**University of Anbar**

**Engineering College**

**Department of Mechanical Engineering**

# ME 3301 - Engineering Analysis (3-3-1-0)

## Third Stage

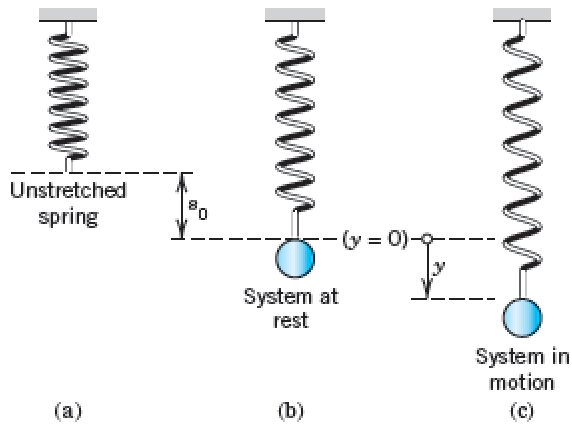
Prepared by: *Dr. Khaldoon F. Brethee*

**Reference:** Dennis G. Zill, Warren S. Wright, (2012)-Advanced Engineering Mathematics

# Chapter 4: Modeling of Free Oscillations of a Mass–Spring System

## 1 Modeling of Free Oscillations of a Mass–Spring System

We take an ordinary coil spring that resists extension as well as compression. We suspend it vertically from a fixed support and attach a body at its lower end, for instance, an iron ball, as shown in Fig. 33. We let y=0 denote the position of the ball when the system is at rest (Fig. 33b). Furthermore, we choose ***the downward direction as positive***, thus regarding downward forces as *positive* and upward forces as *negative*.




We now let the ball move, as follows. We pull it down by an amount $y > 0$ (Fig. 33c). This causes a spring force

**(1)** $$F_1 = -ky$$ **(Hooke's law[2])**

The motion of our mass–spring system is determined by **Newton's second law**

**(2)** $$\text{Mass} \times \text{Acceleration} = my'' = \text{Force}$$

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the ball.

## ODE of the Undamped System

Every system has damping. Otherwise it would keep moving forever. But if the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping. Then Newton's law with $F = -F_1$ gives the model $my'' = -F_1 = -ky$; thus

$$my'' + ky = 0. \tag{3}$$

This is a homogeneous linear ODE with constant coefficients. A general solution is obtained as in Sec. 2.2, namely (see Example 6 in Sec. 2.2)

$$y(t) = A \cos \omega_0 t + B \sin \omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4}$$

This motion is called a **harmonic oscillation** (Fig. 34). Its *frequency* is $f = \omega_0/2\pi$ Hertz[3] (= cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$. The frequency $f$ is called the **natural frequency** of the system. (We write $\omega_0$ to reserve $\omega$ for Sec. 2.8.)

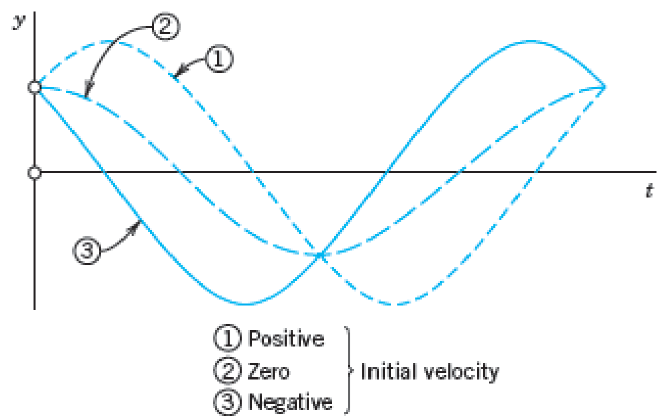



**Fig. 34.** Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

$$y(t) = C \cos(\omega_0 t - \delta) \tag{4*}$$

with $C = \sqrt{A^2 + B^2}$ and phase angle $\delta$, where $\tan \delta = B/A$. This follows from the addition formula (6) in App. 3.1.

## Example

### Harmonic Oscillation of an Undamped Mass–Spring System

If a mass–spring system with an iron ball of weight $W = 98$ nt (about 22 lb) can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m (about 43 in.), how many cycles per minute will the system execute? What will its motion be if we pull the ball down from rest by 16 cm (about 6 in.) and let it start with zero initial velocity?

***Solution.*** Hooke's law (1) with $W$ as the force and 1.09 meter as the stretch gives $W = 1.09k$; thus $k = W/1.09 = 98/1.09 = 90$ [kg/sec$^2$] $= 90$ [nt/meter]. The mass is $m = W/g = 98/9.8 = 10$ [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] $= 29$ [cycles/min].

From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16 \cos 3t \text{ [meter]} \qquad \text{or} \qquad 0.52 \cos 3t \text{ [ft]} \qquad \text{(Fig. 35).}$$

## ODE of the Damped System

To our model $my'' = -ky$ we now add a damping force

$$F_2 = -cy',$$

obtaining $my'' = -ky - cy'$; thus the ODE of the damped mass–spring system is

$$my'' + cy' + ky = 0. \qquad (5) \qquad \text{(Fig. 36)}$$

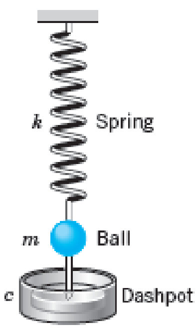



**Fig. 36.** Damped system

Physically this can be done by connecting the ball to a dashpot; see Fig. 36. We assume this damping force to be proportional to the velocity $y' = dy/dt$. This is generally a good approximation for small velocities.

The constant $c$ is called the *damping constant*. Let us show that $c$ is positive. Indeed, the damping force $F_2 = -cy'$ acts *against* the motion; hence for a downward motion we have $y' > 0$ which for positive $c$ makes $F$ negative (an upward force), as it should be. Similarly, for an upward motion we have $y' < 0$ which, for $c > 0$ makes $F_2$ positive (a downward force).

The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by $m$)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$(6) \quad \lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

| | | | |
|---|---|---|---|
| **Case I.** | $c^2 > 4mk$. | *Distinct real roots* $\lambda_1, \lambda_2$. | **(Overdamping)** |
| **Case II.** | $c^2 = 4mk$. | *A real double root.* | **(Critical damping)** |
| **Case III.** | $c^2 < 4mk$. | *Complex conjugate roots.* | **(Underdamping)** |

## Case I. Overdamping

If the damping constant $c$ is so large that $c^2 > 4mk$, then $\lambda_1$ and $\lambda_2$ are distinct real roots. In this case the corresponding general solution of (5) is

$$(7) \qquad y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0, \beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the *static equilibrium position* ($y = 0$). Figure 37 shows (7) for some typical initial conditions.

## Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{8}$$

## Case III. Underdamping

This is the most interesting case. It occurs if the damping constant $c$ is so small that $c^2 < 4mk$. Then $\beta$ in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0). \tag{9}$$

(We now write $\omega^*$ to reserve $\omega$ for driving and electromotive forces in Secs. 2.8 and 2.9.) The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta) \tag{10}$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^* t - \delta$ is an integer multiple of $\pi$ because these are the points at which $\cos (\omega^* t - \delta)$ equals 1 or $-1$.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller $c\ (>0)$ is, the larger is $\omega^*$ and the more rapid the oscillations become. If $c$ approaches 0, then $\omega^*$ approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

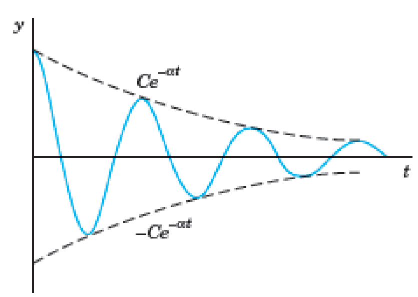



**Fig. 39.** Damped oscillation in Case III [see (10)]

## Example

How does the motion in Example 1 change if we change the damping constant $c$ from one to another of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

$$\text{(I) } c = 100 \text{ kg/sec}, \qquad \text{(II) } c = 60 \text{ kg/sec}, \qquad \text{(III) } c = 10 \text{ kg/sec}.$$

***Solution.*** It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

(I) With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$. It has the roots $-9$ and $-1$. This gives the general solution

$$y = c_1 e^{-9t} + c_2 e^{-t}. \qquad \text{We also need} \qquad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

It approaches 0 as $t \to \infty$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root $-3$. Hence the corresponding general solution is

$$y = (c_1 + c_2 t)e^{-3t}. \qquad \text{We also need} \qquad y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}.$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

**(III)** The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical $c$, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

This gives the general solution

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0, \quad B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 40. ■

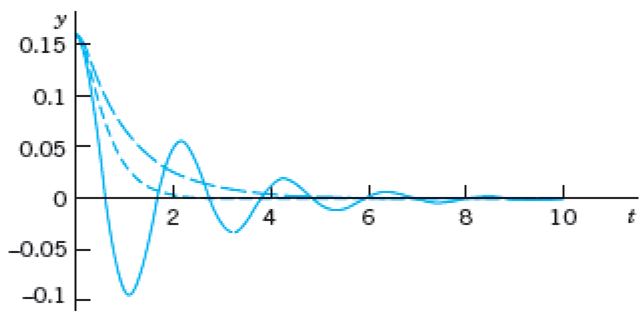



**Fig. 40.** The three solutions in Example 2

## 2 Modeling: Forced Oscillations. Resonance

$$my'' + cy' + ky = 0.$$

We now extend our model by including an additional force, that is, the external force $r(t)$, on the right. Then we have

$$my'' + cy' + ky = r(t). \quad (2^*)$$

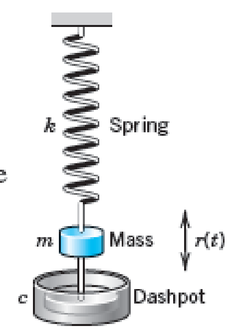



**Fig. 53.** Mass on a spring

Mechanically this means that at each instant $t$ the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the nonhomogeneous ODE

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2}$$

## Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution $y_h$ of the homogeneous ODE (1) plus any solution $y_p$ of (2). To find $y_p$, we use the method of undetermined coefficients (Sec. 2.7), starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{3}$$

By differentiating this function (chain rule!) we obtain

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$

$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting $y_p$, $y_p'$, and $y_p''$ into (2) and collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cosine terms on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right. This gives the two equations

$$\begin{aligned} (k - m\omega^2)a + \quad \omega cb \quad &= F_0 \\ -\omega ca \quad + (k - m\omega^2)b &= 0 \end{aligned} \tag{4}$$

for determining the unknown coefficients $a$ and $b$. This is a linear system. We can solve it by elimination. To eliminate $b$, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate $a$, multiply (the first equation by $\omega c$ and the second by $k - m\omega^2$ and add to get

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(k - m\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for $a$ and $b$,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0\ (> 0)$ as in Sec. 2.4, then $k = m\omega_0^2$ and we obtain

$$\text{(5)} \qquad a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}.$$

We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

$$\text{(6)} \qquad y(t) = y_h(t) + y_p(t).$$

## Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0{}^2 = k/m$)

$$\text{(7)} \qquad y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t.$$

Here we must assume that $\omega^2 \neq \omega_0{}^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4]. From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

$$\text{(8)} \qquad y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

**Resonance.** We discuss (7). We see that the maximum amplitude of $y_p$ is (put $\cos \omega t = 1$)

$$a_0 = \frac{F_0}{k}\rho \qquad \text{where} \qquad \rho = \frac{1}{1-(\omega/\omega_0)^2}. \tag{9}$$

$a_0$ depends on $\omega$ and $\omega_0$. If $\omega \to \omega_0$, then $\rho$ and $a_0$ tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance**. $\rho$ is called the **resonance factor** (Fig. 54), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution $y_p$ and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

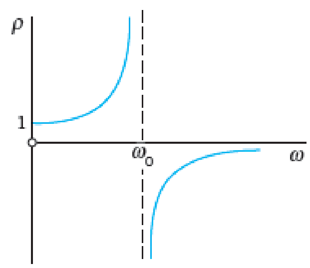
In the case of resonance the nonhomogeneous ODE (2) becomes

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos \omega_0 t. \tag{10}$$

Then (7) is no longer valid, and, from the Modification Rule in Sec. 2.7, we conclude that a particular solution of (10) is of the form

$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

Resonance factor $\rho(\omega)$




By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 55)

$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \tag{11}$$

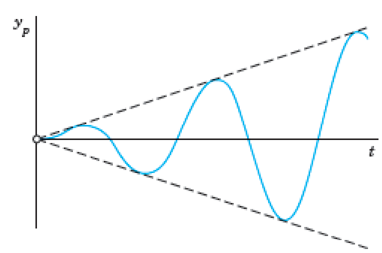



**Fig. 55.** Particular solution in the case of resonance

We see that, because of the factor $t$, the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system. We shall return to this practical aspect of resonance later in this section.

**Beats.** Another interesting and highly important type of oscillation is obtained if $\omega$ is close to $\omega_0$. Take, for example, the particular solution [see (8)]

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t) \qquad (\omega \neq \omega_0). \tag{12}$$

Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

Since $\omega$ is close to $\omega_0$, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 56, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

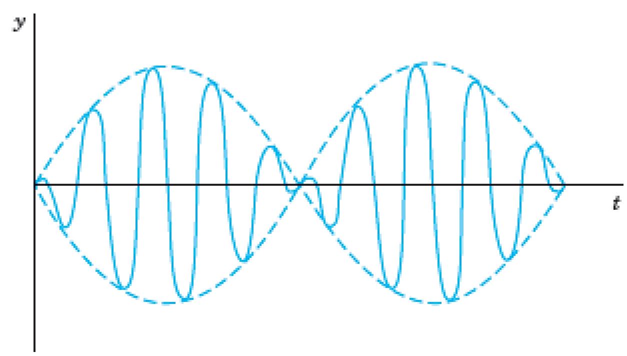



**Fig. 56.** Forced undamped oscillation when the difference of the input and natural frequencies is small (**"beats"**)

## Case 2. Damped Forced Oscillations

If the damping of the mass–spring system is not negligibly small, we have $c > 0$ and a damping term $cy'$ in (1) and (2). Then the general solution $y_h$ of the homogeneous ODE (1) approaches zero as $t$ goes to infinity, as we know from Sec. 2.4. Practically, it is zero after a sufficiently long time. Hence the "**transient solution**" (6) of (2), given by $y = y_h + y_p$, approaches the "**steady-state solution**" $y_p$. This proves the following.

To study the amplitude of $y_p$ as a function of $\omega$, we write (3) in the form

(13) $$y_p(t) = C^* \cos(\omega t - \eta).$$

$C^*$ is called the **amplitude** of $y_p$ and $\eta$ the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

(14)
$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}},$$
$$\tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega^2)}.$$

Let us see whether $C^*(\omega)$ has a maximum and, if so, find its location and then its size. We denote the radicand in the second root in $C^*$ by $R$. Equating the derivative of $C^*$ to zero, we obtain

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2}R^{-3/2}\right)[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2].$$

The expression in the brackets [. . .] is zero if

(15) $$c^2 = 2m^2(\omega_0^2 - \omega^2) \qquad (\omega_0^2 = k/m).$$

By reshuffling terms we have

$$2m^2\omega^2 = 2m^2{\omega_0}^2 - c^2 = 2mk - c^2.$$

The right side of this equation becomes negative if $c^2 > 2mk$, so that then (15) has no real solution and $C^*$ decreases monotone as $\omega$ increases, as the lowest curve in Fig. 57 shows. If $c$ is smaller, $c^2 < 2mk$, then (15) has a real solution $\omega = \omega_{\max}$, where

(15*) $$\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m^2}.$$

From (15*) we see that this solution increases as $c$ decreases and approaches $\omega_0$ as $c$ approaches zero. See also Fig. 57.

The size of $C^*(\omega_{\max})$ is obtained from (14), with $\omega^2 = \omega_{\max}^2$ given by (15*). For this $\omega^2$ we obtain in the second radicand in (14) from (15*)

$$m^2(\omega_0^2 - \omega_{\max}^2)^2 = \frac{c^4}{4m^2} \qquad \text{and} \qquad \omega_{\max}^2 c^2 = \left(\omega_0^2 - \frac{c^2}{2m^2}\right)c^2.$$

The sum of the right sides of these two formulas is

$$(c^4 + 4m^2\omega_0^2c^2 - 2c^4)/(4m^2) = c^2(4m^2\omega_0^2 - c^2)/(4m^2).$$

Substitution into (14) gives

$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}. \tag{16}$$

We see that $C^*(\omega_{\max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2\omega_0^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as $c^2$ ($<2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1. Figure 57 shows the **amplification** $C^*/F_0$ (ratio of the amplitudes of output and input) as a function of $\omega$ for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant $c$.

Figure 58 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.

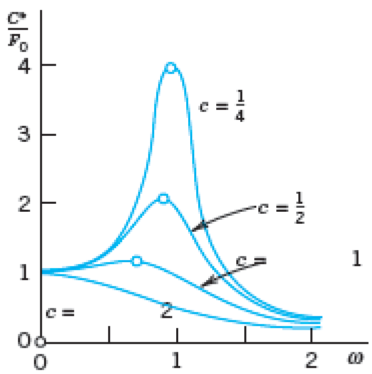



**Fig. 57.** Amplification $C^*/F_0$ as a function of $\omega$ for $m = 1, k = 1$, and various values of the damping constant $c$

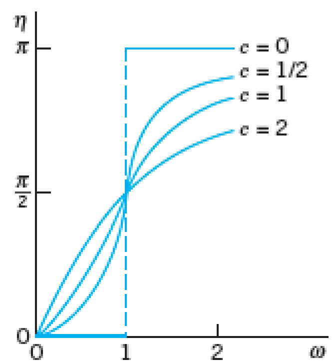



**Fig. 58.** Phase lag $\eta$ as a function of $\omega$ for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant $c$