD when the spring is held in the position shown.



Prob. 3–15

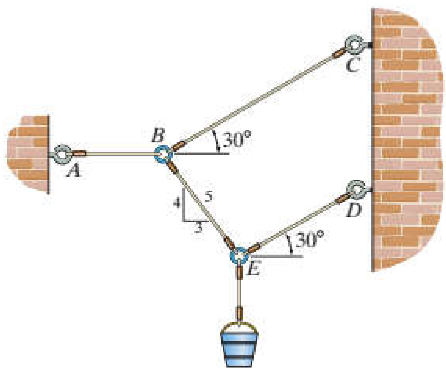
3–30. A 4-kg sphere rests on the smooth parabolic surface. Determine the normal force it exerts on the surface and the mass m_B of block B needed to hold it in the equilibrium position shown.



Prob. 3–30

3–31. If the bucket weighs 50 lb, determine the tension developed in each of the wires.

***3–32.** Determine the maximum weight of the bucket that the wire system can support so that no single wire develops a tension exceeding 100 lb.



Probs. 3–31/32

3.4 Three-Dimensional Force Systems

In Section 3.1 we stated that the necessary and sufficient condition for particle equilibrium is:

$$\Sigma \mathbf{F} = \mathbf{0} \qquad \text{............ (3–4)}$$

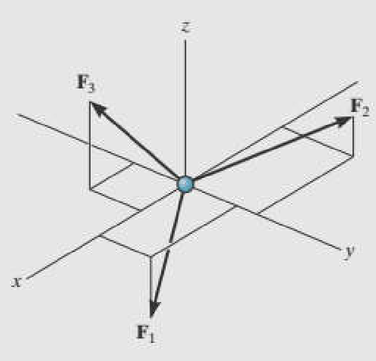


Fig. 3–9

In the case of a three-dimensional force system, as in Fig. 3–9 , we can resolve the forces into their respective **i, j, k** components, so that: $\Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} = \mathbf{0}$. To satisfy this equation we require:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0 \qquad \text{........ (3–5)}$$
$$\Sigma F_z = 0$$

These three equations state that the *algebraic sum* of the components of all the forces acting on the particle along each of the coordinate axes must be zero. Using them we can solve for at most three unknowns, generally represented as coordinate direction angles or magnitudes of forces shown on the particle's free-body diagram.

Procedure for Analysis

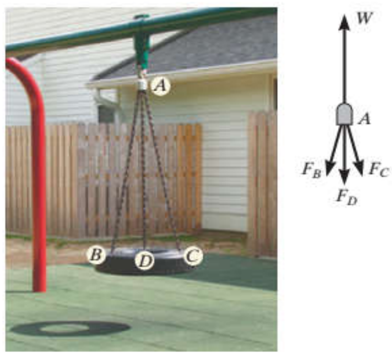
Three-dimensional force equilibrium problems for a particle can be solved using the following procedure.

Free-Body Diagram.

- Establish the *x, y, z* axes in any suitable orientation.
- Label all the known and unknown force magnitudes and directions on the diagram.
- The sense of a force having an unknown magnitude can be assumed.

Equations of Equilibrium.

- Use the scalar equations of equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, in cases where it is easy to resolve each force into its *x, y, z* components.
- If the three-dimensional geometry appears difficult, then first express each force on the free-body diagram as a Cartesian vector, substitute these vectors into $\Sigma \mathbf{F} = \mathbf{0}$, and then set the **i, j, k** components equal to zero.
- If the solution for a force yields a negative result, this indicates that its sense is the reverse of that shown on the free-body diagram.



The joint at A is subjected to the force from the support as well as forces from each of the three chains. If the tire and any load on it have a weight W, then the force at the support will be $\mathbf{W}$, and the three scalar equations of equilibrium can be applied to the free-body diagram of the joint in order to determine the chain forces, $\mathbf{F}_B$, $\mathbf{F}_C$, and $\mathbf{F}_D$.

EXAMPLE 3.6

The 10-kg lamp in Fig. 3–11*a* is suspended from the three equal-length cords. Determine its smallest vertical distance *s* from the ceiling if the force developed in any cord is not allowed to exceed 50 N.

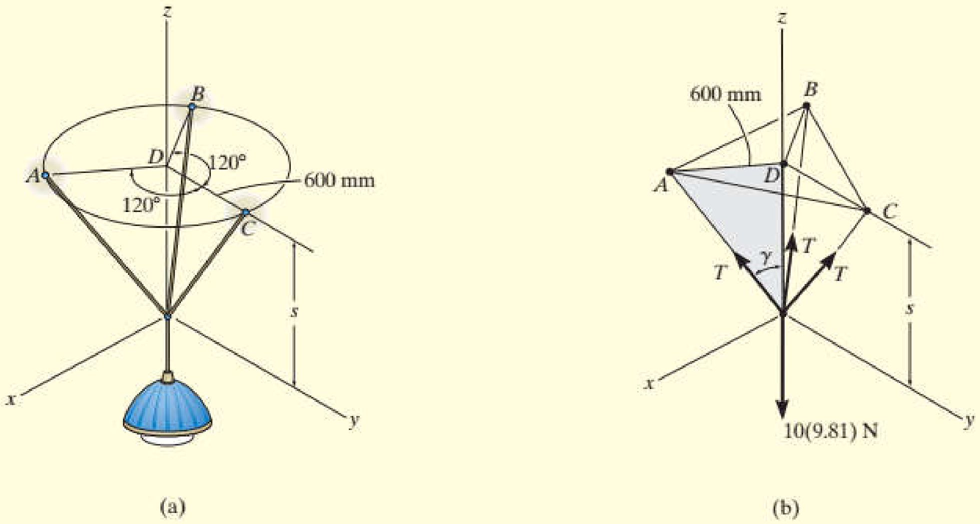


Fig. 3–11

SOLUTION

Free-Body Diagram. Due to symmetry, Fig. 3–11*b*, the distance $DA = DB = DC = 600$ mm. It follows that from $\sum F_x = 0$ and $\sum F_y = 0$, the tension T in each cord will be the same. Also, the angle between each cord and the z axis is γ.

Equation of Equilibrium. Applying the equilibrium equation along the z axis, with $T = 50$ N, we have

$$\sum F_z = 0; \qquad 3[(50\text{ N})\cos\gamma] - 10(9.81)\text{ N} = 0$$

$$\gamma = \cos^{-1}\frac{98.1}{150} = 49.16°$$

From the shaded triangle shown in Fig. 3–11*b*,

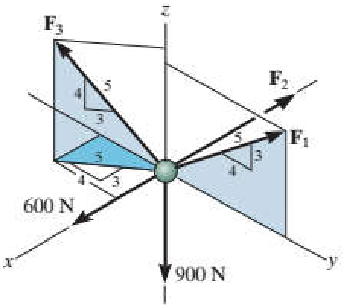
$$\tan 49.16° = \frac{600\text{ mm}}{s}$$

$$s = 519\text{ mm} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

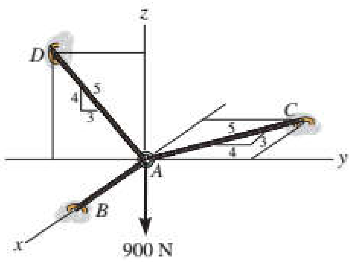
All problem solutions must include an FBD.

F3–7. Determine the magnitude of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, so that the particle is held in equilibrium.



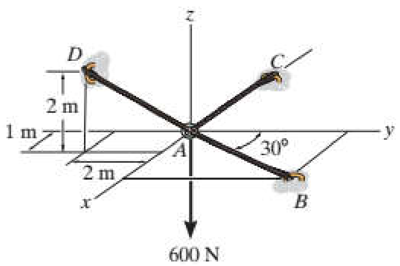
F3–7

F3–8. Determine the tension developed in cables AB, AC, and AD.



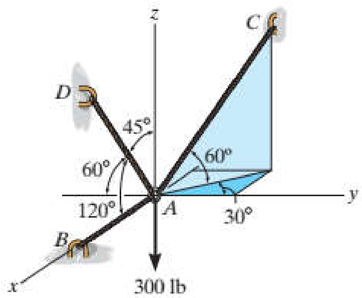
F3–8

F3–9. Determine the tension developed in cables AB, AC, and AD.



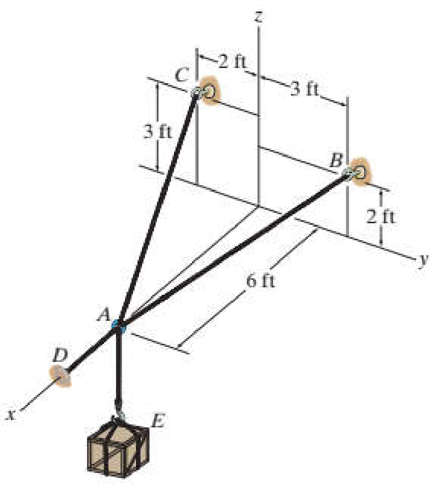
F3–9

F3–10. Determine the tension developed in cables AB, AC, and AD.



F3–10

F3–11. The 150-lb crate is supported by cables AB, AC, and AD. Determine the tension in these wires.



F3–11