University of Anbar
Engineering College
Department of Mechanical Engineering

ME 4309 - Engineering Control and Measurements

(3-3-1-0)

Fourth Stage

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Reference:

1. Automatic Control Engineering, First Edition 1961, by Francis H. Raven, McGraw Hill.
2. Measurement Systems Applications and Design, 5th edition 2003, by E. Doebelin, McGraw Hill.

Chapter 8: State space representation

1.1 State space equations

Transfer function of a 3rd order system whose numerator is a 1st order polynomial,

$$\frac{Y(s)}{U(s)} = \frac{b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Introducing an intermediate variable R(s) into the above equation leaves

$$\frac{Y(s)}{U(s)} \times \frac{R(s)}{R(s)} = \frac{R(s)}{U(s)} \times \frac{Y(s)}{R(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} \times (b_1 s + b_2)$$

1.2 Dealing with $\frac{R(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$

$$(s^3 + a_1 s^2 + a_2 s + a_3)R(s) = U(s)$$

$$\Downarrow$$

$$s^3 R(s) + a_1 s^2 R(s) + a_2 s R(s) + a_3 R(s) = U(s)$$

$$\Downarrow$$

$$\dddot{r} + a_1 \ddot{r} + a_2 \dot{r} + a_3 r = u$$

$$\Downarrow$$

$$\dddot{r} = -a_1 \ddot{r} - a_2 \dot{r} - a_3 r + u$$

Introducing state variables (note that the state space representation using this definition is called the observability form of the state space representation)

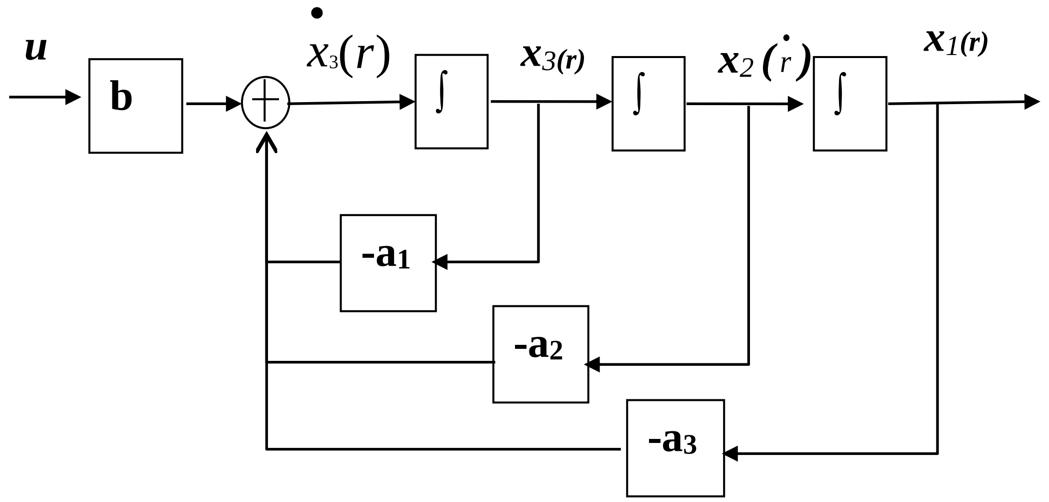
$$\left.\begin{aligned} x_1 &= r \\ x_2 &= \dot{r} \\ x_3 &= \ddot{r} \end{aligned}\right\} \qquad \left.\begin{aligned} \dot{x}_1 &= \dot{r} = x_2 \\ \dot{x}_2 &= \ddot{r} = x_3 \\ \dot{x}_3 &= \dddot{r} \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -a_1\ddot{r} - a_2\dot{r} - a_3 r + u \end{aligned}\right\}$$

State space equations

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -a_1 x_3 - a_2 x_2 - a_3 x_1 + u \end{aligned}\right\}$$

A block diagram of state space equations



1.3 Dealing with

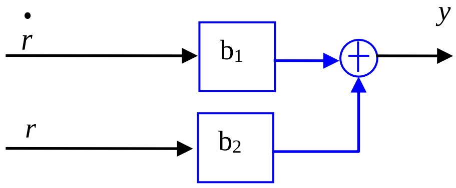
$$\frac{Y(s)}{R(s)} = b_1 s + b_2$$

$$Y(s) = b_1 sR(s) + b_2 R(s)$$

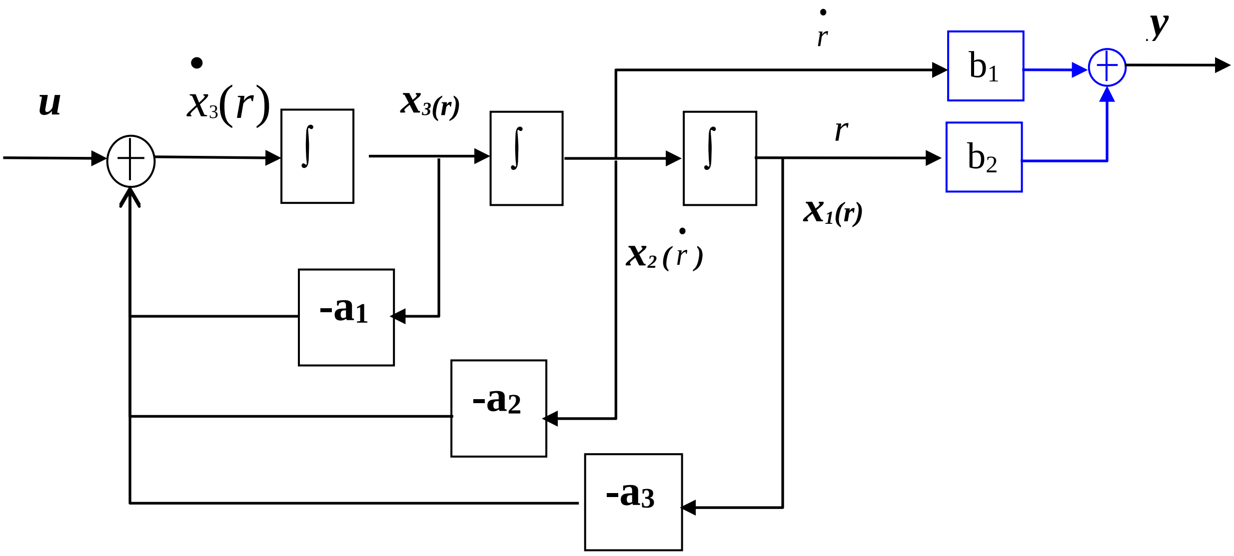
Applying L^{-1} to the above equation gives

$$y = b_1 \dot{r} + b_2 r$$

Block diagram is



1.4 Combination of two block diagrams



1.5 State space equations with matrixes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_2 & b_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{X} = A_O X + B_O u$$

$$y = C_O X$$

$$A_O = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \qquad B_O = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_O = \begin{bmatrix} b_2 & b_1 & 0 \end{bmatrix}$$

The obtained representation is called the observability form of the state space representation

1.6 Controller form of the state space representation

The transfer function of a closed-loop system is $\dfrac{4s-2}{s^3+6s^2+11s+6}$.

Using the definition of $\begin{cases} x_3 = r \\ x_2 = \dot{r} \\ x_1 = \ddot{r} \end{cases}$ to obtain the controller form of a state space representation.

Solution:

(a) **State space equations**

$$\frac{Y(s)}{U(s)} = \frac{4s-2}{s^3+6s^2+11s+6} = \frac{1}{s^3+6s^2+11s+6} \times (4s-2)$$

Let

$$R(s) = \frac{1}{s^3+6s^2+11s+6} U(s) \quad \text{and} \quad Y(s) = (4s-2)R(s)$$

Applying Inverse Laplace transform to the above expressions gives

$$\dddot{r} + 6\ddot{r} + 11\dot{r} + 6r = u \quad \text{and} \quad y = 4\dot{r} - 2r$$

In order to obtain the controller form of a state space representation, we need to define

$$\begin{cases} x_3 = r \\ x_2 = \dot{r} \\ x_1 = \ddot{r} \end{cases}$$

Hence we have

$$\dot{x}_1 = -6x_1 - 11x_2 - 6x_3 + u \quad \text{and} \quad y = 4x_2 - 2x_3$$

$$\left.\begin{array}{l} \dot{X}(t) = A_C X(t) + B_C u(t) \\ y(t) = C_C X(t) \\ \\ A_C = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \\ C_C = \begin{bmatrix} 0 & 4 & -2 \end{bmatrix} \end{array}\right]$$

2 Relationship between the state space equation and the transfer function of a control system

2.1 Establishment of relationship

The general form of state space equations is given by

$$\begin{aligned} \dot{X}(t) &= AX(t) + Bu(t) \\ y(t) &= CX(t) \end{aligned} \qquad (3.2.1)$$

Assuming the initial conditions zero and applying the Laplace transform to the above equations gives

$$\begin{aligned} sX(s) &= AX(s) + BU(s), \\ Y(s) &= CX(s) \end{aligned}$$

Moving *AX(s)* to the left hand side of the equation yields,

$$\begin{aligned} sX(s) - AX(s) &= BU(s), \\ Y(s) &= CX(s) \end{aligned}$$

$$\begin{aligned} (sI - A)X(s) &= BU(s), \\ Y(s) &= CX(s) \end{aligned}$$

$$\begin{aligned} [sI - A]X(s) &= BU(s), \\ Y(s) &= CX(s) \end{aligned} \qquad (3.2.2)$$

Then

$$X(s) = [sI - A]^{-1} BU(s) \qquad (3.2.3)$$

$\Phi(s) = [sI - A]^{-1}$ is called the <u>characteristic matrix</u> of A.

and substituting (3.2.3) into $Y(s) = CX(s)$ gives

$$Y(s) = C\Phi(s)BU(s) \qquad (3.2.4)$$

It follows that transfer function of the system is

$$T(s) = \frac{Y(s)}{U(s)} = C\Phi(s)B \qquad (3.2.6)$$

We know

$$\Phi(s) = [sI - A]^{-1} = \frac{adj[sI - A]}{\det[sI - A]}$$

Where $\det[sI - A]$ is called the determinant of $[sI - A]$, and $adj[sI - A]$ is called the adjoint of $[sI - A]$.

and

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\text{numerator } b(s)}{\text{denominator } a(s)}$$

We have

$$\frac{Y(s)}{U(s)} = \frac{C\, adj[sI - A]\, B}{\det[sI - A]} = \frac{\text{numerator } b(s)}{\text{denominator } a(s)} \qquad (3.2.7)$$

and

$$a(s) = \det[sI - A] \qquad (3.2.7a)$$

We know that $a(s)$ is the characteristics polynomial of the transfer function of a control system. This has proven that the denominator of the closed loop transfer function is the determinant of the matrix $[sI - A]$ of the state space equation. (stop here in week 9)

2.2 Explanation of some terms

(a) $\Phi(s) = [sI - A]^{-1}$ is called the <u>characteristic matrix</u> of A.

(b) $\det[sI - A] = a(s)$ is called the <u>characteristic polynomial</u> of A.

(c) <u>Eigenvalues.</u> Since $a(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$, the roots $\lambda_1, \lambda_2 \cdots \lambda_n$ are called the <u>eigenvalues</u> of the matrix A.

It is evident that the <u>eigenvalues</u> of the matrix A are also the poles of the system transfer functions because a(s) is the characteristics polynomial of the transfer function of the system.

2.3 Worked example

The transfer function of a closed-loop system is $\dfrac{4s-2}{s^3+6s^2+11s+6}$.

(a) Using $\begin{cases} x_3 = r \\ x_2 = \dot{r} \\ x_1 = \ddot{r} \end{cases}$ to obtain the state space equations (a controller form).

(b) Find its determinant.

(c) Given one of the eigenvalues is -1, evaluate the other eigenvalues.

Solution:

(a) State equation

We use the results obtained in 1.6

$$\dot{X}(t) = A_C X(t) + B_C u(t)$$
$$y(t) = C_C X(t)$$

$$A_C = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_C = \begin{bmatrix} 0 & 4 & -2 \end{bmatrix}$$

(b) Find the determinant

$$\det[sI - A_C] = \det\left[s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right] = \det\begin{bmatrix} s+6 & +11 & +6 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix} = s^3 + 6s^2 + 11s + 6$$

(c) Evaluate the eigenvalues

Given s = -1, we can use s+1 to factorize the determinant

$$\det[sI - A] = s^3 + 6s^2 + 11s + 6 = s^3 + s^2 + 5s^2 + 5s + 6s + 6$$
$$= s^2(s+1) + 5s(s+1) + 6(s+1) = (s+1)(s^2 + 5s + 6)$$

The roots of the quadratic equation are obtained with the formula below

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4\times 1\times 6}}{2\times 1} = \begin{cases} -3 \\ -2 \end{cases}.$$

$$\det[sI - A] = s^3 + 6s^2 + 11s + 6 = (s+1)(s^2 + 5s + 6) = (s+1)(s+2)(s+3)$$

So the eigenvalues are (–1, -2, -3).

3 Analytical solution to the state space equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ? \quad and \quad y(t) = ?$$

3.1 Derivation of the solutions of the system

(a) **Derivation of the solutions of state space equations *X(t)***

Assuming that the initial condition are zero.

$$X(t) = L^{-1}[X(s)] = L^{-1}\{\Phi(s)BU(s)\} = L^{-1}\{[sI - A]^{-1} BU(s)\} \tag{3.3.1}$$

Where $\Phi(s) = [sI - A]^{-1}$

Let

$$\Phi(t) = L^{-1}\Phi(s) \tag{3.3.1a}$$

This is called the <u>transition matrix.</u>

Since $$L^{-1}\{\Phi(s)\ BU(s)\} = \Phi(t) * Bu(t)$$

$$X(t) = \Phi(t) * BU(t) \tag{3.3.2}$$

(b) **Derivation of the solution of the system (*y(t)*)**

$$y(t) = CX(t)$$

3.2 Worked example to find the solution of a 2nd system

Given a 2nd order system $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T$, $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(a) Find the state transition matrix Φ(t).

(b) Suppose a unit step input to the system, evaluate the state equation solutions, and
(c) The solution of the system.

Solution:

(a) Find the state transition matrix Φ(t)

$$\Phi(s)=[sI-A]^{-1}=\left[s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}-1&0\\0&-2\end{bmatrix}\right]^{-1}=\begin{bmatrix}s+1&0\\0&s+2\end{bmatrix}^{-1}=$$

$$=\frac{adj\begin{bmatrix}s+1&0\\0&s+2\end{bmatrix}}{\det\begin{vmatrix}s+1&0\\0&s+2\end{vmatrix}}=\frac{\begin{bmatrix}s+2&0\\0&s+1\end{bmatrix}}{(s+1)(s+2)}=\begin{bmatrix}\frac{1}{s+1}&0\\0&\frac{1}{s+2}\end{bmatrix} \qquad (3.3.3)$$

Where

$$adj\begin{bmatrix}s+1&0\\0&s+2\end{bmatrix}=(-1)^{i+j}\times(\text{minor of }\Phi_{ij})=(-1)^{i+j}\times\begin{bmatrix}s+2&0\\0&s+1\end{bmatrix}=$$

$$=\begin{bmatrix}(-1)^{1+1}(s+2)&(-1)^{1+2}\times 0\\(-1)^{2+1}\times 0&(-1)^{2+2}\times(s+1)\end{bmatrix}=\begin{bmatrix}s+2&0\\0&s+1\end{bmatrix}$$

(b) **Solutions to the state space equation**

Since the input is a unit step, we have *U(s)=1/s*,

$$X(t)=\Phi(t)*BU(t)=L^{-1}\{\Phi(s)BU(s)\}=L^{-1}\{\begin{bmatrix}\frac{1}{s+1}&0\\0&\frac{1}{s+2}\end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{s}\}=$$

$$=L^{-1}\{\begin{bmatrix}0\\\frac{1}{s(s+2)}\end{bmatrix}\}=L^{-1}\begin{bmatrix}0\\\frac{0.5}{s}-\frac{0.5}{s+2}\end{bmatrix}=\begin{bmatrix}0\\0.5-0.5e^{-2t}\end{bmatrix} \qquad (3.3.5)$$

Now we can write down the solution to the state space equations subject to the unit step input,

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0.5 - 0.5e^{-2t} \end{bmatrix}. \tag{3.3.6}$$

(c) **Solution of the system *y(t)***

$$y(t) = CX(t) = (1 \quad 1)\begin{pmatrix} 0 \\ 0.5 - 0.5e^{-2t} \end{pmatrix} = 0.5 - 0.5e^{-2t} \tag{3.3.7}$$

You should be aware that if the initial conditions are not zero, then you need to seek the formulas for the solutions of the control systems in the relevant reference books.