٣. قواعد تكامل الدوال المثلثية

إذا كانت u دالة قابلة للاشتقاق في x بتطبيق القانون الأساسي لحساب التكامل للقوانين الأساسية للتفاضل يكون لدينا القوانين التالية:

$$1)\ \int u'\cos u\,dx=\sin u+c \qquad 2)\ \int u'\sin u dx=-\cos u+c$$

$$3)\ \int u'\sec^2 u\,dx=\tan u+c \qquad 4)\ \int u'\csc^2 u\,dx=-\cot u+c$$

$$5)\ \int u'\sec u\tan u dx=\sec u+c \qquad 6)\ \int u'\csc u\cot u\,dx=-\csc u+c$$

$$7)\ \int u'\tan u dx=\ln|\sec u|+c \qquad 8)\ \int u'\cot u dx=-\ln|\csc u|+c$$

مثال ٨: احسب التكاملات التالية:

$$1)\int \sin 4x dx,\quad 2)\int \cos 2x dx,\quad 3)\int x\tan\left(2x^2+1\right)dx,\quad 4)\int \cot\left(7-\frac{x}{2}\right)dx.$$

الحل:

$$1)\int \sin 4x dx=\frac{1}{4}\int 4\sin 4x dx=-\frac{1}{4}\cos 4x+c$$

$$2)\int \cos 2x dx=\frac{1}{2}\int 2\cos 2x dx=\frac{1}{2}\sin 2x+c$$

$$3)\int x\tan\left(2x^2+1\right)dx=\frac{1}{4}\int 4x\tan\left(2x^2+1\right)dx=\frac{1}{4}\ln\left|\sec\left(2x^2+1\right)\right|+c$$

$$4)\int \cot\left(7-\frac{x}{2}\right)dx=-2\int -\frac{1}{2}\cot\left(-\frac{x}{2}\right)dx=2\ln\left|\csc\left(7-\frac{x}{2}\right)\right|+c$$

مثال ٩: احسب التكاملات التالية :

$$1)\int\left[\sin(3x+2)+\cos(2-3x)\right]dx,\qquad 2)\int \sec^2(4x)dx$$

$$3)\int x^2\csc^2\left(3-\frac{x^3}{3}\right)dx,\qquad 4)\int x^2\csc\left(2x^3\right)\cot\left(2x^3\right)dx$$

الحل:

$$1)\ \int[\sin(3x+2)+\cos(2-3x)]dx=\int\sin(3x+2)dx+\int\cos(2-3x)dx.$$

$$=\frac{1}{3}\int 3\sin(3x+2)dx-\frac{1}{3}\int-3\cos(2-3x)dx.$$

$$=-\frac{1}{3}\cos(3x+2)-\frac{1}{3}\sin(2-3x)+c.$$

$$2)\int\sec^2 4x\,dx=\frac{1}{4}\int 4\sec^2 4x\,dx=\frac{1}{4}\tan 4x+c$$

$$3)\int x^2\csc^2\left(3-\frac{x^3}{3}\right)dx=-\int-x^2\csc^2\left(3-\frac{x^3}{3}\right)dx=\cot\left(3-\frac{x^3}{3}\right)+c$$

$$4)\int x^2\csc 2x^3\cot 2x^3dx=\frac{1}{6}\int 6x^2\csc 2x^3\cot 2x^3dx$$

$$=\frac{1}{6}\left(-\csc 2x^3\right)+c=-\frac{1}{6}\csc 2x^3+c$$

مثال ١٠: احسب التكاملات التالية :

$$1)\int\frac{3\sin(2-\sqrt{x})}{\sqrt{x}}dx.,\qquad 2)\int\frac{\cos(3+5\ln 9x).}{7x}dx.$$

$$3)\int\cos 6x\cos(9+4\sin 6x)\,dx,\quad 4)\int\frac{\tan(5-\frac{4}{\sqrt{x}})}{\sqrt{x^3}}dx$$

الحل:

$$1)\int\frac{3\sin(2-\sqrt{x})}{\sqrt{x}}dx.$$

$$u=2-\sqrt{x}\Rightarrow u'=-\frac{1}{2\sqrt{x}}$$

$$\int\frac{3\sin(2-\sqrt{x})}{\sqrt{x}}dx=3(-2)\int-\frac{1}{2}\frac{\sin(2-\sqrt{x})}{\sqrt{x}}dx=-6\int u'\sin u dx$$

$$=6\cos u+c=6\cos(2-\sqrt{x})+c.$$

$$2)\int \frac{\cos(3+5\ln 9x)}{7x}dx$$

$$u=3+5\ln 9x \Rightarrow u'=\frac{5}{x}$$

$$\int \frac{\cos(3+5\ln 9x)}{7x}dx=\frac{1}{5}\frac{1}{7}\int \frac{5}{x}\cos(3+5\ln 9x)dx$$

$$=\frac{1}{35}\int u'\cos\ u\,dx=\frac{1}{35}\sin u+c=\frac{1}{35}\sin(3+5\ln 9x)+c$$

$$3)\int \cos 6x\cos(9+4\sin 6x)\,dx.$$

$$u=9+4\sin 6x \Rightarrow u'=24\cos 6x$$

$$\int \cos 6x\cos(9+4\sin 6x)\,dx=\frac{1}{24}\int 24\cos 6x\cos(9+4\sin 6x)\,dx$$

$$=\frac{1}{24}\sin(9+4\sin 6x)+c$$

$$4)\int \frac{\tan\left(5-\frac{4}{\sqrt{x}}\right)}{\sqrt{x^3}}dx$$

$$u=5-4x^{-\frac{1}{2}} \Rightarrow u'=\frac{4}{2}x^{-\frac{1}{2}-1}=2x^{-\frac{3}{2}}=\frac{2}{\sqrt{x^3}}$$

$$\int \frac{\tan\left(5-\frac{4}{\sqrt{x}}\right)}{\sqrt{x^3}}dx=\frac{1}{2}\int \frac{2}{\sqrt{x^3}}\tan\left(5-\frac{4}{\sqrt{x}}\right)dx=\frac{1}{2}\int u'\tan u\,dx=$$

$$=\frac{1}{2}\ln|\sec u|+c=\frac{1}{2}\ln\left|\sec\left(5-\frac{4}{\sqrt{x}}\right)\right|+c$$

تمرين تدريبي: احسب التكاملات التالية:

1) $\int \cos^3 x \sin x dx$

2) $\int \frac{1}{\sqrt{x}} \sin\sqrt{x} dx$

3) $\int (1+\sin t)^2 \cos t dt$

4) $\int x \cos\left(3x^2\right) dx$

5) $\int x^2 \sec^2 x^3 dx$

6) $\int \cos^3 2t \sin 2t \, dt$

7) $\int \cos 4\theta \sqrt{2-\sin 4\theta} d\theta$

8) $\int \tan^3 5x \sec^2 5x \, dx$

9) $\int \sin 3\theta \sec^2 \cos 3\theta \, d\theta$

10) $\int (1-\cos x)^3 \sin x dx$

11) $\int (1-\sin 2\vartheta)^{\frac{1}{3}} \cos 2\vartheta \, d\vartheta$

12) $\int x^7 \tan(8x^8+6) dx$

13) $\int \sin(7-\cos 3x) \sin 3x dx$

14) $\int te^{t^2} \sec(2+e^{t^2}) \tan(2+$

15) $\int \frac{\cos\sqrt{t}}{\sqrt{t}} dt$

16) $\int \frac{\sec x \tan x}{(3+2\sec x)^2} dx$

17) $\int \frac{\cos x - \sin x}{(\sin x + \cos x)^3} dx$

18) $\int \frac{\sec^2 x}{\sqrt{5-\tan x}} dx$

19) $\int \frac{x+\cos 2x}{\sqrt[3]{x^2+\sin 2x}} dx$

20) $\int \frac{3\cot\sqrt{t}}{\sqrt{t}} dt$

21) $\int (2+\tan^2 x) \sec^2 x dx$