University of Anbar

Engineering College

Department of Mechanical Engineering

ME 3301 - Engineering Analysis (3-3-1-0)

Third Stage

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Reference: Dennis G. Zill, Warren S. Wright, (2012)-Advanced Engineering Mathematics

Chapter 2: Linear Differential Equations

1 Exact ODEs.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'dx$ as in Sec. 1.3)

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2}$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

$$u(x, y) = c. \tag{3}$$

This is called an implicit solution

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

$$\text{(a)}\quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)}\quad \frac{\partial u}{\partial y} = N. \tag{4}$$

The condition to be an exact differential equation is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

$$u = \int M\,dx + k(y); \tag{6}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k. (See Example 1, below.)

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use (4b). Then, instead of (6), we first have by integration with respect to y

$$u = \int N\,dy + l(x). \tag{6*}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

EXAMPLE 1 An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$
$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$
$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \cos(x+y)\,dx + (\cos(x+y) + 3y^2 + 2y)\,dy = 0. \tag{9}$$

2 Linear ODEs.

A first-order ODE is said to be linear if it can be brought into the form

$$y' + p(x)y = r(x), \tag{1}$$

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \tag{2}$$

and is called **homogeneous**. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)\,dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \tag{3}$$

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous**.

The desired solution formula

$$y(x) = e^{-h}\left(\int e^{h} r\,dx + c\right), \qquad h = \int p(x)\,dx. \tag{4}$$

Example 1:

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$h = \int p\,dx = \int \tan x\,dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\,dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$.

Example 2:

Electric Circuit

Model the ***RL*-circuit** in Fig. 19 and solve the resulting ODE for the current $I(t)$ A (amperes), where t is time. Assume that the circuit contains as an **EMF** $E(t)$ (electromotive force) a battery of $E = 48$ V (volts), which is constant, a ***resistor*** of $R = 11\ \Omega$ (ohms), and an ***inductor*** of $L = 0.1$ H (henrys), and that the current is initially zero.

Physical Laws. A current I in the circuit causes a **voltage drop** RI across the resistor (**Ohm's law**) and a voltage drop $LI' = L\,dI/dt$ across the conductor, and the sum of these two voltage drops equals the EMF (**Kirchhoff's Voltage Law, KVL**).

Solution. According to these laws the model of the RL-circuit is $LI' + RI = E(t)$, in standard form

$$I' + \frac{R}{L} I = \frac{E(t)}{L}. \tag{6}$$

We can solve this linear ODE by (4) with $x = t$, $y = I$, $p = R/L$, $h = (R/L)t$, obtaining the general solution

$$I = e^{-(R/L)t}\left(\int e^{(R/L)t}\frac{E(t)}{L}dt + c\right).$$

By integration,

$$I = e^{-(R/L)t}\left(\frac{E}{L}\frac{e^{(R/L)t}}{R/L} + c\right) = \frac{E}{R} + ce^{-(R/L)t}. \tag{7}$$

In our case, $R/L = 11/0.1 = 110$ and $E(t) = 48/0.1 = 480 = \text{const}$; thus,

$$I = \tfrac{48}{11} + ce^{-110t}.$$

In modeling, one often gets better insight into the nature of a solution (and smaller roundoff errors) by inserting given numeric data only near the end. Here, the general solution (7) shows that the current approaches the limit $E/R = 48/11$ faster the larger R/L is, in our case, $R/L = 11/0.1 = 110$, and the approach is very fast, from below if $I(0) < 48/11$ or from above if $I(0) > 48/11$. If $I(0) = 48/11$, the solution is constant (48/11 A). See Fig. 19.

The initial value $I(0) = 0$ gives $I(0) = E/R + c = 0$, $c = -E/R$ and the particular solution

$$I = \frac{E}{R}(1 - e^{-(R/L)t}), \qquad \text{thus} \qquad I = \frac{48}{11}(1 - e^{-110t}). \tag{8}$$