University of Anbar
Engineering College
Department of Mechanical Engineering

ME 4309 - Engineering Control and Measurements

(3-3-1-0)

Fourth Stage

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Reference:

1. Automatic Control Engineering, First Edition 1961, by Francis H. Raven, McGraw Hill.
2. Measurement Systems Applications and Design, 5th edition 2003, by E. Doebelin, McGraw Hill.

Chapter 7: State Space Method

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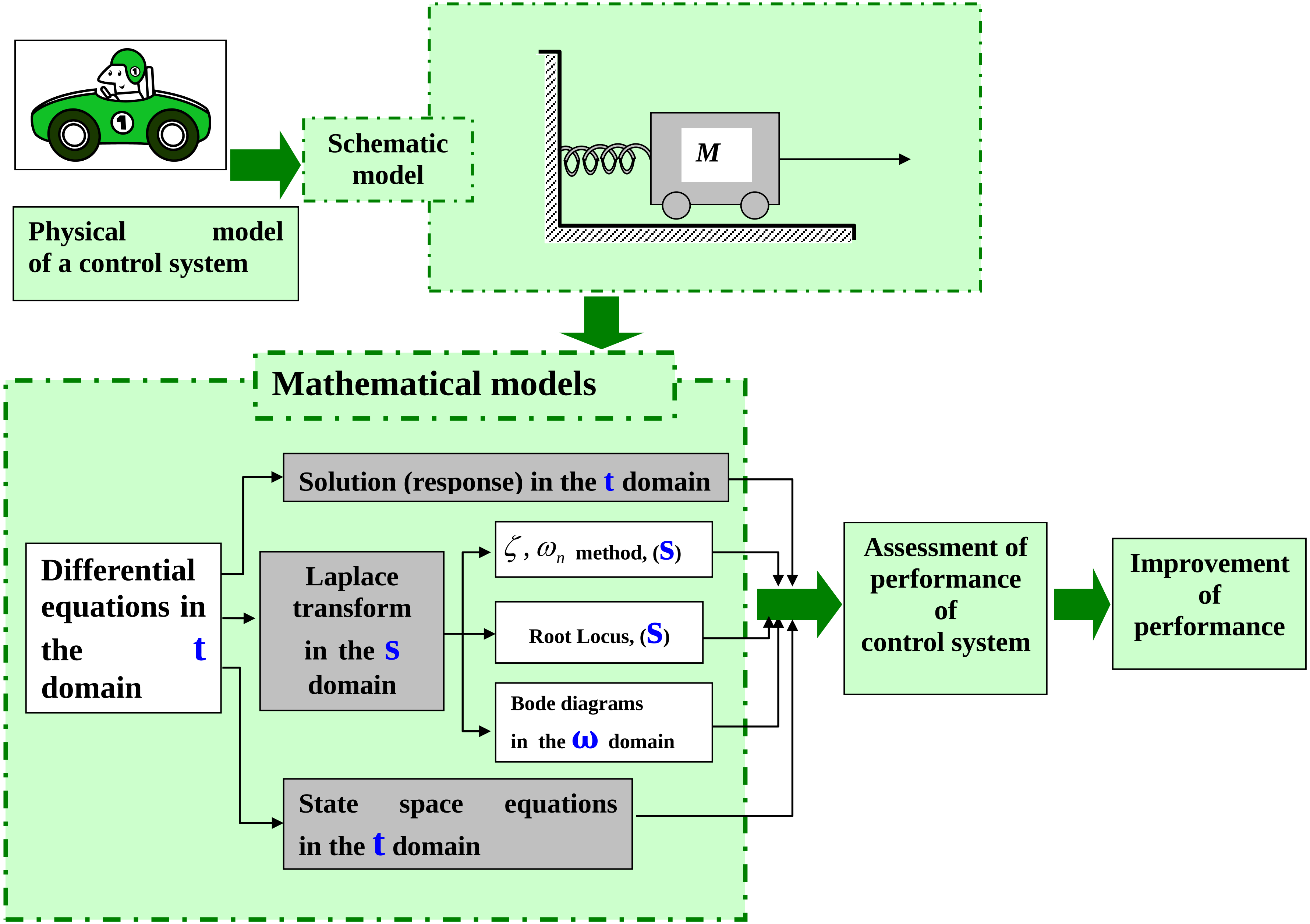
1. Introduction to state space method

Historic development of mathematical models: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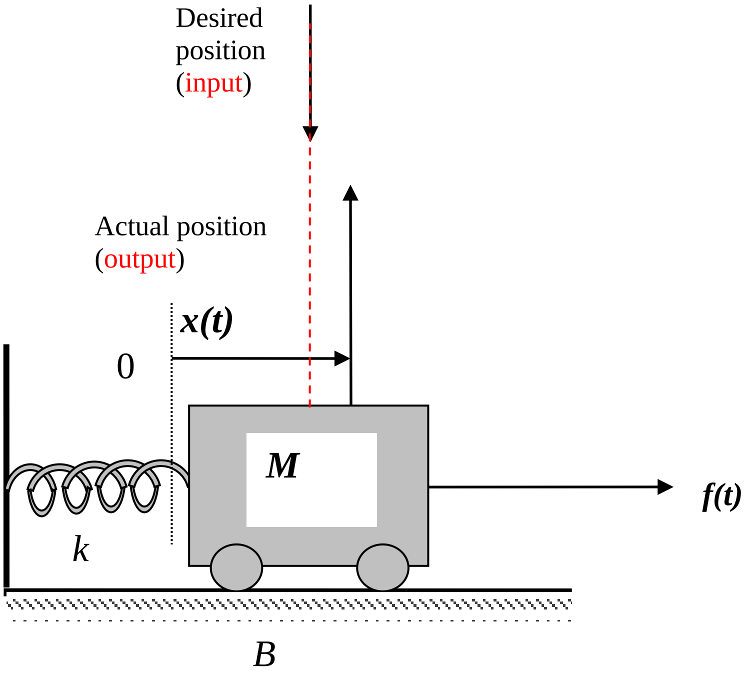
Reasons for using state space method

(1) It is suitable for non-linear time varying systems, e.g., dealing with a spring-cart system with B(t) instead of B as a constant.

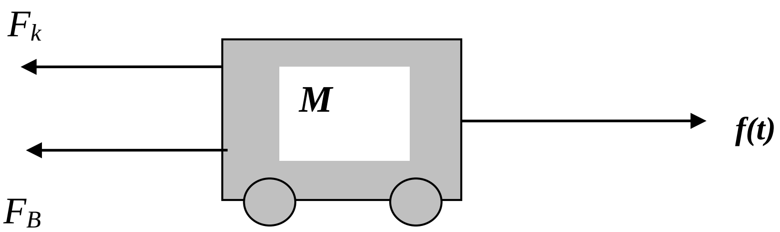
(2) It can give the direct solutions to differential equations with computers in the time domain.

2. State representation of a spring-cart system

a. Diagrammatic model



b. Diagrammatic free body



$$f(t) - F_k - F_B = Ma$$

$$M\ddot{x}(t) + B\dot{x}(t) + k\,x(t) = f(t)$$

c. Laplace transform

$$Ms^2X(s) + BsX(s) + kX(s) = F(s)$$

$$(Ms^2 + Bs + k)X(s) = F(s)$$

d. Transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{k}{M}}$$

(c). State representation of the differential equations

Conventionally we use ***y*** as an output and ***u*** as an input in analysis of control systems in the state space, so $M\ddot{x}(t) + B\dot{x}(t) + k\,x(t) = f(t)$ is converted to

$$M\ddot{y} + B\dot{y} + k\,y = u$$

The above equation can be written as

$$M\ddot{y} = -B\dot{y} - k\,y + u$$

or

$$\ddot{y} = -\frac{B}{M}\dot{y} - \frac{k}{M}y + \frac{1}{M}u$$

Define

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}\right\} \quad \text{hence} \quad \left.\begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{B}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \end{aligned}\right\}$$

Where x_1 and x_2 are called state variables while $\dot{x}_1$ and $\dot{x}_2$ called state phase variables.

(d). State equation represented with matrix form

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{B}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{x}_1 &= 1\ x_1 + 0x_2 + 0\,u \\ \dot{x}_2 &= -\frac{k}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}u \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \right\}$$

$$y = x_1 \quad \Rightarrow \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}\right\}$$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is defined as the state variable vector, and

$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ is defined as the state phase variable vector

Another form is

$$\left\{ \begin{array}{l} \dot{X} = AX + Bu \\ \\ y = CX \end{array} \right. \qquad \text{where} \qquad \left. \begin{array}{l} A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{B}{M} \end{bmatrix} \\ \\ B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array} \right\}$$

(e). Block diagram representation of the state equation

Step 1: a block diagram element for $\dot{x}_1 = x_2$

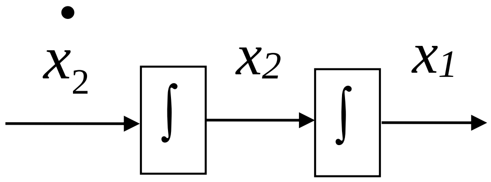
$$\dot{x}_1 = x_2 \quad \Rightarrow \quad x_1 = \int \dot{x}_1\, dt = \int x_2 dt \quad \Rightarrow \quad x_1 = \int x_2 dt$$

$$x_2 \longrightarrow \boxed{\int} \longrightarrow x_1$$

Step 2: A block element for $\dot{x}_2$

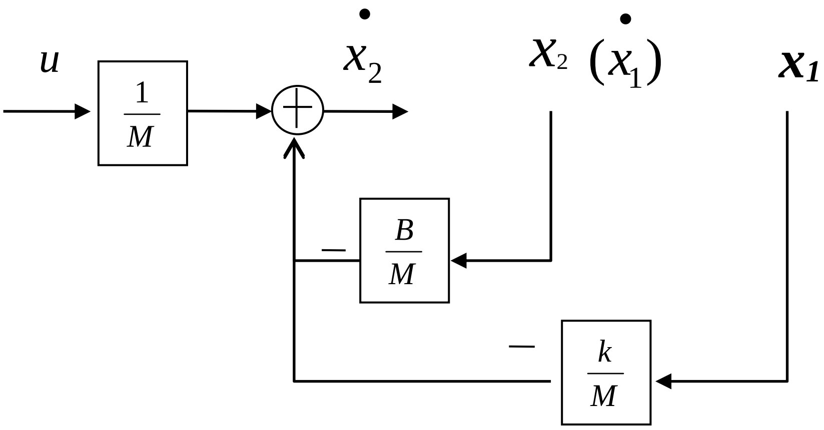
$$\dot{x}_2 \longrightarrow \boxed{\int} \longrightarrow x_2$$

Step 3: Combination of these blocks above



Step 4: A block element for the second equation

$$\dot{x}_2 = -\frac{B}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$



Step 5: Complete block diagram