CHAPTER - 4

Force System Resultants

CHAPTER OBJECTIVES:

- ***To discuss the concept of the moment of a force and show how to calculate it in two and three dimensions.***
- ***To provide a method for finding the moment of a force about a specified axis.***
- ***To define the moment of a couple.***
- ***To present methods for determining the resultants of non-concurrent force systems.***
- ***To indicate how to reduce a simple distributed loading to a resultant force having a specified location.***

4.1 Moment of a Force - Scalar Formulation:

In addition to the tendency to move a body in the direction of its application, a force can also tend to rotate a body about an axis. The axis may be any line which neither intersects nor is parallel to the line of action of the force. This rotational tendency is known as the *moment* **M** of the force. Moment is also referred to as *torque*.

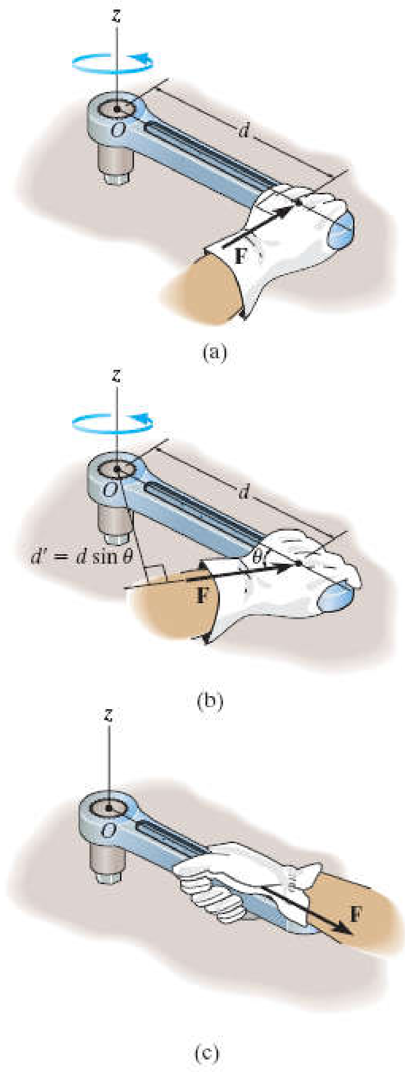


Fig. 4–1

Consider the force **F** and point *O* which lie in the shaded plane as shown in Fig. 4–2 *a*. The moment $\mathbf{M}_O$ about point *O*, or about an axis passing through *O* and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

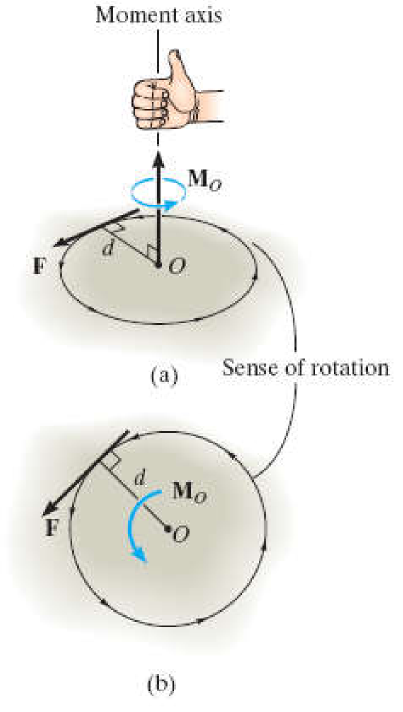


Fig. 4–2

Magnitude: The magnitude of $\mathbf{M}_O$ is:

$$M_O = Fd \qquad (4\text{–}1)$$

Where ***d* is the moment arm** or ***perpendicular distance from the axis at point O*** to the line of action of the force. **Units** of moment are **N.m** or **lb.ft**.

Direction: The direction of $\mathbf{M}_O$ is defined by its *moment axis,* which is perpendicular to the plane that contains the force **F** and its moment arm ***d*** . The right-hand rule is used to establish the sense of direction of $\mathbf{M}_O$.

Resultant Moment: For two-dimensional problems, where all the forces lie within the *x–y* plane, Fig. 4–3 , the resultant moment $(\mathbf{M}_R)_O$ about point *O* (the *z* axis) can be determined by *finding the algebraic sum* of the moments caused by all the forces in the system. As a convention, we will generally consider *positive moments* as *counterclockwise* since they are directed along the positive *z* axis (out of the page). *Clockwise moments* will be *negative*. Therefore:

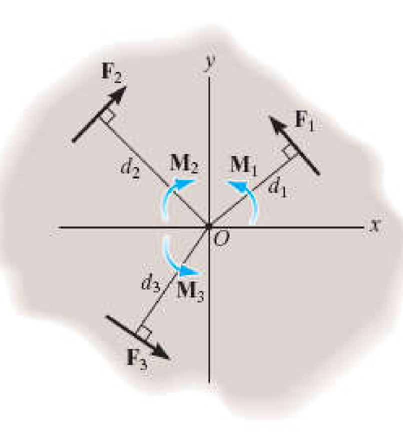


Fig. 4–3

$$\circlearrowleft +(M_R)_o = \Sigma Fd; \qquad (M_R)_o = F_1d_1 - F_2d_2 + F_3d_3$$

If the numerical result of this sum is a positive scalar, $(\mathbf{M}_R)_O$ will be a Counterclockwise moment (out of the page); and if the result is negative, $(\mathbf{M}_R)_O$ will be a clockwise moment (into the page).

EXAMPLE 4.1

For each case illustrated in Fig. 4–4, determine the moment of the force about point O.

SOLUTION *(SCALAR ANALYSIS)*

The line of action of each force is extended as a dashed line in order to establish the moment arm d. Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about O is shown as a colored curl. Thus,

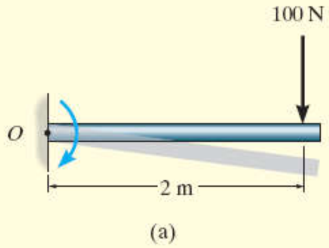
Fig. 4–4*a* $M_O = (100\text{ N})(2\text{ m}) = 200\text{ N}\cdot\text{m} \circlearrowright$ *Ans.*

Fig. 4–4*b* $M_O = (50\text{ N})(0.75\text{ m}) = 37.5\text{ N}\cdot\text{m} \circlearrowright$ *Ans.*

Fig. 4–4*c* $M_O = (40\text{ lb})(4\text{ ft} + 2\cos 30^\circ\text{ ft}) = 229\text{ lb}\cdot\text{ft} \circlearrowright$ *Ans.*

Fig. 4–4*d* $M_O = (60\text{ lb})(1\sin 45^\circ\text{ ft}) = 42.4\text{ lb}\cdot\text{ft} \circlearrowleft$ *Ans.*

Fig. 4–4*e* $M_O = (7\text{ kN})(4\text{ m} - 1\text{ m}) = 21.0\text{ kN}\cdot\text{m} \circlearrowleft$ *Ans.*



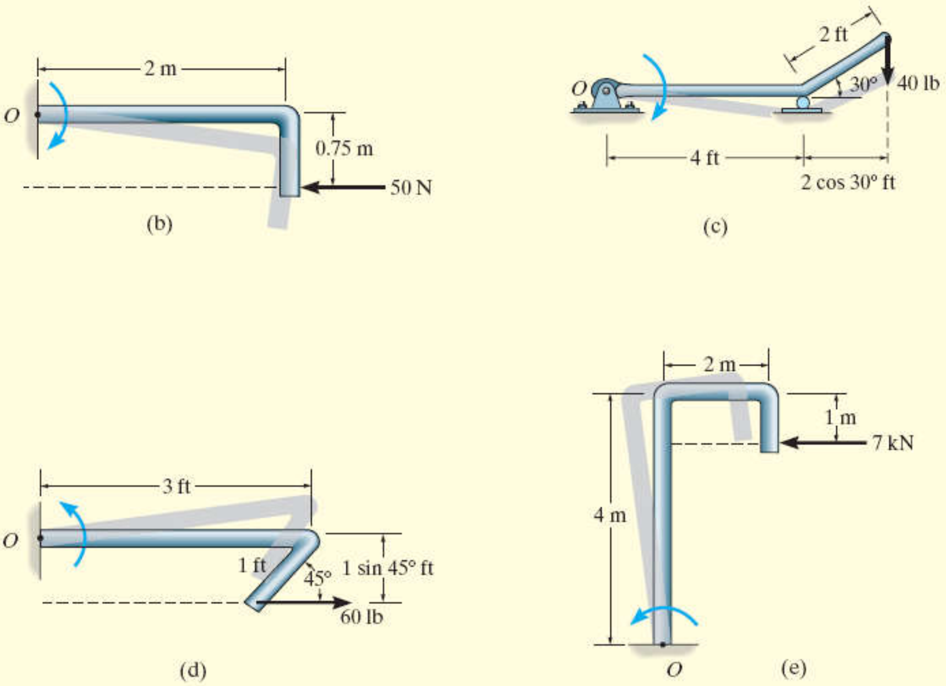


Fig. 4–4

EXAMPLE 4.2

Determine the resultant moment of the four forces acting on the rod shown in Fig. 4–5 about point O.

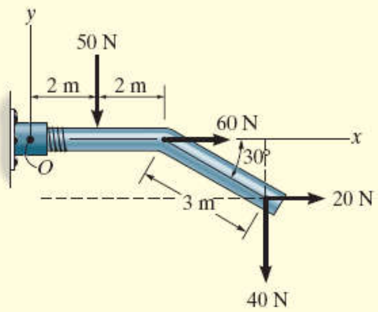


Fig. 4–5

SOLUTION

Assuming that positive moments act in the $+\mathbf{k}$ direction, i.e., counterclockwise, we have

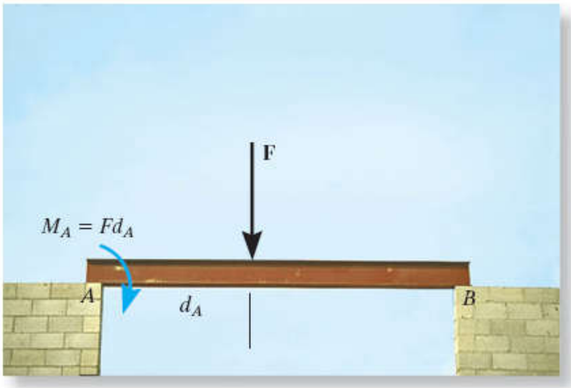
$$\circlearrowleft + \quad (M_R)_o = \Sigma Fd;$$

$$(M_R)_o = -50\text{ N}(2\text{ m}) + 60\text{ N}(0) + 20\text{ N}(3 \sin 30^\circ\text{ m})$$

$$-40\text{ N}(4\text{ m} + 3 \cos 30^\circ\text{ m})$$

$$(M_R)_o = -334\text{ N}\cdot\text{m} = 334\text{ N}\cdot\text{m}\circlearrowright \qquad \textit{Ans.}$$

For this calculation, note how the moment-arm distances for the 20-N and 40-N forces are established from the extended (dashed) lines of action of each of these forces.



As illustrated by the example problems, the moment of a force does not always cause a rotation. For example, the force **F** tends to rotate the beam clockwise about its support at A with a moment $M_A = Fd_A$. The actual rotation would occur if the support at B were removed.

The ability to remove the nail will require the moment of $\mathbf{F}_H$ about point O to be larger than the moment of the force $\mathbf{F}_N$ about O that is needed to pull the nail out.

4.2 Cross Product:

The moment of a force will be formulated using Cartesian vectors in the next section. Before doing this, however, it is first necessary to expand our knowledge of vector algebra and introduce the cross-product method of vector multiplication.

The *cross product* of two vectors **A** and **B** yields the vector **C** , which is written as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad (4\text{–}2)$$

and is read " **C** equals **A** cross **B** ."

Magnitude: The *magnitude* of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle u between their tails (0°≤ θ ≤ 180°). Thus, **C**=$A\,B\sin\theta$.

Direction: Vector **C** has a ***direction*** that is perpendicular to the plane containing **A** and **B** such that **C** is specified by the right-hand rule; i.e., curling the fingers of the right hand from vector **A** (cross) to vector **B**, the thumb points in the direction of **C** , as shown in Fig. 4–6 .

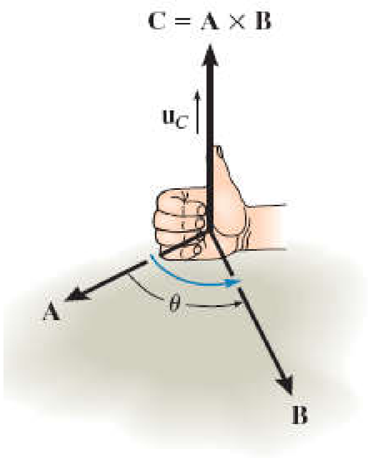


Fig. 4–6

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB\sin\theta)\mathbf{u}_C \qquad (4\text{–}3)$$

Laws of Operation:

• The commutative law is *not* valid; i.e. , **A** x **B** ≠ **B** x **A** Rather, **A** x **B** = -**B** x **A.** This is shown in Fig. 4–7 by using the right-hand rule. The cross product **B** x **A** yields a vector that has the same magnitude but acts in the opposite direction to **C** ; i.e., **B** x **A** = - **C**.

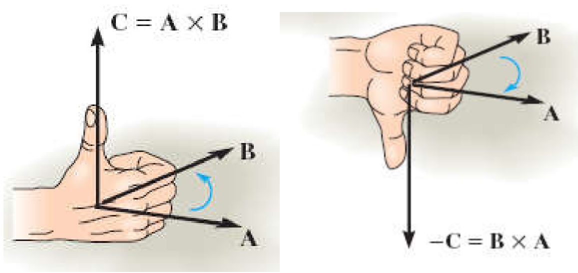


Fig. 4–7

• If the cross product is multiplied by a scalar ***a***, it obeys the associative law;

$$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$$

• The distributive law of addition,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$$

Cartesian Vector Formulation: Equation 4–3 may be used to find the cross product of any pair of Cartesian unit vectors. For example, to find **i** x **j**, the magnitude of the resultant vector is $(i)(j)(\sin 90^0) = (1)(1)(1) = 1$, and its direction is determined using the right-hand rule. As shown in Fig. 4–8 , the resultant vector points in the +**k** direction. Thus, **i** x **j** = (1)**k**. In a similar manner,

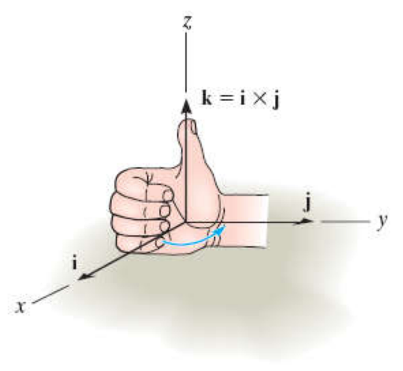


Fig. 4–8

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{i} \times \mathbf{i} = \mathbf{0}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

These results should *not* be memorized; rather, it should be clearly understood how each is obtained by using the right-hand rule and the definition of the cross product. A simple scheme shown in Fig. 4–9 is helpful for obtaining the same results when the need arises. If the circle is constructed as shown, then "crossing" two unit vectors in a *counterclockwise* fashion around the circle yields the *positive* third unit vector; e.g., **k** x **i** = **j**. "Crossing" *clockwise* , a *negative* unit vector is obtained; e.g., **i** x **k** = **-j**.

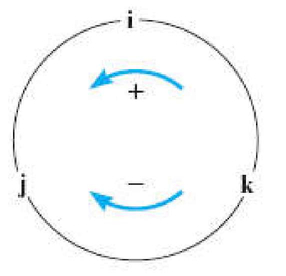


Fig. 4–9

Let us now consider the cross product of two general vectors **A** and **B**,

$$\mathbf{A}\times\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$\mathbf{A}\times\mathbf{B} = A_xB_x(\mathbf{i}\times\mathbf{i}) + A_xB_y(\mathbf{i}\times\mathbf{j}) + A_xB_z(\mathbf{i}\times\mathbf{k}) + A_yB_x(\mathbf{j}\times\mathbf{i}) + A_yB_y(\mathbf{j}\times\mathbf{j}) + A_yB_z(\mathbf{j}\times\mathbf{k}) + A_zB_x(\mathbf{k}\times\mathbf{i}) + A_zB_y(\mathbf{k}\times\mathbf{j}) + A_zB_z(\mathbf{k}\times\mathbf{k})$$

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

This equation may also be written in a more compact determinant form as:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (4\text{–}5)$$

4.3 Moment of a Force - Vector Formulation

The moment of a force **F** about point *O* , or actually about the moment axis passing through *O* and perpendicular to the plane containing *O* and **F** , Fig. 4–10 *a* , can be expressed using the vector cross product, namely,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \quad (4\text{–}6)$$

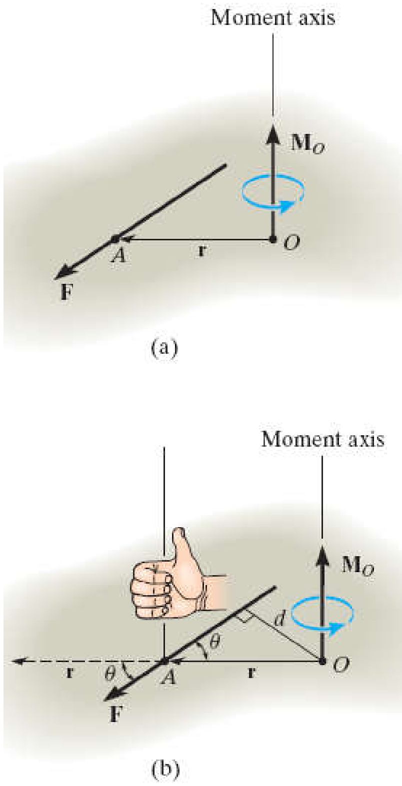


Fig. 4–10

Here **r** represents a position vector directed *from O* to *any point* on the line of action of **F** .

The magnitude of the cross product is defined from Eq.4–3 as $M_O = rF\sin\theta$, where the angle θ is measured between the *tails* of **r** and **F**. From Fig. 4–10 *b,* since the moment arm $d = r\sin\theta$, then:

$$M_O = rF\sin\theta = F(r\sin\theta) = Fd$$

The direction and ***sense*** of M_O in Eq. 4–6 are determined by the right-hand rule as it applies to the cross product.

Principle of Transmissibility: The cross product operation is often used in three dimensions since the perpendicular distance or moment arm from point *O* to the line of action of the force is not needed. In other words, we can use any position vector **r** measured from point *O* to any point on the line of action of the force **F**, Fig. 4–11 . Thus,

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

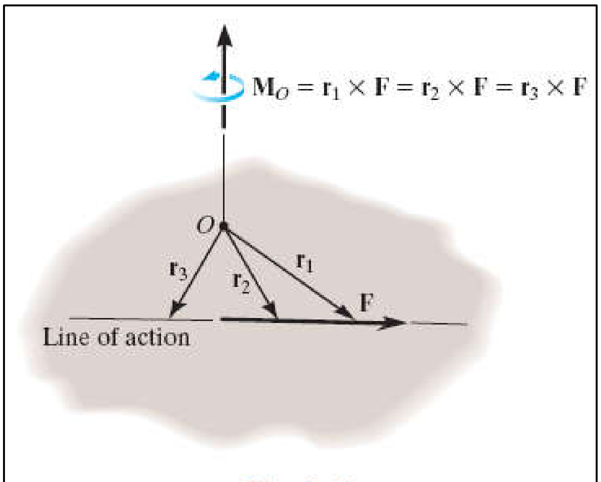
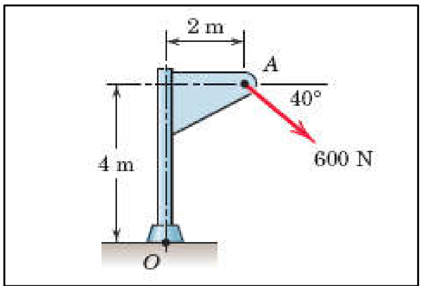


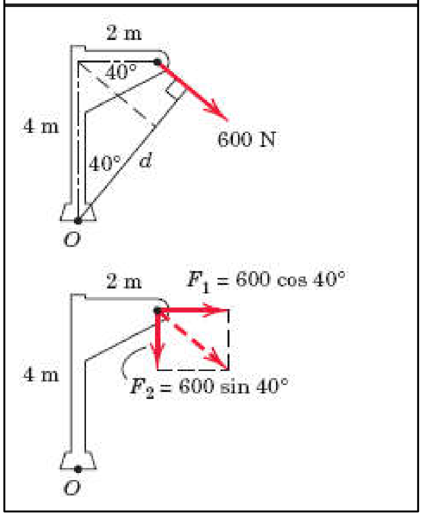
Fig. 4–11

Since **F** can be applied at any point along its line of action and still create this ***same moment*** about point *O* , then **F** can be considered a ***sliding vector*** . This property is called the ***principle of transmissibility*** of a force.

Example -1

Calculate the magnitude of the moment about the base point *O* of the 600-N force in five different ways.





Solution:

(I) The moment arm to the 600-N force is $d = 4 \cos 40^o + 2 \sin 40^o = 4.35$ m

By $M = Fd$ the moment is clockwise and has the magnitude:

$M_O = 600(4.35) = 2610$ N.m *Ans.*

(II) Replace the force by its rectangular components at *A*,

$F_1 = 600 \cos 40^o = 460$ N, $F_2 = 600 \sin 40^o = 386$ N

The moment becomes:

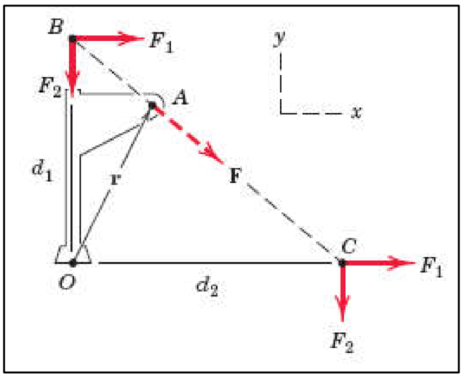
$M_O = 460(4) + 386(2) = 2610$ N.m *Ans.*

(III) By the principle of transmissibility, move the 600-N force along its line of action to point *B*, which eliminates the moment of the component F_2. The moment arm of F_1 becomes: $d_1 = 4 + 2 \tan 40^o = 5.68$ m and the moment is:

$M_O = 460(5.68) = 2610$ N.m *Ans.*

(IV) Moving the force to point *C* eliminates the moment of the component F_1. The moment arm of F_2 becomes: $d_2 = 2 + 4 \cot 40^o = 6.77\ m$
and the moment is:

$$M_O = 386(6.77) = 2610\ N.m \qquad Ans.$$



(IV) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have:

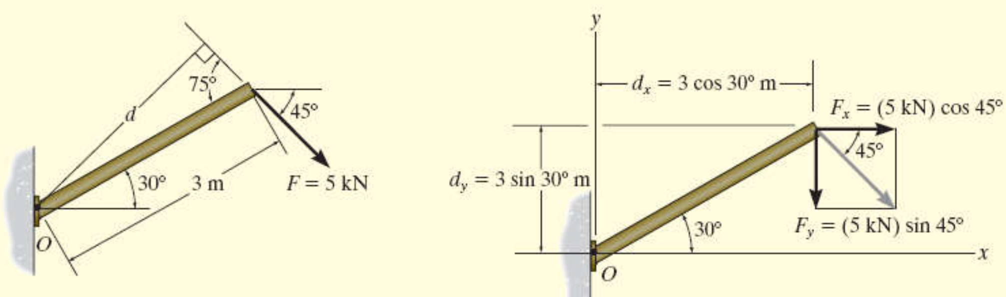
$$M_O = \mathbf{r} \times \mathbf{F} = (2i + 4j) \times 600(i \cos 40^o - j \sin 40^o)$$
$$= -2610\mathbf{k}\ N.m$$

The minus sign indicates that the vector is in the negative *z*-direction. The magnitude of the vector expression is:

$$M_O = 2610\ N.m \qquad Ans.$$

EXAMPLE 4.5

Determine the moment of the force in Fig. 4–18*a* about point *O*.



(a) (b)

SOLUTION I

The moment arm *d* in Fig. 4–18*a* can be found from trigonometry.

$$d = (3\text{ m}) \sin 75^\circ = 2.898\text{ m}$$

Thus,

$$M_O = Fd = (5\text{ kN})(2.898\text{ m}) = 14.5\text{ kN}\cdot\text{m} \circlearrowright \qquad Ans.$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

continue Example (4.5)

SOLUTION II

The x and y components of the force are indicated in Fig. 4–18b. Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$\circlearrowleft + M_O = -F_x d_y - F_y d_x$$
$$= -(5 \cos 45° \text{ kN})(3 \sin 30° \text{ m}) - (5 \sin 45° \text{ kN})(3 \cos 30° \text{ m})$$
$$= -14.5 \text{ kN}\cdot\text{m} = 14.5 \text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

SOLUTION III

The x and y axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 4–18c. Here $\mathbf{F}_x$ produces no moment about point O since its line of action passes through this point. Therefore,

$$\circlearrowleft + M_O = -F_y d_x$$
$$= -(5 \sin 75° \text{ kN})(3 \text{ m})$$
$$= -14.5 \text{ kN}\cdot\text{m} = 14.5 \text{ kN}\cdot\text{m} \circlearrowright \qquad \textit{Ans.}$$

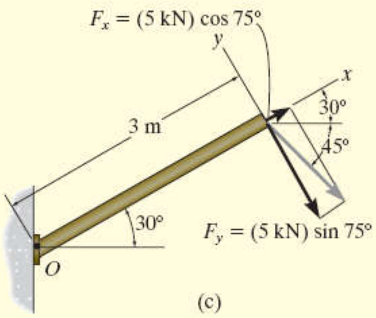


Fig. 4–18

Cartesian vector formulation:

If we establish *x, y, z* coordinate axes, then the position vector **r** and force **F** can be expressed as Cartesian vectors (Fig 4-12-a) then we can write:

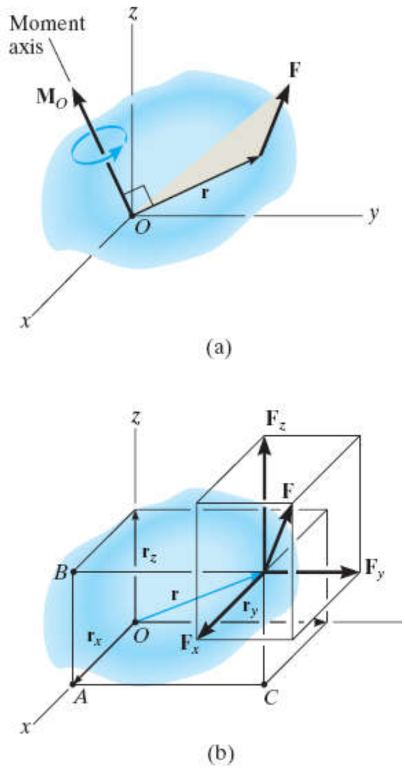


Fig. 4–12

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \qquad (4\text{–}7)$$

Where r_x, r_y,r_z represent the *x, y, z* components of the position vector drawn from point O to any point on the line of action of the force. F_x, F_y, F_z represent the *x, y, z* ***components*** of the force vector. If the determinant is expanded, then like Eq. 4–4 we have:

$$\mathbf{M}_O = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k} \qquad (4\text{–}8)$$

Resultant Moment of a system of forces:

If a body is acted upon by a system of forces (Fig 4-13), the resultant moment of the forces about point O can be determined by vector addition of the moment of each force. This resultant can be written symbolically as:

$$(\mathbf{M}_R)_o = \Sigma\,(\mathbf{r} \times \mathbf{F})$$

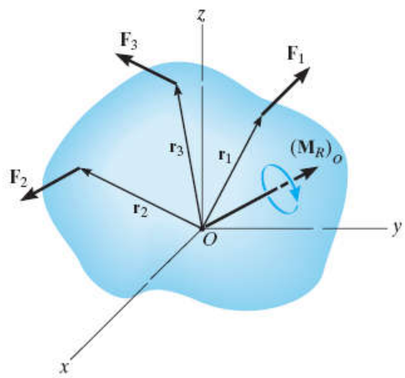


Fig. 4–13