17.1 Complex Numbers

Introduction You have undoubtedly encountered complex numbers in your earlier courses in mathematics. When you first learned to solve a quadratic equation $ax^2 + bx + c = 0$ by the quadratic formula, you saw that the roots of the equation are not real; that is, complex, whenever the discriminant $b^2 - 4ac$ is negative. So, for example, simple equations such as $x^2 + 5 = 0$ and $x^2 + x + 1 = 0$ have no real solutions. For example, the roots of the last equation are $-\frac{1}{2} + \frac{\sqrt{-3}}{2}$ and $-\frac{1}{2} + \frac{\sqrt{-3}}{2}$. If it is assumed that $\sqrt{-3} = \sqrt{3}\sqrt{-1}$, then the roots are written $-\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}$.

A Definition Two hundred years ago, around the time that complex numbers were gaining some respectability in the mathematical community, the symbol i was originally used as a disguise for the embarrassing symbol $\sqrt{-1}$. We now simply say that i is the **imaginary unit** and define it by the property $i^2 = -1$. Using the imaginary unit, we build a general complex number out of two real numbers.

Definition 17.1.1 Complex Number

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

Terminology The number i in Definition 17.1.1 is called the **imaginary unit**. The real number x in $z = x + iy$ is called the **real part** of z; the real number y is called the **imaginary part** of z. The real and imaginary parts of a complex number z are abbreviated $\text{Re}(z)$ and $\text{Im}(z)$, respectively. For example, if $z = 4 - 9i$, then $\text{Re}(z) = 4$ and $\text{Im}(z) = -9$. A real constant multiple of the imaginary unit is called a **pure imaginary number**. For example, $z = 6i$ is a pure imaginary number. Two complex numbers are **equal** if their real and imaginary parts are equal. Since this simple concept is sometimes useful, we formalize the last statement in the next definition.

Note: The imaginary part of $z = 4 - 9i$ is -9 not $-9i$.

Definition 17.1.2 Equality

Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are **equal**, $z_1 = z_2$, if

$$\text{Re}(z_1) = \text{Re}(z_2) \quad \text{and} \quad \text{Im}(z_1) = \text{Im}(z_2).$$

A complex number $x + iy = 0$ if $x = 0$ and $y = 0$.

Arithmetic Operations Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, these operations are defined as follows.

Addition: $$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction: $$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Multiplication: $$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2) \end{aligned}$$

Division: $$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \end{aligned}$$

The familiar commutative, associative, and distributive laws hold for complex numbers.

Commutative laws: $\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$

Associative laws: $\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

In view of these laws, there is no need to memorize the definitions of addition, subtraction, and multiplication. To add (subtract) two complex numbers, we simply add (subtract) the corresponding real and imaginary parts. To multiply two complex numbers, we use the distributive law and the fact that $i^2 = -1$.

EXAMPLE 1 Addition and Multiplication

If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find **(a)** $z_1 + z_2$ and **(b)** $z_1 z_2$.

SOLUTION **(a)** By adding the real and imaginary parts of the two numbers, we get

$$(2 + 4i) + (-3 + 8i) = (2 - 3) + (4 + 8)i = -1 + 12i.$$

(b) Using the distributive law, we have

$$\begin{aligned} (2 + 4i)(-3 + 8i) &= (2 + 4i)(-3) + (2 + 4i)(8i) \\ &= -6 - 12i + 16i + 32i^2 \\ &= (-6 - 32) + (16 - 12)i = -38 + 4i. \end{aligned}$$

≡

There is also no need to memorize the definition of division, but before discussing that we need to introduce another concept.

Conjugate If z is a complex number, then the number obtained by changing the sign of its imaginary part is called the **complex conjugate** or, simply, the **conjugate** of z. If $z = x + iy$, then its conjugate is

$$\bar{z} = x - iy.$$

For example, if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$; if $z = -5 - i$, then $\bar{z} = -5 + i$. If z is a real number, say $z = 7$, then $\bar{z} = 7$. From the definition of addition it can be readily shown that the conjugate of a sum of two complex numbers is the sum of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

Moreover, we have the additional three properties

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

The definitions of addition and multiplication show that the sum and product of a complex number z and its conjugate $\bar{z}$ are also real numbers:

$$z + \bar{z} = (x + iy) + (x - iy) = 2x \tag{1}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2. \tag{2}$$

The difference between a complex number z and its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy. \tag{3}$$

Since $x = \text{Re}(z)$ and $y = \text{Im}(z)$, (1) and (3) yield two useful formulas:

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

However, (2) is the important relationship that enables us to approach division in a more practical manner: To divide z_1 by z_2, we multiply both numerator and denominator of z_1/z_2 by the conjugate of z_2. This procedure is illustrated in the next example.

EXAMPLE 2 Division

If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find **(a)** $\dfrac{z_1}{z_2}$ and **(b)** $\dfrac{1}{z_1}$.

SOLUTION In both parts of this example we shall multiply both numerator and denominator by the conjugate of the denominator and then use (2).

(a)
$$\frac{2-3i}{4+6i} = \frac{2-3i}{4+6i}\,\frac{4-6i}{4-6i} = \frac{8-12i-12i+18i^2}{16+36}$$
$$= \frac{-10-24i}{52} = -\frac{5}{26} - \frac{6}{13}i.$$

(b)
$$\frac{1}{2-3i} = \frac{1}{2-3i}\,\frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2}{13} + \frac{3}{13}i.$$
≡

Geometric Interpretation

A complex number $z = x + iy$ is uniquely determined by an *ordered pair* of real numbers (x, y). The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. For example, the ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordinate plane. But, as we saw in Section 7.1, an ordered pair of real numbers can be interpreted as the components of a vector. Thus, a complex number $z = x + iy$ can also be viewed as a *vector* whose initial point is the origin and whose terminal point is (x, y). The coordinate plane illustrated in FIGURE 17.1.1 is called the **complex plane** or simply the ***z*-plane**. The horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. The length of a vector z, or the distance from the origin to the point (x, y), is clearly $\sqrt{x^2 + y^2}$. This real number is given a special name.

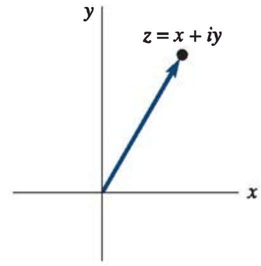

FIGURE 17.1.1 z as a position vector

Definition 17.1.3 Modulus or Absolute Value

The **modulus** or **absolute value** of $z = x + iy$, denoted by $|z|$, is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{4}$$

EXAMPLE 3 Modulus of a Complex Number

If $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$.
≡

As FIGURE 17.1.2 shows, the sum of the vectors z_1 and z_2 is the vector $z_1 + z_2$. For the triangle given in the figure, we know that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be longer than the sum of the remaining two sides. In symbols this is

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{5}$$

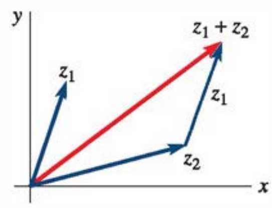

FIGURE 17.1.2 Sum of vectors

The result in (5) is known as the **triangle inequality** and extends to any finite sum:

$$|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|. \tag{6}$$

Using (5) on $z_1 + z_2 + (-z_2)$, we obtain another important inequality:

$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{7}$$

Remarks

Many of the properties of the real system hold in the complex number system, but there are some remarkable differences as well. For example, we cannot compare two complex numbers $z_1 = x_1 + iy_1, y_1 \neq 0$, and $z_2 = x_2 + iy_2, y_2 \neq 0$, by means of inequalities. In other words, statements such as $z_1 < z_2$ and $z_2 \geq z_1$ have no meaning except in the case when the two numbers z_1 and z_2 are real. We can, however, compare the absolute values of two complex numbers. Thus, if $z_1 = 3 + 4i$ and $z_2 = 5 - i$, then $|z_1| = 5$ and $|z_2| = \sqrt{26}$, and consequently $|z_1| < |z_2|$. This last inequality means that the point (3, 4) is closer to the origin than is the point $(5, -1)$.

17.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-38.

In Problems 1–26, write the given number in the form $a + ib$.

1. $2i^3 - 3i^2 + 5i$

2. $3i^5 - i^4 + 7i^3 - 10i^2 - 9$

3. i^8

4. i^{11}

5. $(5 - 9i) + (2 - 4i)$

6. $3(4 - i) - 3(5 + 2i)$

7. $i(5 + 7i)$

8. $i(4 - i) + 4i(1 + 2i)$

9. $(2 - 3i)(4 + i)$

10. $(\frac{1}{2} - \frac{1}{4}i)(\frac{2}{3} + \frac{5}{3}i)$

11. $(2 + 3i)^2$

12. $(1 - i)^3$

13. $\dfrac{2}{i}$

14. $\dfrac{i}{1 + i}$

15. $\dfrac{2 - 4i}{3 + 5i}$

16. $\dfrac{10 - 5i}{6 + 2i}$

17. $\dfrac{(3 - i)(2 + 3i)}{1 + i}$

18. $\dfrac{(1 + i)(1 - 2i)}{(2 + i)(4 - 3i)}$

19. $\dfrac{(5 - 4i) - (3 + 7i)}{(4 + 2i) + (2 - 3i)}$

20. $\dfrac{(4 + 5i) + 2i^3}{(2 + i)^2}$

21. $i(1 - i)(2 - i)(2 + 6i)$

22. $(1 + i)^2(1 - i)^3$

23. $(3 + 6i) + (4 - i)(3 + 5i) + \dfrac{1}{2 - i}$

24. $(2 + 3i)\left(\dfrac{2 - i}{1 + 2i}\right)^2$

25. $\left(\dfrac{i}{3 - i}\right)\left(\dfrac{1}{2 + 3i}\right)$

26. $\dfrac{1}{(1 + i)(1 - 2i)(1 + 3i)}$

In Problems 27–32, let $z = x + iy$. Find the indicated expression.

27. $\text{Re}(1/z)$

28. $\text{Re}(z^2)$

29. $\text{Im}(2z + 4\bar{z} - 4i)$

30. $\text{Im}(\bar{z}^2 + z^2)$

31. $|z - 1 - 3i|$

32. $|z + 5\bar{z}|$

In Problems 33–38, use Definition 17.1.2 to find a complex number z satisfying the given equation.

33. $2z = i(2 + 9i)$

34. $z - 2\bar{z} + 7 - 6i = 0$

35. $z^2 = i$

36. $\bar{z}^2 = 4z$

37. $z + 2\bar{z} = \dfrac{2 - i}{1 + 3i}$

38. $\dfrac{z}{1 + \bar{z}} = 3 + 4i$

In Problems 39 and 40, determine which complex number is closer to the origin.

39. $10 + 8i, \quad 11 - 6i$

40. $\frac{1}{2} - \frac{1}{4}i, \quad \frac{2}{3} + \frac{1}{6}i$

41. Prove that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane.

42. Show for all complex numbers z on the circle $x^2 + y^2 = 4$ that $|z + 6 + 8i| \leq 12$.

For Discussion

43. For n a nonnegative integer, i^n can be one of four values: i, -1, $-i$, and 1. In each of the following four cases express the integer exponent n in terms of the symbol k, where $k = 0, 1, 2, \ldots$.
(a) $i^n = i$ **(b)** $i^n = -1$ **(c)** $i^n = -i$ **(d)** $i^n = 1$

44. **(a)** Without doing any significant work such as multiplying out or using the binomial theorem, think of an easy way of evaluating $(1 + i)^8$.
(b) Use your method in part (a) to evaluate $(1 + i)^{64}$.

17.2 Powers and Roots

Introduction Recall from calculus that a point (x, y) in rectangular coordinates can also be expressed in terms of polar coordinates (r, θ). We shall see in this section that the ability to express a complex number z in terms of r and θ greatly facilitates finding powers and roots of z.

Polar Form Rectangular coordinates (x, y) and polar coordinates (r, θ) are related by the equations $x = r\cos\theta$ and $y = r\sin\theta$ (see Section 14.1). Thus a nonzero complex number $z = x + iy$ can be written as $z = (r\cos\theta) + i(r\sin\theta)$ or

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

We say that (1) is the **polar form** of the complex number z. We see from **FIGURE 17.2.1** that the polar coordinate r can be interpreted as the distance from the origin to the point (x, y). In other words, we adopt the convention that r is never negative so that we can take r to be the modulus of z; that is, $r = |z|$. The angle θ of inclination of the vector z measured in radians from the positive real axis is positive when measured counterclockwise and negative when measured clockwise. The angle θ is called an **argument** of z and is written $\theta = \arg z$. From Figure 17.2.1 we see that an argument of a complex number must satisfy the equation $\tan\theta = y/x$. The solutions of this equation are not unique, since if θ_0 is an argument of z, then necessarily the angles $\theta_0 \pm 2\pi$, $\theta_0 \pm 4\pi, \ldots$, are also arguments. The argument of a complex number in the interval $-\pi < \theta \le \pi$ is called the **principal argument** of z and is denoted by Arg z. For example, $\text{Arg}(i) = \pi/2$.

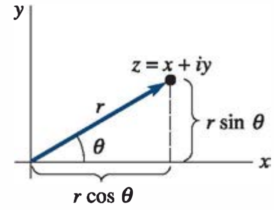

FIGURE 17.2.1 Polar coordinates

EXAMPLE 1 A Complex Number in Polar Form

Express $1 - \sqrt{3}i$ in polar form.

SOLUTION With $x = 1$ and $y = -\sqrt{3}$, we obtain $r = |z| = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$. Now since the point $(1, -\sqrt{3})$ lies in the fourth quadrant, we can take the solution of $\tan\theta = -\sqrt{3}/1 = -\sqrt{3}$ to be $\theta = \arg z = 5\pi/3$. It follows from (1) that a polar form of the number is

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right).$$

As we see in **FIGURE 17.2.2**, the argument of $1 - \sqrt{3}i$ that lies in the interval $(-\pi, \pi]$, the principal argument of z, is $\text{Arg}\, z = -\pi/3$. Thus, an alternative polar form of the complex number is

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$

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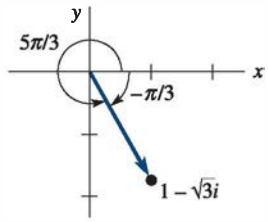


FIGURE 17.2.2 Two arguments of $z = 1 - \sqrt{3}i$ in Example 1

Multiplication and Division

The polar form of a complex number is especially convenient to use when multiplying or dividing two complex numbers. Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \tag{2}$$

and for $z_2 \ne 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]. \tag{3}$$

From the addition formulas from trigonometry, (2) and (3) can be rewritten, in turn, as

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{4}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \tag{5}$$

Inspection of (4) and (5) shows that

$$|z_1z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{6}$$

and

$$\arg(z_1z_2) = \arg z_1 + \arg z_2, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{7}$$

EXAMPLE 2 Argument of a Product and of a Quotient

We have seen that $\text{Arg}\, z_1 = \pi/2$ for $z_1 = i$. In Example 1 we saw that $\text{Arg}\, z_2 = -\pi/3$ for $z_2 = 1 - \sqrt{3}i$. Thus, for

$$z_1z_2 = i(1 - \sqrt{3}i) = \sqrt{3} + i \quad \text{and} \quad \frac{z_1}{z_2} = \frac{i}{1 - \sqrt{3}i} = -\frac{\sqrt{3}}{4} + \frac{1}{4}i$$

it follows from (7) that

$$\arg(z_1z_2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} - \left(-\frac{\pi}{3}\right) = \frac{5\pi}{6}.$$

≡

In Example 2 we used the principal arguments of z_1 and z_2 and obtained $\arg(z_1z_2) = \operatorname{Arg}(z_1z_2)$ and $\arg(z_1/z_2) = \operatorname{Arg}(z_1/z_2)$. It should be observed, however, that this was a coincidence. Although (7) is true for any arguments of z_1 and z_2, *it is not true*, in general, that $\operatorname{Arg}(z_1z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ and $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$. See Problem 39 in Exercises 17.2.

Powers of z We can find integer powers of the complex number z from the results in (4) and (5). For example, if $z = r(\cos\theta + i\sin\theta)$, then with $z_1 = z$ and $z_2 = z$, (4) gives

$$z^2 = r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] = r^2(\cos 2\theta + i\sin 2\theta).$$

Since $z^3 = z^2z$, it follows that

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta).$$

Moreover, since $\arg(1) = 0$, it follows from (5) that

$$\frac{1}{z^2} = z^{-2} = r^{-2}[\cos(-2\theta) + i\sin(-2\theta)].$$

Continuing in this manner, we obtain a formula for the nth power of z for any integer n:

$$z^n = r^n(\cos n\theta + i\sin n\theta). \tag{8}$$

EXAMPLE 3 Power of a Complex Number

Compute z^3 for $z = 1 - \sqrt{3}i$.

SOLUTION In Example 1 we saw that

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$

Hence from (8) with $r = 2$, $\theta = -\pi/3$, and $n = 3$, we get

$$(1 - \sqrt{3}i)^3 = 2^3\left[\cos\left(3\left(-\frac{\pi}{3}\right)\right) + i\sin\left(3\left(-\frac{\pi}{3}\right)\right)\right]$$
$$= 8[\cos(-\pi) + i\sin(-\pi)] = -8.$$

≡

DeMoivre's Formula When $z = \cos\theta + i\sin\theta$, we have $|z| = r = 1$ and so (8) yields

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{9}$$

This last result is known as **DeMoivre's formula** and is useful in deriving certain trigonometric identities. See Problems 37 and 38 in Exercises 17.2.

Roots A number w is said to be an ***n*th root** of a nonzero complex number z if $w^n = z$. If we let $w = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$ be the polar forms of w and z, then in view of (8), $w^n = z$ becomes

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta).$$

From this we conclude that $\rho^n = r$ or $\rho = r^{1/n}$ and

$$\cos n\phi + i\sin n\phi = \cos\theta + i\sin\theta.$$

By equating the real and imaginary parts, we get from this equation

$$\cos n\phi = \cos\theta \quad \text{and} \quad \sin n\phi = \sin\theta.$$

These equalities imply that $n\phi = \theta + 2k\pi$, where k is an integer. Thus,

$$\phi = \frac{\theta + 2k\pi}{n}.$$

As k takes on the successive integer values $k = 0, 1, 2, \ldots, \; n - 1$, we obtain n *distinct* roots with the same modulus but different arguments. But for $k \geq n$ we obtain the same roots because the sine and cosine are 2π-periodic. To see this, suppose $k = n + m$, where $m = 0, 1, 2, \ldots$. Then

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

and so $$\sin\phi = \sin\left(\frac{\theta + 2m\pi}{n}\right), \qquad \cos\phi = \cos\left(\frac{\theta + 2m\pi}{n}\right).$$

We summarize this result. The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \tag{10}$$

where $k = 0, 1, 2, \ldots, \; n - 1$.

EXAMPLE 4 Roots of a Complex Number

Find the three cube roots of $z = i$.

SOLUTION With $r = 1$, $\theta = \arg z = \pi/2$, the polar form of the given number is $z = \cos(\pi/2) + i\sin(\pi/2)$. From (10) with $n = 3$ we obtain

$$w_k = (1)^{1/3}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right], \quad k = 0, 1, 2.$$

Hence, the three roots are

$$k = 0, \quad w_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 1, \quad w_1 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 2, \quad w_2 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

≡

The root w of a complex number z obtained by using the principal argument of z with $k = 0$ is sometimes called the **principal nth root** of z. In Example 4, since $\text{Arg}(i) = \pi/2$, $w_0 = \sqrt{3}/2 + (1/2)i$ is the principal third root of i.

Since the roots given by (8) have the same modulus, the n roots of a nonzero complex number z lie on a circle of radius $r^{1/n}$ centered at the origin in the complex plane. Moreover, since the difference between the arguments of any two successive roots is $2\pi/n$, the nth roots of z are equally spaced on this circle. **FIGURE 17.2.3** shows the three cube roots of i equally spaced on a unit circle; the angle between roots (vectors) w_k and w_{k+1} is $2\pi/3$.

As the next example will show, the roots of a complex number do not have to be "nice" numbers as in Example 3.

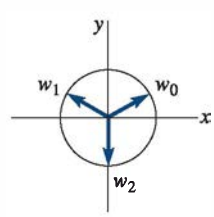


FIGURE 17.2.3 Three cube roots of i

EXAMPLE 5 Roots of a Complex Number

Find the four fourth roots of $z = 1 + i$.

SOLUTION In this case, $r = \sqrt{2}$ and $\theta = \arg z = \pi/4$. From (10) with $n = 4$, we obtain

$$w_k = (\sqrt{2})^{1/4}\left[\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{4}\right)\right], \quad k = 0, 1, 2, 3.$$

The roots, rounded to four decimal places, are

$$k = 0, \quad w_0 = (\sqrt{2})^{1/4}\left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right] = 1.0696 + 0.2127i$$

$$k = 1, \quad w_1 = (\sqrt{2})^{1/4}\left[\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right] = -0.2127 + 1.0696i$$

$$k = 2, \quad w_2 = (\sqrt{2})^{1/4}\left[\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right] = -1.0696 - 0.2127i$$

$$k = 3, \quad w_3 = (\sqrt{2})^{1/4}\left[\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right] = 0.2127 - 1.0696i. \equiv$$

17.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-38.

In Problems 1–10, write the given complex number in polar form.

1. 2

2. -10

3. $-3i$

4. $6i$

5. $1 + i$

6. $5 - 5i$

7. $-\sqrt{3} + i$

8. $-2 - 2\sqrt{3}i$

9. $\dfrac{3}{-1 + i}$

10. $\dfrac{12}{\sqrt{3} + i}$

In Problems 11–14, write the number given in polar form in the form $a + ib$.

11. $z = 5\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

12. $z = 8\sqrt{2}\left(\cos\frac{11\pi}{4} + i\sin\frac{11\pi}{4}\right)$

13. $z = 6\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

14. $z = 10\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

In Problems 15 and 16, find z_1z_2 and z_1/z_2. Write the number in the form $a + ib$.

15. $z_1 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$, $z_2 = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

16. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$,

$z_2 = \sqrt{3}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

In Problems 17–20, write each complex number in polar form. Then use either (4) or (5) to obtain a polar form of the given number. Write the polar form in the form $a + ib$.

17. $(3 - 3i)(5 + 5\sqrt{3}i)$

18. $(4 + 4i)(-1 + i)$

19. $\dfrac{-i}{2 - 2i}$

20. $\dfrac{\sqrt{2} + \sqrt{6}i}{-1 + \sqrt{3}i}$

In Problems 21–26, use (8) to compute the indicated power.

21. $(1 + \sqrt{3}i)^9$

22. $(2 - 2i)^5$

23. $(\frac{1}{2} + \frac{1}{2}i)^{10}$

24. $(-\sqrt{2} + \sqrt{6}i)^4$

25. $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^{12}$

26. $\left[\sqrt{3}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)\right]^6$

In Problems 27–32, use (10) to compute all roots. Sketch these roots on an appropriate circle centered at the origin.

27. $(8)^{1/3}$

28. $(1)^{1/8}$

29. $(i)^{1/2}$

30. $(-1 + i)^{1/3}$

31. $(-1 + \sqrt{3}i)^{1/2}$

32. $(-1 - \sqrt{3}i)^{1/4}$

In Problems 33 and 34, find all solutions of the given equation.

33. $z^4 + 1 = 0$

34. $z^8 - 2z^4 + 1 = 0$

In Problems 35 and 36, express the given complex number first in polar form and then in the form $a + ib$.

35. $\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5$

36. $\dfrac{\left[8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)\right]^{10}}$

37. Use the result $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$ to find trigonometric identities for $\cos 2\theta$ and $\sin 2\theta$.

38. Use the result $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ to find trigonometric identities for $\cos 3\theta$ and $\sin 3\theta$.

39. (a) If $z_1 = -1$ and $z_2 = 5i$, verify that

$$\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2).$$

(b) If $z_1 = -1$ and $z_2 = -5i$, verify that

$$\text{Arg}(z_1/z_2) \neq \text{Arg}(z_1) - \text{Arg}(z_2).$$

40. For the complex numbers given in Problem 39, verify in both parts (a) and (b) that

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

17.3 Sets in the Complex Plane

Introduction In the preceding sections we examined some rudiments of the algebra and geometry of complex numbers. But we have barely scratched the surface of the subject known as complex analysis; the main thrust of our study lies ahead. Our goal in the sections and chapters that follow is to examine functions of a single complex variable $z = x + iy$ and the calculus of these functions.

Before introducing the notion of a function of a complex variable, we need to state some essential definitions and terminology about sets in the complex plane.

Terminology Before discussing the concept of functions of a complex variable, we need to introduce some essential terminology about sets in the complex plane.

Suppose $z_0 = x_0 + iy_0$. Since $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho,$$

$\rho > 0$, lie on a **circle** of radius ρ centered at the point z_0. See FIGURE 17.3.1.

FIGURE 17.3.1 Circle of radius ρ

EXAMPLE 1 Circles

(a) $|z| = 1$ is the equation of a unit circle centered at the origin.

(b) $|z - 1 - 2i| = 5$ is the equation of a circle of radius 5 centered at $1 + 2i$.

The points z satisfying the inequality $|z - z_0| < \rho$, $\rho > 0$, lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0 or an **open disk**. A point z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an **open set**. See FIGURE 17.3.2. For example, the inequality $\text{Re}(z) > 1$ defines a *right half-plane*, which is an open set. All complex numbers $z = x + iy$ for which $x > 1$ are in this set. If we choose, for example, $z_0 = 1.1 + 2i$, then a neighborhood of z_0 lying entirely in the set is defined by $|z - (1.1 + 2i)| < 0.05$. See FIGURE 17.3.3. On the other hand, the set S of points in the complex plane defined by $\text{Re}(z) \geq 1$ is *not* open, since every neighborhood of a point on the line $x = 1$ must contain points in S and points not in S. See FIGURE 17.3.4.

FIGURE 17.3.2 Open set

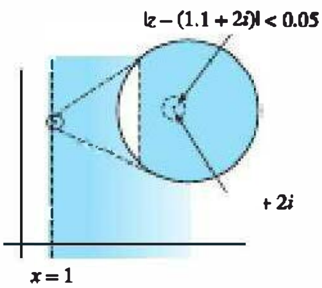


FIGURE 17.3.3 Open set magnified view of a point near $x = 1$

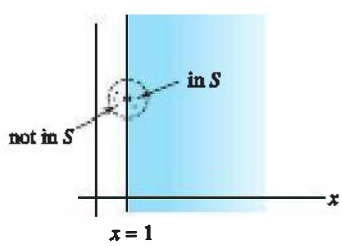


FIGURE 17.3.4 Set S is not open

EXAMPLE 2 Open Sets

FIGURE 17.3.5 illustrates some additional open sets.

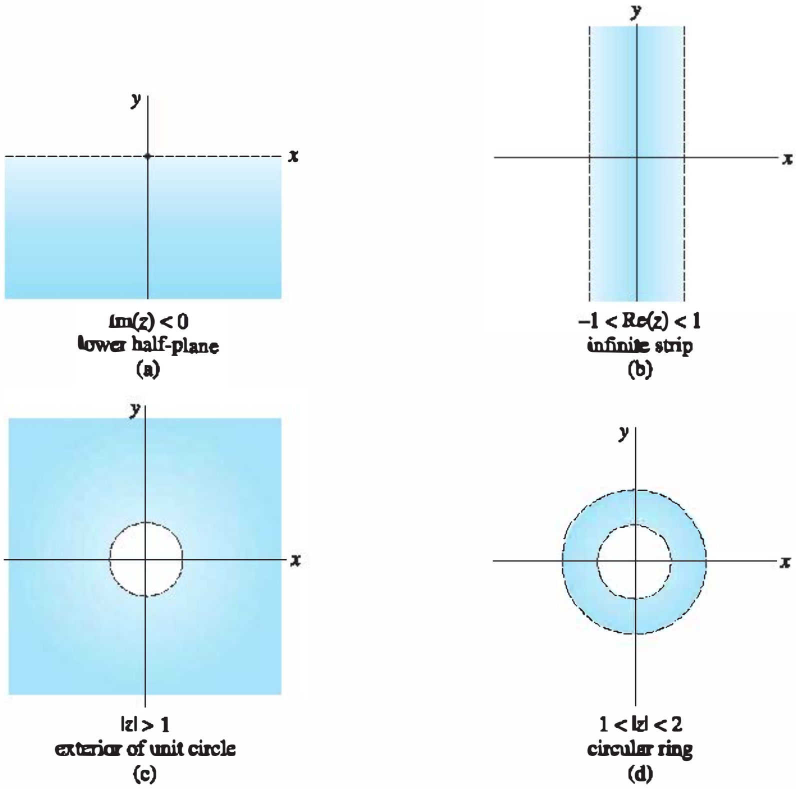


FIGURE 17.3.5 Four examples of open sets ≡

The set of numbers satisfying the inequality

$$\rho_1 < |z - z_0| < \rho_2,$$

such as illustrated in Figure 17.3.5(d), is called an **open annulus**.

If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a **boundary point** of S. The **boundary** of a set S is the set of all boundary points of S. For the set of points defined by $\text{Re}(z) \geq 1$, the points on the line $x = 1$ are boundary points. The points on the circle $|z - i| = 2$ are boundary points for the disk $|z - i| \leq 2$.

If any pair of points z_1 and z_2 in an open set S can be connected by a polygonal line that lies entirely in the set, then the open set S is said to be **connected**. See FIGURE 17.3.6. An open connected set is called a **domain**. All the open sets in Figure 17.3.5 are connected and so are domains. The set of numbers satisfying $\text{Re}(z) \neq 4$ is an open set but is not connected, since it is not possible to join points on either side of the vertical line $x = 4$ by a polygonal line without leaving the set (bear in mind that the points on $x = 4$ are not in the set).

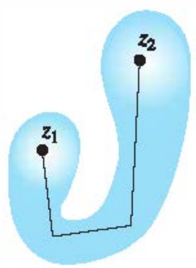


FIGURE 17.3.6 Connected set

A **region** is a domain in the complex plane with all, some, or none of its boundary points. Since an open connected set does not contain any boundary points, it is automatically a region. A region containing all its boundary points is said to be **closed**. The disk defined by $|z - i| \leq 2$ is an example of a closed region and is referred to as a **closed disk**. A region may be neither open nor closed; the annular region defined by $1 \leq |z - 5| < 3$ contains only some of its boundary points and so is neither open nor closed.

Remarks

Often in mathematics the same word is used in entirely different contexts. Do not confuse the concept of "domain" defined in this section with the concept of the "domain of a function."

17.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-39.

In Problems 1–8, sketch the graph of the given equation.

1. $\text{Re}(z) = 5$
2. $\text{Im}(z) = -2$
3. $\text{Im}(\bar{z} + 3i) = 6$
4. $\text{Im}(z - i) = \text{Re}(z + 4 - 3i)$
5. $|z - 3i| = 2$
6. $|2z + 1| = 4$
7. $|z - 4 + 3i| = 5$
8. $|z + 2 + 2i| = 2$

In Problems 9–22, sketch the set of points in the complex plane satisfying the given inequality. Determine whether the set is a domain.

9. $\text{Re}(z) < -1$
10. $|\text{Re}(z)| > 2$
11. $\text{Im}(z) > 3$
12. $\text{Im}(z - i) < 5$
13. $2 < \text{Re}(z - 1) < 4$
14. $-1 \leq \text{Im}(z) < 4$
15. $\text{Re}(z^2) > 0$
16. $\text{Im}(1/z) < \frac{1}{2}$
17. $0 \leq \arg(z) \leq 2\pi/3$
18. $|\arg(z)| < \pi/4$
19. $|z - i| > 1$
20. $|z - i| > 0$
21. $2 < |z - i| < 3$
22. $1 \leq |z - 1 - i| < 2$
23. Describe the set of points in the complex plane that satisfies $|z + 1| = |z - i|$.
24. Describe the set of points in the complex plane that satisfies $|\text{Re}(z)| \leq |z|$.
25. Describe the set of points in the complex plane that satisfies $z^2 + \bar{z}^2 = 2$.
26. Describe the set of points in the complex plane that satisfies $|z - i| + |z + i| = 1$.

17.4 Functions of a Complex Variable

Introduction One of the most important concepts in mathematics is that of a function. You may recall from previous courses that a function is a certain kind of correspondence between two sets; more specifically: *A* ***function*** *f from a set A to a set B is a rule of correspondence that assigns to each element in A one and only one element in B.* If b is the element in the set B assigned to the element a in the set A by f, we say that b is the **image** of a and write $b = f(a)$. The set A is called the **domain** of the function f (but is not necessarily a domain in the sense defined in Section 17.3). The set of all images in B is called the **range** of the function. For example, suppose the set A is a set of real numbers defined by $3 \leq x < \infty$ and the function is given by $f(x) = \sqrt{x - 3}$; then $f(3) = 0$, $f(4) = 1$, $f(8) = \sqrt{5}$, and so on. In other words, the range of f is the set given by $0 \leq y < \infty$. Since A is a set of real numbers, we say f is a **function of a real variable x**.

Functions of a Complex Variable When the domain A in the foregoing definition of a function is a set of complex numbers z, we naturally say that f is a **function of a complex variable z** or a **complex function** for short. The image w of a complex number z will be some complex number $u + iv$; that is,

$$w = f(z) = u(x, y) + iv(x, y), \tag{1}$$

where u and v are the real and imaginary parts of w and are real-valued functions. Inherent in the mathematical statement (1) is the fact that *we cannot draw a graph of a complex function $w = f(z)$* since a graph would require four axes in a four-dimensional coordinate system.

Some examples of functions of a complex variable are

$$f(z) = z^2 - 4z, \qquad z \text{ any complex number}$$

$$f(z) = \frac{z}{z^2 + 1}, \qquad z \neq i \text{ and } z \neq -i$$

$$f(z) = z + \text{Re}(z), \qquad z \text{ any complex number.}$$

Each of these functions can be expressed in form (1). For example,

$$f(z) = z^2 - 4z = (x + iy)^2 - 4(x + iy) = (x^2 - y^2 - 4x) + i(2xy - 4y).$$

Thus, $u(x, y) = x^2 - y^2 - 4x$, and $v(x, y) = 2xy - 4y$.

Although we cannot draw a graph, a complex function $w = f(z)$ can be interpreted as a **mapping** or **transformation** from the z-plane to the w-plane. See **FIGURE 17.4.1**.

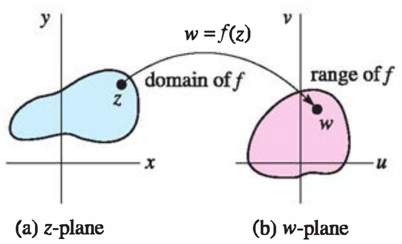
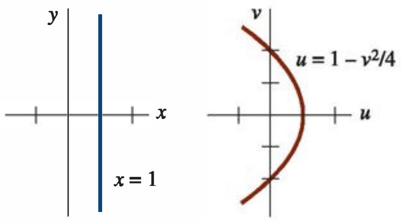


FIGURE 17.4.1 Mapping from z-plane to w-plane



(a) z-plane (b) w-plane

FIGURE 17.4.2 Image of $x = 1$ is a parabola

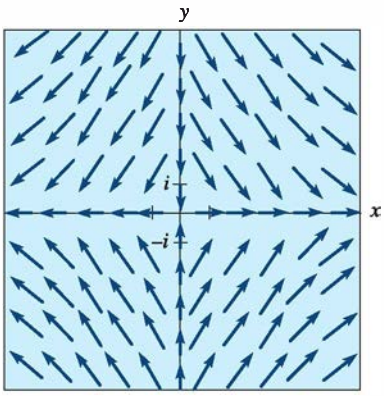


FIGURE 17.4.3 $f_1(z) = \bar{z}$ (normalized)

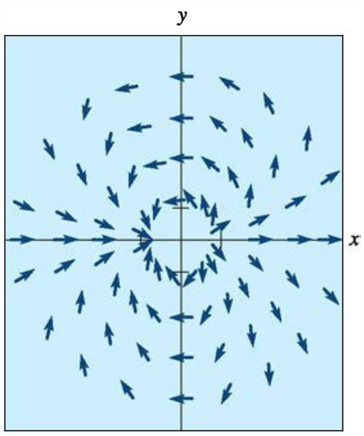


FIGURE 17.4.4 $f_2(z) = z^2$ (normalized)

EXAMPLE 1 Image of a Vertical Line

Find the image of the line $\text{Re}(z) = 1$ under the mapping $f(z) = z^2$.

SOLUTION For the function $f(z) = z^2$ we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Now, $\text{Re}(z) = x$ and so by substituting $x = 1$ into the functions u and v, we obtain $u = 1 - y^2$ and $v = 2y$. These are parametric equations of a curve in the w-plane. Substituting $y = v/2$ into the first equation eliminates the parameter y to give $u = 1 - v^2/4$. In other words, the image of the line in **FIGURE 17.4.2(a)** is the parabola shown in Figure 17.4.2(b). ≡

We shall pursue the idea of $f(z)$ as a mapping in greater detail in Chapter 20.

It should be noted that a complex function is completely determined by the real-valued functions u and v. This means a complex function $w = f(z)$ can be defined by arbitrarily specifying $u(x, y)$ and $v(x, y)$, even though $u + iv$ may not be obtainable through the familiar operations on the symbol z alone. For example, if $u(x, y) = xy^2$ and $v(x, y) = x^2 - 4y^3$, then $f(z) = xy^2 + i(x^2 - 4y^3)$ is a function of a complex variable. To compute, say, $f(3 + 2i)$, we substitute $x = 3$ and $y = 2$ into u and v to obtain $f(3 + 2i) = 12 - 23i$.

Complex Functions as Flows We also may interpret a complex function $w = f(z)$ as a **two-dimensional fluid flow** by considering the complex number $f(z)$ as a vector based at the point z. The vector $f(z)$ specifies the speed and direction of the flow at a given point z. **FIGURES 17.4.3** and **17.4.4** show the flows corresponding to the complex functions $f_1(z) = \bar{z}$ and $f_2(z) = z^2$, respectively.

If $x(t) + iy(t)$ is a parametric representation for the path of a particle in the flow, the tangent vector $\mathbf{T} = x'(t) + iy'(t)$ must coincide with $f(x(t) + iy(t))$. When $f(z) = u(x, y) + iv(x, y)$, it follows that the path of the particle must satisfy the system of differential equations

$$\frac{dx}{dt} = u(x, y)$$

$$\frac{dy}{dt} = v(x, y).$$

We call the family of solutions of this system the **streamlines** of the flow associated with $f(z)$.

EXAMPLE 2 Streamlines

Find the streamlines of the flows associated with the complex functions **(a)** $f_1(z) = \bar{z}$ and **(b)** $f_2(z) = z^2$.

SOLUTION **(a)** The streamlines corresponding to $f_1(z) = x - iy$ satisfy the system

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -y$$

and so $x(t) = c_1e^t$ and $y(t) = c_2e^{-t}$. By multiplying these two parametric equations, we see that the point $x(t) + iy(t)$ lies on the hyperbola $xy = c_1c_2$.

(b) To find the streamlines corresponding to $f_2(z) = (x^2 - y^2) + i2xy$, note that $dx/dt = x^2 - y^2$, $dy/dt = 2xy$, and so

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}.$$

This homogeneous differential equation has the solution $x^2 + y^2 = c_2y$, which represents a family of circles that have centers on the y-axis and pass through the origin. ≡

Limits and Continuity The definition of a limit of a complex function $f(z)$ as $z \to z_0$ has the same outward appearance as the limit in real variables.

Definition 17.4.1 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a **limit** at z_0, written

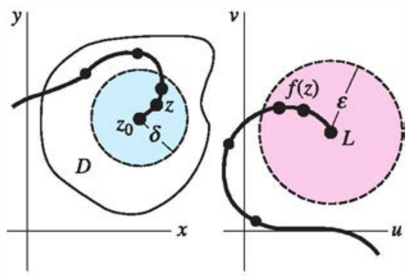
$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

In words, $\lim_{z \to z_0} f(z) = L$ means that the points $f(z)$ can be made arbitrarily close to the point L if we choose the point z sufficiently close to, but not equal to, the point z_0. As shown in FIGURE 17.4.5, for each ε-neighborhood of L (defined by $|f(z) - L| < \varepsilon$) there is a δ-neighborhood of z_0 (defined by $|z - z_0| < \delta$) so that the images of all points $z \neq z_0$ in this neighborhood lie in the ε-neighborhood of L.

The fundamental difference between this definition and the limit concept in real variables lies in the understanding of $z \to z_0$. For a function f of a single real variable x, $\lim_{x \to x_0} f(x) = L$ means $f(x)$ approaches L as x approaches x_0 either from the right of x_0 or from the left of x_0 on the real number line. But since z and z_0 are points in the complex plane, when we say that $\lim_{z \to z_0} f(z)$ exists, we mean that $f(z)$ approaches L as the point z approaches z_0 from *any* direction.

The following theorem summarizes some properties of limits:



(a) δ-neighborhood (b) ε-neighborhood

FIGURE 17.4.5 Geometric meaning of a complex limit

Theorem 17.4.1 Limit of Sum, Product, Quotient

Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

(*i*) $\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2$

(*ii*) $\lim_{z \to z_0} f(z)g(z) = L_1 L_2$

(*iii*) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_1}{L_2}, \quad L_2 \neq 0.$

Definition 17.4.2 Continuity at a Point

A function f is **continuous** at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

As a consequence of Theorem 17.4.1, it follows that if two functions f and g are continuous at a point z_0, then their sum and product are continuous at z_0. The quotient of the two functions is continuous at z_0 provided $g(z_0) \neq 0$.

A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0, \tag{2}$$

where n is a nonnegative integer and the coefficients a_i, $i = 0, 1, \ldots, n$, are complex constants, is called a **polynomial** of degree n. Although we shall not prove it, the limit result $\lim_{z \to z_0} z = z_0$ indicates that the simple polynomial function $f(z) = z$ is continuous everywhere—that is, on the entire z-plane. With this result in mind and with repeated applications of Theorem 17.4.1 (*i*) and (*ii*), it follows that a polynomial function (2) is continuous everywhere. A **rational function**

$$f(z) = \frac{g(z)}{h(z)},$$

where g and h are polynomial functions, is continuous except at those points at which $h(z)$ is zero.

Derivative The derivative of a complex function is defined in terms of a limit. The symbol Δz used in the following definition is the complex number $\Delta x + i\Delta y$.

Definition 17.4.3 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{3}$$

provided this limit exists.

If the limit in (3) exists, the function f is said to be **differentiable** at z_0. The derivative of a function $w = f(z)$ is also written dw/dz.

As in real variables, differentiability implies continuity:

If f is differentiable at z_0, then f is continuous at z_0.

Moreover, the **rules of differentiation** are the same as in the calculus of real variables. If f and g are differentiable at a point z, and c is a complex constant, then

Constant Rules: $$\frac{d}{dz} c = 0, \quad \frac{d}{dz} cf(z) = cf'(z) \tag{4}$$

Sum Rule: $$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z) \tag{5}$$

Product Rule: $$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + g(z)f'(z) \tag{6}$$

Quotient Rule: $$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2} \tag{7}$$

Chain Rule: $$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z). \tag{8}$$

The usual **Power Rule** for differentiation of powers of z is also valid:

$$\frac{d}{dz} z^n = nz^{n-1}, \quad n \text{ an integer.} \tag{9}$$

EXAMPLE 3 Using the Rules of Differentiation

Differentiate **(a)** $f(z) = 3z^4 - 5z^3 + 2z$ and **(b)** $f(z) = \dfrac{z^2}{4z + 1}$.

SOLUTION **(a)** Using the Power Rule (9) along with the Sum Rule (5), we obtain

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 = 12z^3 - 15z^2 + 2.$$

(b) From the Quotient Rule (7),

$$f'(z) = \frac{(4z + 1) \cdot 2z - z^2 \cdot 4}{(4z + 1)^2} = \frac{4z^2 + 2z}{(4z + 1)^2}.$$

≡

In order for a complex function f to be differentiable at a point z_0, $\lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ must approach the same complex number from any direction. Thus in the study of complex variables, to require the differentiability of a function is a greater demand than in real variables. If a complex function is made up, such as $f(z) = x + 4iy$, there is a good chance that it is not differentiable.

EXAMPLE 4 A Function That Is Nowhere Differentiable

Show that the function $f(z) = x + 4iy$ is nowhere differentiable.

SOLUTION With $\Delta z = \Delta x + i\Delta y$, we have

$$f(z + \Delta z) - f(z) = (x + \Delta x) + 4i(y + \Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

and so
$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}. \tag{10}$$

Now, if we let $\Delta z \to 0$ along a line parallel to the x-axis, then $\Delta y = 0$ and the value of (10) is 1. On the other hand, if we let $\Delta z \to 0$ along a line parallel to the y-axis, then $\Delta x = 0$ and the value of (10) is seen to be 4. Therefore, $f(z) = x + 4iy$ is not differentiable at any point z. ≡

Analytic Functions While the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called **analytic functions**.

Definition 17.4.4 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

A function f is analytic in a domain D if it is analytic at every point in D.

The student should read Definition 17.4.4 carefully. Analyticity *at a point* is a neighborhood property. Analyticity at a point is, therefore, not the same as differentiability at a point. It is left as an exercise to show that the function $f(z) = |z|^2$ is differentiable at $z = 0$ but is differentiable nowhere else. Hence, $f(z) = |z|^2$ is nowhere analytic. In contrast, the simple polynomial $f(z) = z^2$ is differentiable at every point z in the complex plane. Hence, $f(z) = z^2$ is analytic everywhere. A function that is analytic at every point z is said to be an **entire function**. Polynomial functions are differentiable at every point z and so are entire functions.

Remarks

Recall from algebra that a number c is a zero of a polynomial function if and only if $x - c$ is a factor of $f(x)$. The same result holds in complex analysis. For example, since $f(z) = z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$, the zeros of f are $-i$, i, $-2i$, and $2i$. Hence, $f(z) = (z + i)(z - i)(z + 2i)(z - 2i)$. Moreover, the quadratic formula is also valid. For example, using this formula, we can write

$$\begin{aligned} f(z) = z^2 - 2z + 2 &= (z - (1 + i))(z - (1 - i)) \\ &= (z - 1 - i)(z - 1 + i). \end{aligned}$$

See Problems 21 and 22 in Exercises 17.4.

17.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-39.

In Problems 1–6, find the image of the given line under the mapping $f(z) = z^2$.

1. $y = 2$
2. $x = -3$
3. $x = 0$
4. $y = 0$
5. $y = x$
6. $y = -x$

In Problems 7–14, express the given function in the form $f(z) = u + iv$.

7. $f(z) = 6z - 5 + 9i$
8. $f(z) = 7z - 9i\bar{z} - 3 + 2i$
9. $f(z) = z^2 - 3z + 4i$
10. $f(z) = 3\bar{z}^2 + 2z$
11. $f(z) = z^3 - 4z$
12. $f(z) = z^4$
13. $f(z) = z + 1/z$
14. $f(z) = \dfrac{z}{z + 1}$

In Problems 15–18, evaluate the given function at the indicated points.

15. $f(z) = 2x - y^2 + i(xy^3 - 2x^2 + 1)$
  (a) $2i$ (b) $2 - i$ (c) $5 + 3i$

16. $f(z) = (x + 1 + 1/x) + i(4x^2 - 2y^2 - 4)$
  (a) $1 + i$ (b) $2 - i$ (c) $1 + 4i$

17. $f(z) = 4z + i\bar{z} + \operatorname{Re}(z)$
  (a) $4 - 6i$ (b) $-5 + 12i$ (c) $2 - 7i$

18. $f(z) = e^x \cos y + ie^x \sin y$
  (a) $\pi i/4$ (b) $-1 - \pi i$ (c) $3 + \pi i/3$

In Problems 19–22, the given limit exists. Find its value.

19. $\lim_{z \to i} (4z^3 - 5z^2 + 4z + 1 - 5i)$

20. $\lim_{z \to 1-i} \dfrac{5z^2 - 2z + 2}{z + 1}$

21. $\lim_{z \to i} \dfrac{z^4 - 1}{z - i}$

22. $\lim_{z \to 1+i} \dfrac{z^2 - 2z + 2}{z^2 - 2i}$

In Problems 23 and 24, show that the given limit does not exist.

23. $\lim_{z \to 0} \dfrac{z}{\bar{z}}$

24. $\lim_{z \to 1} \dfrac{x + y - 1}{z - 1}$

In Problems 25 and 26, use (3) to obtain the indicated derivative of the given function.

25. $f(z) = z^2, \;\; f'(z) = 2z$

26. $f(z) = 1/z, \;\; f'(z) = -1/z^2$

In Problems 27–34, use (4)–(8) to find the derivative $f'(z)$ for the given function.

27. $f(z) = 4z^3 - (3 + i)z^2 - 5z + 4$

28. $f(z) = 5z^4 - iz^3 + (8 - i)z^2 - 6i$

29. $f(z) = (2z + 1)(z^2 - 4z + 8i)$

30. $f(z) = (z^5 + 3iz^3)(z^4 + iz^3 + 2z^2 - 6iz)$

31. $f(z) = (z^2 - 4i)^3$

32. $f(z) = (2z - 1/z)^6$

33. $f(z) = \dfrac{3z - 4 + 8i}{2z + i}$

34. $f(z) = \dfrac{5z^2 - z}{z^3 + 1}$

In Problems 35–38, give the points at which the given function will not be analytic.

35. $f(z) = \dfrac{z}{z - 3i}$

36. $f(z) = \dfrac{2i}{z^2 - 2z + 5iz}$

37. $f(z) = \dfrac{z^3 + z}{z^2 + 4}$

38. $f(z) = \dfrac{z - 4 + 3i}{z^2 - 6z + 25}$

39. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

40. The function $f(z) = |z|^2$ is continuous throughout the entire complex plane. Show, however, that f is differentiable only at the point $z = 0$. [*Hint*: Use (3) and consider two cases: $z = 0$ and $z \neq 0$. In the second case let Δz approach zero along a line parallel to the x-axis and then let Δz approach zero along a line parallel to the y-axis.]

In Problems 41–44, find the streamlines of the flow associated with the given complex function.

41. $f(z) = 2z$

42. $f(z) = iz$

43. $f(z) = 1/\bar{z}$

44. $f(z) = x^2 - iy^2$

In Problems 45 and 46, use a graphics calculator or computer to obtain the image of the given parabola under the mapping $f(z) = z^2$.

45. $y = \frac{1}{2}x^2$

46. $y = (x - 1)^2$

17.5 Cauchy–Riemann Equations

Introduction In the preceding section we saw that a function f of a complex variable z is *analytic at a point* z when f is differentiable at z and differentiable at every point in *some* neighborhood of z. This requirement is more stringent than simply *differentiability at a point* because a complex function can be differentiable at a point z but yet be differentiable nowhere else. A function f is *analytic in a domain* D if f is differentiable at all points in D. We shall now develop a test for analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$.

A Necessary Condition for Analyticity In the next theorem we see that if a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point z, then the functions u and v must satisfy a pair of equations that relate their first-order partial derivatives. This result is a necessary condition for analyticity.

Theorem 17.5.1 Cauchy–Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

PROOF: Since $f'(z)$ exists, we know that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

By writing $f(z) = u(x, y) + iv(x, y)$ and $\Delta z = \Delta x + i\Delta y$, we get from (2)

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}. \tag{3}$$

Since this limit exists, Δz can approach zero from any convenient direction. In particular, if $\Delta z \to 0$ horizontally, then $\Delta z = \Delta x$ and so (3) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}. \tag{4}$$

Since $f'(z)$ exists, the two limits in (4) exist. But by definition the limits in (4) are the first partial derivatives of u and v with respect to x. Thus, we have shown that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{5}$$

Now if we let $\Delta z \to 0$ vertically, then $\Delta z = i\Delta y$ and (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}, \tag{6}$$

which is the same as

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{7}$$

Equating the real and imaginary parts of (5) and (7) yields the pair of equations in (1). ≡

If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D, then the real functions u and v must satisfy the Cauchy–Riemann equations (1) at every point in D.

EXAMPLE 1 Using the Cauchy–Riemann Equations

The polynomial $f(z) = z^2 + z$ is analytic for all z and $f(z) = x^2 - y^2 + x + i(2xy + y)$. Thus, $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$. For any point (x, y), we see that the Cauchy–Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

≡

EXAMPLE 2 Using the Cauchy–Riemann Equations

Show that the function $f(z) = (2x^2 + y) + i(y^2 - x)$ is not analytic at any point.

SOLUTION We identify $u(x, y) = 2x^2 + y$ and $v(x, y) = y^2 - x$. Now from

$$\frac{\partial u}{\partial x} = 4x \quad \text{and} \quad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1$$

we see that $\partial u/\partial y = -\partial v/\partial x$ but that the equality $\partial u/\partial x = \partial v/\partial y$ is satisfied only on the line $y = 2x$. However, for any point z on the line, there is no neighborhood or open disk about z in which f is differentiable. We conclude that f is nowhere analytic. ≡

Important.

By themselves, the Cauchy–Riemann equations are not sufficient to ensure analyticity. However, when we add the condition of continuity to u and v *and* the four partial derivatives, the Cauchy–Riemann equations can be shown to imply analyticity. The proof is long and complicated and so we state only the result.

Theorem 17.5.2 Criterion for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy–Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

EXAMPLE 3 Using Theorem 17.5.2

For the function $f(z) = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$ we have

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.$$

In other words, the Cauchy–Riemann equations are satisfied except at the point where $x^2 + y^2 = 0$; that is, at $z = 0$. We conclude from Theorem 17.5.2 that f is analytic in any domain not containing the point $z = 0$. ≡

The results in (5) and (7) were obtained under the basic assumption that f was differentiable at the point z. In other words, (5) and (7) give us a formula for computing $f'(z)$:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{8}$$

For example, we know that $f(z) = z^2$ is differentiable for all z. With $u(x, y) = x^2 - y^2$, $\partial u/\partial x = 2x$, $v(x, y) = 2xy$, and $\partial v/\partial x = 2y$, we see that

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Recall that analyticity implies differentiability but not vice versa. Theorem 17.5.2 has an analogue that gives sufficient conditions for differentiability:

If the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a neighborhood of z, and if u and v satisfy the Cauchy–Riemann equations at the point z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f'(z)$ is given by (8).

The function $f(z) = x^2 - y^2 i$ is nowhere analytic. With the identifications $u(x, y) = x^2$ and $v(x, y) = -y^2$, we see from

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial v}{\partial y} = -2y \quad \text{and} \quad \frac{\partial u}{\partial y} = 0, \ \frac{\partial v}{\partial x} = 0$$

that the Cauchy–Riemann equations are satisfied only when $y = -x$. But since the functions u, $\partial u/\partial x$, $\partial u/\partial y$, v, $\partial v/\partial x$, and $\partial v/\partial y$ are continuous at every point, it follows that f is differentiable on the line $y = -x$ and on that line (8) gives the derivative $f'(z) = 2x = -2y$.

Harmonic Functions We saw in Chapter 13 that Laplace's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ occurs in certain problems involving steady-state temperatures. This partial differential equation also plays an important role in many areas of applied mathematics. Indeed, as we now see, the real and imaginary parts of an analytic function cannot be chosen arbitrarily, since both u and v must satisfy Laplace's equation. It is this link between analytic functions and Laplace's equation that makes complex variables so essential in the serious study of applied mathematics.

Definition 17.5.1 Harmonic Functions

A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Theorem 17.5.3 A Source of Harmonic Functions

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

PROOF: In this proof we shall assume that u and v have continuous second-order partial derivatives. Since f is analytic, the Cauchy–Riemann equations are satisfied. Differentiating both sides of $\partial u/\partial x = \partial v/\partial y$ with respect to x and differentiating both sides of $\partial u/\partial y = -\partial v/\partial x$ with respect to y then give

We will see in Chapter 18 that an analytic function possesses derivatives of all orders.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}.$$

With the assumption of continuity, the mixed partials are equal. Hence, adding these two equations gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This shows that $u(x, y)$ is harmonic.

Now differentiating both sides of $\partial u/\partial x = \partial v/\partial y$ with respect to y and differentiating both sides of $\partial u/\partial y = -\partial v/\partial x$ with respect to x and subtracting yield

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$ ≡

Harmonic Conjugate Functions

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D. Now suppose $u(x, y)$ is a given function that is harmonic in D. It is then sometimes possible to find another function $v(x, y)$ that is harmonic in D so that $u(x, y) + iv(x, y)$ is an analytic function in D. The function v is called a **harmonic conjugate function** of u.

EXAMPLE 4 Harmonic Function/ Harmonic Conjugate Function

(a) Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.
(b) Find the harmonic conjugate function of u.

SOLUTION **(a)** From the partial derivatives

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial u}{\partial y} = -6xy - 5, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

we see that u satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

(b) Since the harmonic conjugate function v must satisfy the Cauchy–Riemann equations, we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 5. \tag{9}$$

Partial integration of the first equation in (9) with respect to y gives $v(x, y) = 3x^2y - y^3 + h(x)$. From this we get

$$\frac{\partial v}{\partial x} = 6xy + h'(x).$$

Substituting this result into the second equation in (9) gives $h'(x) = 5$, and so $h(x) = 5x + C$. Therefore, the harmonic conjugate function of u is $v(x, y) = 3x^2y - y^3 + 5x + C$. The analytic function is $f(z) = x^3 - 3xy^2 - 5y + i(3x^2y - y^3 + 5x + C)$. ≡

Remarks

Suppose u and v are the harmonic functions that comprise the real and imaginary parts of an analytic function $f(z)$. The level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ defined by these functions form two **orthogonal families of curves**. See Problem 32 in Exercises 17.5. For example, the level curves generated by the simple analytic function $f(z) = z = x + iy$ are $x = c_1$ and $y = c_2$. The family of vertical lines defined by $x = c_1$ is clearly orthogonal to the family of horizontal lines defined by $y = c_2$. In electrostatics, if $u(x, y) = c_1$ defines the **equipotential curves**, then the other, and orthogonal, family $v(x, y) = c_2$ defines the lines of force.

17.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-39.

In Problems 1 and 2, the given function is analytic for all z. Show that the Cauchy–Riemann equations are satisfied at every point.

1. $f(z) = z^3$

2. $f(z) = 3z^2 + 5z - 6i$

In Problems 3–8, show that the given function is not analytic at any point.

3. $f(z) = \text{Re}(z)$

4. $f(z) = y + ix$

5. $f(z) = 4z - 6\bar{z} + 3$

6. $f(z) = \bar{z}^2$

7. $f(z) = x^2 + y^2$

8. $f(x) = \dfrac{x}{x^2 + y^2} + i\dfrac{y}{x^2 + y^2}$

In Problems 9–14, use Theorem 17.5.2 to show that the given function is analytic in an appropriate domain.

9. $f(z) = e^x \cos y + ie^x \sin y$

10. $f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$

11. $f(z) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2} \sin 2xy$

12. $f(z) = 4x^2 + 5x - 4y^2 + 9 + i(8xy + 5y - 1)$

13. $f(z) = \dfrac{x - 1}{(x - 1)^2 + y^2} - i\dfrac{y}{(x - 1)^2 + y^2}$

14. $f(x) = \dfrac{x^3 + xy^2 + x}{x^2 + y^2} + i\dfrac{x^2y + y^3 - y}{x^2 + y^2}$

In Problems 15 and 16, find real constants a, b, c, and d so that the given function is analytic.

15. $f(z) = 3x - y + 5 + i(ax + by - 3)$

16. $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

In Problems 17–20, show that the given function is not analytic at any point, but is differentiable along the indicated curve(s).

17. $f(z) = x^2 + y^2 + 2xyi$; x-axis

18. $f(z) = 3x^2y^2 - 6x^2y^2i$; coordinate axes

19. $f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y)$; coordinate axes

20. $f(z) = x^2 - x + y + i(y^2 - 5y - x)$; $y = x + 2$

21. Use (8) to find the derivative of the function in Problem 9.

22. Use (8) to find the derivative of the function in Problem 11.

In Problems 23–28, verify that the given function u is harmonic. Find v, the harmonic conjugate function of u. Form the corresponding analytic function $f(z) = u + iv$.

23. $u(x, y) = x$

24. $u(x, y) = 2x - 2xy$

25. $u(x, y) = x^2 - y^2$

26. $u(x, y) = 4xy^3 - 4x^3y + x$

27. $u(x, y) = \log_e(x^2 + y^2)$

28. $u(x, y) = e^x(x \cos y - y \sin y)$

29. Sketch the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the analytic function $f(z) = z^2$.

30. Consider the function $f(z) = 1/z$. Describe the level curves.

31. Consider the function $f(z) = z + 1/z$. Describe the level curve $v(x, y) = 0$.

32. Suppose u and v are the harmonic functions forming the real and imaginary parts of an analytic function. Show that the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. [*Hint*: Consider the gradient of u and the gradient of v. Ignore the case where a gradient vector is the zero vector.]

17.6 Exponential and Logarithmic Functions

Introduction In this and the next section, we shall examine the exponential, logarithmic, trigonometric, and hyperbolic functions of a complex variable z. Although the definitions of these complex functions are motivated by their real variable analogues, the properties of these complex functions will yield some surprises.

Exponential Function Recall that in real variables the exponential function $f(x) = e^x$ has the properties

$$f'(x) = f(x) \quad \text{and} \quad f(x_1 + x_2) = f(x_1)f(x_2). \tag{1}$$

We certainly want the definition of the complex function $f(z) = e^z$, where $z = x + iy$, to reduce e^x for $y = 0$ and to possess the same properties as in (1).

We have already used an exponential function with a pure imaginary exponent. Euler's formula,

$$e^{iy} = \cos y + i \sin y, \qquad y \text{ a real number}, \tag{2}$$

played an important role in Section 3.3. We can formally establish the result in (2) by using the Maclaurin series for e^x and replacing x by iy and rearranging terms:

$$\begin{aligned} e^{iy} &= \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

Maclaurin series for $\cos y$ and $\sin y$.

For $z = x + iy$, it is natural to expect that

$$e^{x+iy} = e^x e^{iy}$$

and so by (2), $$e^{x+iy} = e^x(\cos y + i \sin y).$$

Inspired by this formal result, we make the following definition.

Definition 17.6.1 Exponential Function

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y). \tag{3}$$

The exponential function e^z is also denoted by the symbol $\exp z$. Note that (3) reduces to e^x when $y = 0$.

EXAMPLE 1 Complex Value of the Exponential Function

Evaluate $e^{1.7+4.2i}$.

SOLUTION With the identifications $x = 1.7$ and $y = 4.2$ and the aid of a calculator, we have, to four rounded decimal places,

$$e^{1.7} \cos 4.2 = -2.6837 \qquad \text{and} \qquad e^{1.7} \sin 4.2 = -4.7710.$$

It follows from (3) that $e^{1.7+4.2i} = -2.6837 - 4.7710i$. ≡

The real and imaginary parts of e^z, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$, are continuous and have continuous first partial derivatives at every point z of the complex plane. Moreover, the Cauchy–Riemann equations are satisfied at all points of the complex plane:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

It follows from Theorem 17.5.2 that $f(z) = e^z$ is analytic for all z; in other words, f is an entire function.

Properties We shall now demonstrate that e^z possesses the two desired properties given in (1). First, the derivative of f is given by (5) of Section 17.5:

$$f'(z) = e^x \cos y + i(e^x \sin y) = e^x(\cos y + i \sin y) = f(z).$$

As desired, we have established that

$$\frac{d}{dz} e^z = e^z.$$

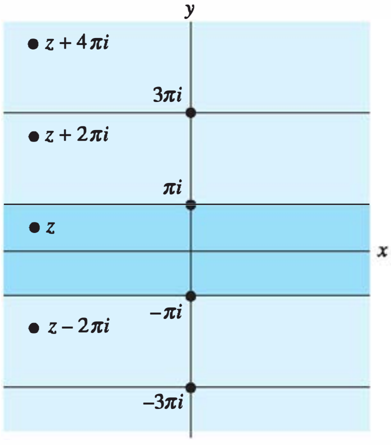


FIGURE 17.6.1 Values of $f(z) = e^z$ at the four points are the same

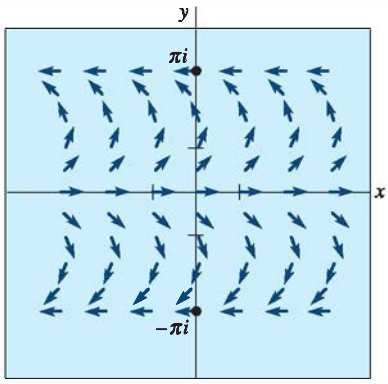


FIGURE 17.6.2 Flow over the fundamental region

Second, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then by multiplication of complex numbers and the addition formulas of trigonometry, we obtain

$$\begin{aligned} f(z_1)f(z_2) &= e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)] \\ &= e^{x_1+x_2}[(\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = f(z_1 + z_2). \end{aligned}$$

In other words,

$$e^{z_1}e^{z_2} = e^{z_1+z_2}. \tag{4}$$

It is left as an exercise to prove that

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

Periodicity

Unlike the real function e^x, the complex function $f(z) = e^z$ is **periodic** with the complex period $2\pi i$. Since $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$ and, in view of (4), $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$ for all z, it follows that $f(z + 2\pi i) = f(z)$. Because of this complex periodicity, all possible functional values of $f(z) = e^z$ are assumed in any infinite horizontal strip of width 2π. Thus, if we divide the complex plane into horizontal strips defined by $(2n - 1)\pi < y \le (2n + 1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$, then, as shown in **FIGURE 17.6.1**, for any point z in the strip $-\pi < y \le \pi$, the values $f(z), f(z + 2\pi i), f(z - 2\pi i), f(z + 4\pi i)$, and so on, are the same. The strip $-\pi < y \le \pi$ is called the **fundamental region** for the exponential function $f(z) = e^z$. The corresponding flow over the fundamental region is shown in **FIGURE 17.6.2**.

Polar Form of a Complex Number

In Section 17.2, we saw that the complex number z could be written in polar form as $z = r(\cos\theta + i \sin\theta)$. Since $e^{i\theta} = \cos\theta + i \sin\theta$, we can now write the polar form of a complex number as

$$z = re^{i\theta}.$$

For example, in polar form $z = 1 + i$ is $z = \sqrt{2}e^{\pi i/4}$.

Circuits

In applying mathematics, mathematicians and engineers often approach the same problem in completely different ways. Consider, for example, the solution of Example 10 in Section 3.8. In this example we used strictly real analysis to find the steady-state current $i_p(t)$ in an LRC-series circuit described by the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0 \sin \gamma t.$$

Electrical engineers often solve circuit problems such as this using complex analysis. To illustrate, let us first denote the imaginary unit $\sqrt{-1}$ by the symbol j to avoid confusion with the current i. Since current i is related to charge q by $i = dq/dt$, the differential equation is the same as

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E_0 \sin \gamma t.$$

Moreover, the impressed voltage $E_0 \sin \gamma t$ can be replaced by $\text{Im}(E_0 e^{j\gamma t})$, where Im means the "imaginary part of." Because of this last form, the method of undetermined coefficients suggests that we assume a solution in the form of a constant multiple of complex exponential—that is, $i_p(t) = \text{Im}(Ae^{j\gamma t})$. We substitute this expression into the last differential equation, use the fact that q is an antiderivative of i, and equate coefficients of $e^{j\gamma t}$:

$$\left(jL\gamma + R + \frac{1}{jC\gamma}\right)A = E_0 \quad \text{gives} \quad A = \frac{E_0}{R + j\left(L\gamma - \dfrac{1}{C\gamma}\right)}.$$

The quantity $Z = R + j(L\gamma - 1/C\gamma)$ is called the **complex impedance** of the circuit. Note that the modulus of the complex impedance, $|Z| = \sqrt{R^2 + (L\gamma - 1/C\gamma)^2}$, was denoted in Example 10 of Section 3.8 by the letter Z and called the impedance.

Now, in polar form the complex impedance is

$$Z = |Z|e^{j\theta} \quad \text{where} \quad \tan\theta = \frac{L\gamma - \dfrac{1}{C\gamma}}{R}.$$

Hence, $A = E_0/Z = E_0/(|Z|e^{j\theta})$, and so the steady-state current can be written as

$$i_p(t) = \operatorname{Im} \frac{E_0}{|Z|} e^{-j\theta} e^{j\gamma t}.$$

The reader is encouraged to verify that this last expression is the same as (35) in Section 3.8.

Logarithmic Function

The logarithm of a complex number $z = x + iy$, $z \neq 0$, is defined as the inverse of the exponential function—that is,

$$w = \ln z \quad \text{if} \quad z = e^w. \tag{5}$$

In (5) we note that $\ln z$ is not defined for $z = 0$, since there is no value of w for which $e^w = 0$. To find the real and imaginary parts of $\ln z$, we write $w = u + iv$ and use (3) and (5):

$$x + iy = e^{u+iv} = e^u(\cos v + i \sin v) = e^u \cos v + ie^u \sin v.$$

The last equality implies $x = e^u \cos v$ and $y = e^u \sin v$. We can solve these two equations for u and v. First, by squaring and adding the equations, we find

$$e^{2u} = x^2 + y^2 = r^2 = |z|^2 \quad \text{and so} \quad u = \log_e|z|,$$

where $\log_e|z|$ denotes the *real natural logarithm* of the modulus of z. Second, to solve for v, we divide the two equations to obtain

$$\tan v = \frac{y}{x}.$$

This last equation means that v is an argument of z; that is, $v = \theta = \arg z$. But since there is no unique argument of a given complex number $z = x + iy$, if θ is an argument of z, then so is $\theta + 2n\pi$, $n = 0, \pm1, \pm2, \ldots$.

Definition 17.6.2 Logarithm of a Complex Number

For $z \neq 0$, and $\theta = \arg z$,

$$\ln z = \log_e|z| + i(\theta + 2n\pi), \quad n = 0, \pm1, \pm2, \ldots. \tag{6}$$

As is clearly indicated in (6), there are infinitely many values of the logarithm of a complex number z. This should not be any great surprise since the exponential function is periodic.

In real calculus, logarithms of negative numbers are not defined. As the next example will show, this is not the case in complex calculus.

EXAMPLE 2 Complex Values of the Logarithmic Function

Find the values of **(a)** $\ln(-2)$, **(b)** $\ln i$, and **(c)** $\ln(-1 - i)$.

SOLUTION **(a)** With $\theta = \arg(-2) = \pi$ and $\log_e|-2| = 0.6932$, we have from (6)

$$\ln(-2) = 0.6932 + i(\pi + 2n\pi).$$

(b) With $\theta = \arg(i) = \pi/2$ and $\log_e|i| = \log_e 1 = 0$, we have from (6)

$$\ln i = i\left(\frac{\pi}{2} + 2n\pi\right).$$

In other words, $\ln i = \pi i/2, -3\pi i/2, 5\pi i/2, -7\pi i/2$, and so on.

(c) With $\theta = \arg(-1 - i) = 5\pi/4$ and $\log_e|-1 - i| = \log_e \sqrt{2} = 0.3466$, we have from (6)

$$\ln(-1 - i) = 0.3466 + i\left(\frac{5\pi}{4} + 2n\pi\right).$$ ≡

EXAMPLE 3 Solving an Exponential Equation

Find all values of z such that $e^z = \sqrt{3} + i$.

SOLUTION From (5), with the symbol w replaced by z, we have $z = \ln(\sqrt{3} + i)$. Now $|\sqrt{3} + i| = 2$ and $\tan\theta = 1/\sqrt{3}$ imply that $\arg(\sqrt{3} + i) = \pi/6$, and so (6) gives

$$z = \log_e 2 + i\left(\frac{\pi}{6} + 2n\pi\right) \quad \text{or} \quad z = 0.6931 + i\left(\frac{\pi}{6} + 2n\pi\right).$$ ≡

Principal Value It is interesting to note that as a consequence of (6), the logarithm of a positive real number has many values. For example, in real calculus, $\log_e 5$ has only one value: $\log_e 5 = 1.6094$, whereas in complex calculus, $\ln 5 = 1.6094 + 2n\pi i$. The value of $\ln 5$ corresponding to $n = 0$ is the same as the real logarithm $\log_e 5$ and is called the **principal value** of $\ln 5$. Recall that in Section 17.2 we stipulated that the principal argument of a complex number, written $\text{Arg } z$, lies in the interval $(-\pi, \pi]$. In general, we define the **principal value** of $\ln z$ as that complex logarithm corresponding to $n = 0$ and $\theta = \text{Arg } z$. To emphasize the principal value of the logarithm, we shall adopt the notation $\text{Ln } z$. In other words,

$$\text{Ln } z = \log_e|z| + i \text{ Arg } z. \tag{7}$$

Since $\text{Arg } z$ is unique, there is only one value of $\text{Ln } z$ for each $z \neq 0$.

EXAMPLE 4 Principal Values

The principal values of the logarithms in Example 2 are as follows:

(a) Since $\text{Arg}(-2) = \pi$, we need only set $n = 0$ in the result given in part (a) of Example 2:

$$\text{Ln}(-2) = 0.6932 + \pi i.$$

(b) Similarly, since $\text{Arg}(i) = \pi/2$, we set $n = 0$ in the result in part (b) of Example 2 to obtain

$$\text{Ln } i = \frac{\pi}{2} i.$$

(c) In part (c) of Example 2, $\arg(-1 - i) = 5\pi/4$ is not the principal argument of $z = -1 - i$. The argument of z that lies in the interval $(-\pi, \pi]$ is $\text{Arg}(-1 - i) = -3\pi/4$. Hence, it follows from (7) that

$$\text{Ln}(-1 - i) = 0.3466 - \frac{3\pi}{4} i.$$ ≡

Up to this point we have avoided the use of the word *function* for the obvious reason that $\ln z$ defined in (6) is not a function in the strictest interpretation of that word. Nonetheless, it is customary to write $f(z) = \ln z$ and to refer to $f(z) = \ln z$ by the seemingly contradictory phrase *multiple-valued function*. Although we shall not pursue the details, (6) can be interpreted as an infinite collection of logarithmic functions (standard meaning of the word). Each function in the collection is called a **branch** of $\ln z$. The function $f(z) = \text{Ln } z$ is then called the **principal branch** of $\ln z$, or the **principal logarithmic function**. To minimize the confusion, we shall hereafter simply use the words *logarithmic function* when referring to either $f(z) = \ln z$ or $f(z) = \text{Ln } z$.

Some familiar properties of the logarithmic function hold in the complex case:

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad \text{and} \quad \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2. \tag{8}$$

Equations (8) and (9) are to be interpreted in the sense that if values are assigned to two of the terms, then a correct value is assigned to the third term.

EXAMPLE 5 Properties of Logarithms

Suppose $z_1 = 1$ and $z_2 = -1$. Then if we take $\ln z_1 = 2\pi i$ and $\ln z_2 = \pi i$, we get

$$\ln(z_1 z_2) = \ln(-1) = \ln z_1 + \ln z_2 = 2\pi i + \pi i = 3\pi i$$

$$\ln\left(\frac{z_1}{z_2}\right) = \ln(-1) = \ln z_1 - \ln z_2 = 2\pi i - \pi i = \pi i.$$

≡

Just as (7) of Section 17.2 was not valid when arg z was replaced with Arg z, so too (8) is not true, in general, when $\ln z$ is replaced by Ln z. See Problems 45 and 46 in Exercises 17.6.

Analyticity

The logarithmic function $f(z) = \text{Ln}\, z$ is not continuous at $z = 0$ since $f(0)$ is not defined. Moreover, $f(z) = \text{Ln}\, z$ is discontinuous at all points of the negative real axis. This is because the imaginary part of the function, $v = \text{Arg}\, z$, is discontinuous only at these points. To see this, suppose x_0 is a point on the negative real axis. As $z \to x_0$ from the upper half-plane, Arg $z \to \pi$, whereas if $z \to x_0$ from the lower half-plane, then Arg $z \to -\pi$. This means that $f(z) = \text{Ln}\, z$ is not analytic on the nonpositive real axis. However, $f(z) = \text{Ln}\, z$ is analytic throughout the domain D consisting of all the points in the complex plane except those on the nonpositive real axis. It is convenient to think of D as the complex plane from which the nonpositive real axis has been cut out. Since $f(z) = \text{Ln}\, z$ is the principal branch of $\ln z$, the nonpositive real axis is referred to as a **branch cut** for the function. See FIGURE 17.6.3. It is left as exercises to show that the Cauchy–Riemann equations are satisfied throughout this cut plane and that the derivative of Ln z is given by

$$\frac{d}{dz}\text{Ln}\, z = \frac{1}{z} \tag{9}$$

for all z in D.

FIGURE 17.6.4 shows $w = \text{Ln}\, z$ as a flow. Note that the vector field is not continuous along the branch cut.

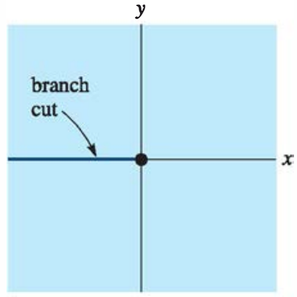


FIGURE 17.6.3 Branch cut for Ln z

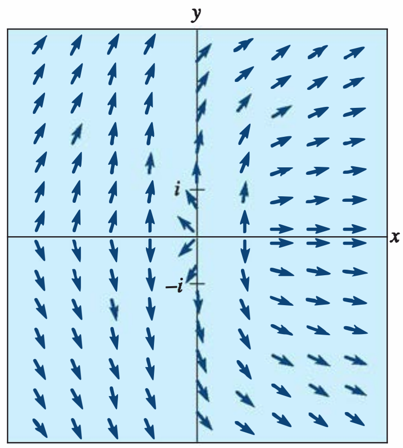


FIGURE 17.6.4 $w = \text{Ln}\, z$ as a flow

Complex Powers

Inspired by the identity $x^a = e^{a \ln x}$ in real variables, we can define complex powers of a complex number. If α is a complex number and $z = x + iy$, then z^α is defined by

$$z^\alpha = e^{\alpha \ln z}, \quad z \neq 0. \tag{10}$$

In general, z^α is multiple-valued since $\ln z$ is multiple-valued. However, in the special case when $\alpha = n$, $n = 0, \pm 1, \pm 2, \ldots$, (10) is single-valued since there is only one value for z^2, z^3, z^{-1}, and so on. To see that this is so, suppose $\alpha = 2$ and $z = re^{i\theta}$, where θ is any argument of z. Then

$$e^{2 \ln z} = e^{2(\log_e r + i\theta)} = e^{2\log_e r + 2i\theta} = e^{2\log_e r} e^{2i\theta} = r^2 e^{i\theta} e^{i\theta} = (re^{i\theta})(re^{i\theta}) = z^2.$$

If we use Ln z in place of $\ln z$, then (10) gives the **principal value** of z^α.

EXAMPLE 6 Complex Power

Find the value of i^{2i}.

SOLUTION With $z = i$, arg $z = \pi/2$, and $\alpha = 2i$, it follows from (10) that

$$i^{2i} = e^{2i[\log_e 1 + i(\pi/2 + 2n\pi)]} = e^{-(1+4n)\pi}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Inspection of the equation shows that i^{2i} is real for every value of n. Since $\pi/2$ is the principal argument of $z = i$, we obtain the principal value of i^{2i} for $n = 0$. To four rounded decimal places, this principal value is $i^{2i} = e^{-\pi} = 0.0432$. ≡

17.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-40.

In Problems 1–10, express e^z in the form $a + ib$.

1. $z = \frac{\pi}{6}i$
2. $z = -\frac{\pi}{3}i$
3. $z = -1 + \frac{\pi}{4}i$
4. $z = 2 - \frac{\pi}{2}i$
5. $z = \pi + \pi i$
6. $z = -\pi + \frac{3\pi}{2}i$
7. $z = 1.5 + 2i$
8. $z = -0.3 + 0.5i$
9. $z = 5i$
10. $z = -0.23 - i$

In Problems 11 and 12, express the given number in the form $a + ib$.

11. $e^{1+5\pi i/4}e^{-1-\pi i/3}$
12. $\dfrac{e^{2+3\pi i}}{e^{-3+\pi i/2}}$

In Problems 13–16, use Definition 17.6.1 to express the given function in the form $f(z) = u + iv$.

13. $f(z) = e^{-iz}$
14. $f(z) = e^{2\bar{z}}$
15. $f(z) = e^{z^2}$
16. $f(z) = e^{1/z}$

In Problems 17–20, verify the given result.

17. $|e^z| = e^x$
18. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
19. $e^{z+\pi i} = e^{z-\pi i}$
20. $(e^z)^n = e^{nz}$, n an integer
21. Show that $f(z) = e^{\bar{z}}$ is nowhere analytic.
22. (a) Use the result in Problem 15 to show that $f(z) = e^{z^2}$ is an entire function.
    (b) Verify that $u(x, y) = \text{Re}(e^{z^2})$ is a harmonic function.

In Problems 23–28, express $\ln z$ in the form $a + ib$.

23. $z = -5$
24. $z = -ei$
25. $z = -2 + 2i$
26. $z = 1 + i$
27. $z = \sqrt{2} + \sqrt{6}i$
28. $z = -\sqrt{3} + i$

In Problems 29–34, express $\text{Ln}\, z$ in the form $a + ib$.

29. $z = 6 - 6i$
30. $z = -e^3$
31. $z = -12 + 5i$
32. $z = 3 - 4i$
33. $z = (1 + \sqrt{3}i)^5$
34. $z = (1 + i)^4$

In Problems 35–38, find all values of z satisfying the given equation.

35. $e^z = 4i$
36. $e^{1/z} = -1$
37. $e^{z-1} = -ie^2$
38. $e^{2z} + e^z + 1 = 0$

In Problems 39–42, find all values of the given quantity.

39. $(-i)^{4i}$
40. $3^{i/\pi}$
41. $(1 + i)^{(1+i)}$
42. $(1 + \sqrt{3}i)^{3i}$

In Problems 43 and 44, find the principal value of the given quantity. Express answers in the form $a + ib$.

43. $(-1)^{(-2i/\pi)}$
44. $(1 - i)^{2i}$
45. If $z_1 = i$ and $z_2 = -1 + i$, verify that
$$\text{Ln}(z_1z_2) \neq \text{Ln}\, z_1 + \text{Ln}\, z_2.$$
46. Find two complex numbers z_1 and z_2 such that
$$\text{Ln}(z_1/z_2) \neq \text{Ln}\, z_1 - \text{Ln}\, z_2.$$
47. Determine whether the given statement is true.
    (a) $\text{Ln}(-1 + i)^2 = 2\,\text{Ln}(-1 + i)$
    (b) $\text{Ln}\, i^3 = 3\,\text{Ln}\, i$
    (c) $\ln i^3 = 3 \ln i$
48. The laws of exponents hold for complex numbers α and β:
$$z^\alpha z^\beta = z^{\alpha+\beta}, \quad \frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}, \quad (z^\alpha)^n = z^{n\alpha}, \quad n \text{ an integer.}$$
However, the last law is not valid if n is a complex number. Verify that $(i^i)^2 = i^{2i}$, but $(i^2)^i \neq i^{2i}$.
49. For complex numbers z satisfying $\text{Re}(z) > 0$, show that (7) can be written as
$$\text{Ln}\, z = \frac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}\frac{y}{x}.$$
50. The function given in Problem 49 is analytic.
    (a) Verify that $u(x, y) = \log_e(x^2 + y^2)$ is a harmonic function.
    (b) Verify that $v(x, y) = \tan^{-1}(y/x)$ is a harmonic function.

17.7 Trigonometric and Hyperbolic Functions

Introduction In this section we define the complex trigonometric and hyperbolic functions. Analogous to the complex functions e^z and $\text{Ln}\, z$ defined in the previous section, these functions will agree with their real counterparts for real values of z. In addition, we will show that the complex trigonometric and hyperbolic functions have the same derivatives and satisfy many of the same identities as the real trigonometric and hyperbolic functions.

Trigonometric Functions If x is a real variable, then Euler's formula gives
$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$

By subtracting and then adding these equations, we see that the real functions sin x and cos x can be expressed as a combination of exponential functions:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{1}$$

Using (1) as a model, we now define the sine and cosine of a complex variable:

Definition 17.7.1 Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{2}$$

As in trigonometry, we define four additional trigonometric functions in terms of sin z and cos z:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}. \tag{3}$$

When $y = 0$, each function in (2) and (3) reduces to its real counterpart.

Analyticity Since the exponential functions e^{iz} and e^{-iz} are entire functions, it follows that sin z and cos z are entire functions. Now, as we shall see shortly, $\sin z = 0$ only for the real numbers $z = n\pi$, n an integer, and $\cos z = 0$ only for the real numbers $z = (2n + 1)\pi/2$, n an integer. Thus, tan z and sec z are analytic except at the points $z = (2n + 1)\pi/2$, and cot z and csc z are analytic except at the points $z = n\pi$.

Derivatives Since $(d/dz)e^z = e^z$, it follows from the Chain Rule that $(d/dz)e^{iz} = ie^{iz}$ and $(d/dz)e^{-iz} = -ie^{-iz}$. Hence,

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

In fact, it is readily shown that the forms of the derivatives of the complex trigonometric functions are the same as the real functions. We summarize the results:

$$\begin{aligned}
&\frac{d}{dz}\sin z = \cos z && \frac{d}{dz}\cos z = -\sin z \\
&\frac{d}{dz}\tan z = \sec^2 z && \frac{d}{dz}\cot z = -\csc^2 z \\
&\frac{d}{dz}\sec z = \sec z \tan z && \frac{d}{dz}\csc z = -\csc z \cot z.
\end{aligned} \tag{4}$$

Identities The familiar trigonometric identities are also the same in the complex case:

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2\sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z.$$

Zeros To find the zeros of sin z and cos z we need to express both functions in the form $u + iv$. Before proceeding, recall from calculus that if y is real, then the hyperbolic sine and hyperbolic cosine are defined in terms of the real exponential functions e^y and e^{-y}:

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}. \tag{5}$$

Now from Definition 17.7.1 and Euler's formula we find, after simplifying,

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin x\left(\frac{e^y + e^{-y}}{2}\right) + i\cos x\left(\frac{e^y - e^{-y}}{2}\right).$$

Thus from (5) we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y. \tag{6}$$

It is left as an exercise to show that

$$\cos z = \cos x \cosh y - i \sin x \sinh y. \tag{7}$$

From (6), (7), and $\cosh^2 y = 1 + \sinh^2 y$, we find

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \tag{8}$$

and

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{9}$$

Now a complex number z is zero if and only if $|z|^2 = 0$. Thus, if $\sin z = 0$, then from (8) we must have $\sin^2 x + \sinh^2 y = 0$. This implies that $\sin x = 0$ and $\sinh y = 0$, and so $x = n\pi$ and $y = 0$. Thus the only **zeros** of $\sin z$ are the real numbers $z = n\pi + 0i = n\pi$, $n = 0, \pm1, \pm2, \ldots$. Similarly, it follows from (9) that $\cos z = 0$ only when $z = (2n + 1)\pi/2$, $n = 0, \pm1, \pm2, \ldots$.

EXAMPLE 1 Complex Value of the Sine Function

From (6) we have, with the aid of a calculator,

$$\sin(2 + i) = \sin 2 \cosh 1 + i \cos 2 \sinh 1 = 1.4031 - 0.4891i. \equiv$$

In ordinary trigonometry we are accustomed to the fact that $|\sin x| \le 1$ and $|\cos x| \le 1$. Inspection of (8) and (9) shows that these inequalities do not hold for the complex sine and cosine, since $\sinh y$ can range from $-\infty$ to ∞. In other words, it is perfectly feasible to have solutions for equations such as $\cos z = 10$.

EXAMPLE 2 Solving a Trigonometric Equation

Solve the equation $\cos z = 10$.

SOLUTION From (2), $\cos z = 10$ is equivalent to $(e^{iz} + e^{-iz})/2 = 10$. Multiplying the last equation by e^{iz} then gives the quadratic equation in e^{iz}:

$$e^{2iz} - 20e^{iz} + 1 = 0.$$

From the quadratic formula we find $e^{iz} = 10 \pm 3\sqrt{11}$. Thus, for $n = 0, \pm1, \pm2, \ldots$, we have $iz = \log_e(10 \pm 3\sqrt{11}) + 2n\pi i$. Dividing by i and utilizing $\log_e(10 - 3\sqrt{11}) = -\log_e(10 + 3\sqrt{11})$, we can express the solutions of the given equation as $z = 2n\pi \pm i\log_e(10 + 3\sqrt{11})$. $\equiv$

Hyperbolic Functions We define the complex hyperbolic sine and cosine in a manner analogous to the real definitions given in (5).

Definition 17.7.2 Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{10}$$

The hyperbolic tangent, cotangent, secant, and cosecant functions are defined in terms of $\sinh z$ and $\cosh z$:

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{1}{\tanh z}, \quad \text{sech}\, z = \frac{1}{\cosh z}, \quad \text{csch}\, z = \frac{1}{\sinh z}. \tag{11}$$

The hyperbolic sine and cosine are entire functions, and the functions defined in (11) are analytic except at points where the denominators are zero. It is also easy to see from (10) that

$$\frac{d}{dz}\sinh z = \cosh z \quad \text{and} \quad \frac{d}{dz}\cosh z = \sinh z. \tag{12}$$

It is interesting to observe that, in contrast to real calculus, the trigonometric and hyperbolic functions are related in complex calculus. If we replace z by iz everywhere in (10) and compare the results with (2), we see that $\sinh(iz) = i\sin z$ and $\cosh(iz) = \cos z$. These equations enable us to express $\sin z$ and $\cos z$ in terms of $\sinh(iz)$ and $\cosh(iz)$, respectively. Similarly, by replacing z by iz in (2) we can express, in turn, $\sinh z$ and $\cosh z$ in terms of $\sin(iz)$ and $\cos(iz)$. We summarize the results:

$$\sin z = -i\sinh(iz), \quad \cos z = \cosh(iz) \tag{13}$$

$$\sinh z = -i\sin(iz), \quad \cosh z = \cos(iz). \tag{14}$$

Zeros The relationships given in (14) enable us to derive identities for the hyperbolic functions utilizing results for the trigonometric functions. For example, to express $\sinh z$ in the form $u + iv$ we write $\sinh z = -i\sin(iz)$ in the form $\sinh z = -i\sin(-y + ix)$ and use (6):

$$\sinh z = -i[\sin(-y)\cosh x + i\cos(-y)\sinh x].$$

Since $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$, the foregoing expression simplifies to

$$\sinh z = \sinh x\cos y + i\cosh x\sin y. \tag{15}$$

Similarly,
$$\cosh z = \cosh x\cos y + i\sinh x\sin y. \tag{16}$$

It also follows directly from (14) that the **zeros** of $\sinh z$ and $\cosh z$ are pure imaginary and are, respectively,

$$z = n\pi i \quad \text{and} \quad z = (2n+1)\frac{\pi i}{2}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Periodicity Since $\sin x$ and $\cos x$ are 2π-periodic, we can easily demonstrate that $\sin z$ and $\cos z$ are also periodic with the same real period 2π. For example, from (6), note that

$$\begin{aligned}\sin(z + 2\pi) &= \sin(x + 2\pi + iy)\\ &= \sin(x + 2\pi)\cosh y + i\cos(x + 2\pi)\sinh y\\ &= \sin x\cosh y + i\cos x\sinh y;\end{aligned}$$

that is, $\sin(z + 2\pi) = \sin z$. In exactly the same manner, it follows from (7) that $\cos(z + 2\pi) = \cos z$. In addition, the hyperbolic functions $\sinh z$ and $\cosh z$ have the imaginary period $2\pi i$. This last result follows from either Definition 17.7.2 and the fact that e^z is periodic with period $2\pi i$, or from (15) and (16) and replacing z by $z + 2\pi i$.

17.7 Exercises

Answers to selected odd-numbered problems begin on page ANS-40.

In Problems 1–12, express the given quantity in the form $a + ib$.

1. $\cos(3i)$
2. $\sin(-2i)$
3. $\sin\left(\frac{\pi}{4} + i\right)$
4. $\cos(2 - 4i)$
5. $\tan(i)$
6. $\cot\left(\frac{\pi}{2} + 3i\right)$
7. $\sec(\pi + i)$
8. $\csc(1 + i)$
9. $\cosh(\pi i)$
10. $\sinh\left(\frac{3\pi}{2}i\right)$
11. $\sinh\left(1 + \frac{\pi}{3}i\right)$
12. $\cosh(2 + 3i)$

In Problems 13 and 14, verify the given result.

13. $\sin\left(\frac{\pi}{2} + i\ln 2\right) = \frac{5}{4}$
14. $\cos\left(\frac{\pi}{2} + i\ln 2\right) = -\frac{3}{4}i$

In Problems 15–20, find all values of z satisfying the given equation.

15. $\sin z = 2$
16. $\cos z = -3i$
17. $\sinh z = -i$
18. $\sinh z = -1$
19. $\cos z = \sin z$
20. $\cos z = i\sin z$

In Problems 21 and 22, use the definition of equality of complex numbers to find all values of z satisfying the given equation.

21. $\cos z = \cosh 2$

22. $\sin z = i \sinh 2$

23. Prove that $\cos z = \cos x \cosh y - i \sin x \sinh y$.

24. Prove that $\sinh z = \sinh x \cos y + i \cosh x \sin y$.

25. Prove that $\cosh z = \cosh x \cos y + i \sinh x \sin y$.

26. Prove that $|\sinh z|^2 = \sin^2 y + \sinh^2 x$.

27. Prove that $|\cosh z|^2 = \cos^2 y + \sinh^2 x$.

28. Prove that $\cos^2 z + \sin^2 z = 1$.

29. Prove that $\cosh^2 z - \sinh^2 z = 1$.

30. Show that $\tan z = u + iv$, where

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y} \quad \text{and} \quad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

31. Prove that $\tanh z$ is periodic with period πi.

32. Prove that **(a)** $\overline{\sin z} = \sin \bar{z}$ and **(b)** $\overline{\cos z} = \cos \bar{z}$.

17.8 Inverse Trigonometric and Hyperbolic Functions

Introduction As functions of a complex variable z, we have seen that both the trigonometric and hyperbolic functions are periodic. Consequently, these functions do not possess inverses that are functions in the strictest interpretation of that word. The inverses of these analytic functions are multiple-valued functions. As we did in Section 17.6, in the examination of the logarithmic function, we shall drop the adjective "multiple-valued" throughout the discussion that follows.

Inverse Sine The **inverse sine** function, written as $\sin^{-1} z$ or $\arcsin z$, is defined by

$$w = \sin^{-1} z \quad \text{if} \quad z = \sin w. \tag{1}$$

The inverse sine can be expressed in terms of the logarithmic function. To see this we use (1) and the definition of the sine function:

$$\frac{e^{iw} - e^{-iw}}{2i} = z \quad \text{or} \quad e^{2iw} - 2ize^{iw} - 1 = 0.$$

From the last equation and the quadratic formula, we then obtain

$$e^{iw} = iz + (1 - z^2)^{1/2}. \tag{2}$$

Note in (2) we did not use the customary symbolism $\pm\sqrt{1 - z^2}$, since we know from Section 17.2 that $(1 - z^2)^{1/2}$ is two-valued. Solving (2) for w then gives

$$\sin^{-1} z = -i \ln[iz + (1 - z^2)^{1/2}]. \tag{3}$$

Proceeding in a similar manner, we find the inverses of the cosine and tangent to be

$$\cos^{-1} z = -i \ln[z + i(1 - z^2)^{1/2}] \tag{4}$$

$$\tan^{-1} z = \frac{i}{2} \ln \frac{i + z}{i - z}. \tag{5}$$

EXAMPLE 1 Values of an Inverse Sine

Find all values of $\sin^{-1}\sqrt{5}$.

SOLUTION From (3) we have

$$\sin^{-1}\sqrt{5} = -i \ln[\sqrt{5}i + (1 - (\sqrt{5})^2)^{1/2}].$$

With $(1 - (\sqrt{5})^2)^{1/2} = (-4)^{1/2} = \pm 2i$, the preceding expression becomes

$$\begin{aligned} \sin^{-1}\sqrt{5} &= -i \ln[(\sqrt{5} \pm 2)i] \\ &= -i\left[\log_e(\sqrt{5} \pm 2) + \left(\frac{\pi}{2} + 2n\pi\right)i\right], \quad n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

The foregoing result can be simplified a little by noting that $\log_e(\sqrt{5}-2)=\log_e(1/(\sqrt{5}+2))=-\log_e(\sqrt{5}+2)$. Thus for $n = 0, \pm1, \pm2, \ldots,$

$$\sin^{-1}\sqrt{5} = \frac{\pi}{2} + 2n\pi \pm i\log_e(\sqrt{5}+2). \tag{6}$$ ≡

To obtain particular values of, say, $\sin^{-1}z$, we must choose a specific root of $1 - z^2$ and a specific branch of the logarithm. For example, if we choose $(1-(\sqrt{5})^2)^{1/2} = (-4)^{1/2} = 2i$ and the principal branch of the logarithm, then (6) gives the single value

$$\sin^{-1}\sqrt{5} = \frac{\pi}{2} - i\log_e(\sqrt{5}+2).$$

Derivatives The derivatives of the three inverse trigonometric functions considered above can be found by implicit differentiation. To find the derivative of the inverse sine function $w = \sin^{-1}z$, we begin by differentiating $z = \sin w$:

$$\frac{d}{dz}z = \frac{d}{dz}\sin w \quad \text{gives} \quad \frac{dw}{dz} = \frac{1}{\cos w}.$$

Using the trigonometric identity $\cos^2 w + \sin^2 w = 1$ (see Problem 28 in Exercises 17.7) in the form $\cos w = (1-\sin^2 w)^{1/2} = (1-z^2)^{1/2}$, we obtain

$$\frac{d}{dz}\sin^{-1}z = \frac{1}{(1-z^2)^{1/2}}. \tag{7}$$

Similarly, we find that

$$\frac{d}{dz}\cos^{-1}z = \frac{1}{(1-z^2)^{1/2}} \tag{8}$$

$$\frac{d}{dz}\tan^{-1}z = \frac{1}{1+z^2}. \tag{9}$$

It should be noted that the square roots used in (7) and (8) must be consistent with the square roots used in (3) and (4).

EXAMPLE 2 Evaluating a Derivative

Find the derivative of $w = \sin^{-1}z$ at $z = \sqrt{5}$.

SOLUTION In Example 1, if we use $(1-(\sqrt{5})^2)^{1/2} = (-4)^{1/2} = 2i$, then that same root must be used in (7). The value of the derivative consistent with this choice is given by

$$\left.\frac{dw}{dz}\right|_{z=\sqrt{5}} = \frac{1}{(1-(\sqrt{5})^2)^{1/2}} = \frac{1}{(-4)^{1/2}} = \frac{1}{2i} = -\frac{1}{2}i.$$ ≡

Inverse Hyperbolic Functions The inverse hyperbolic functions can also be expressed in terms of the logarithm. We summarize these results for the inverse hyperbolic sine, cosine, and tangent along with their derivatives:

$$\sinh^{-1}z = \ln\left[z + (z^2+1)^{1/2}\right] \tag{10}$$

$$\cosh^{-1}z = \ln\left[z + (z^2-1)^{1/2}\right] \tag{11}$$

$$\tanh^{-1}z = \frac{1}{2}\ln\frac{1+z}{1-z} \tag{12}$$

$$\frac{d}{dz}\sinh^{-1}z = \frac{1}{(z^2+1)^{1/2}} \tag{13}$$

$$\frac{d}{dz}\cosh^{-1}z = \frac{1}{(z^2-1)^{1/2}} \tag{14}$$

$$\frac{d}{dz}\tanh^{-1}z = \frac{1}{1-z^2}. \tag{15}$$

EXAMPLE 3 Values of an Inverse Hyperbolic Cosine

Find all values of $\cosh^{-1}(-1)$.

SOLUTION From (11) with $z = -1$, we get

$$\cosh^{-1}(-1) = \ln(-1) = \log_e 1 + (\pi + 2n\pi)i.$$

Since $\log_e 1 = 0$ we have for $n = 0, \pm 1, \pm 2, \dots,$

$$\cosh^{-1}(-1) = (2n+1)\pi i.$$

≡

17.8 Exercises

Answers to selected odd-numbered problems begin on page ANS-40.

In Problems 1–14, find all values of the given quantity.

1. $\sin^{-1}(-i)$
2. $\sin^{-1}\sqrt{2}$
3. $\sin^{-1} 0$
4. $\sin^{-1}\frac{13}{5}$
5. $\cos^{-1} 2$
6. $\cos^{-1} 2i$
7. $\cos^{-1}\frac{1}{2}$
8. $\cos^{-1}\frac{5}{3}$
9. $\tan^{-1} 1$
10. $\tan^{-1} 3i$
11. $\sinh^{-1}\frac{4}{3}$
12. $\cosh^{-1} i$
13. $\tanh^{-1}(1+2i)$
14. $\tanh^{-1}(-\sqrt{3}i)$

17 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-40.

Answer Problems 1–16 without referring back to the text. Fill in the blank or answer true/false.

1. $\text{Re}(1+i)^{10} =$ _____ and $\text{Im}(1+i)^{10} =$ _____.
2. If z is a point in the third quadrant, then $i\bar{z}$ is in the _____ quadrant.
3. If $z = 3+4i$, then $\text{Re}\left(\dfrac{z}{\bar{z}}\right) =$ _____.
4. $i^{127} - 5i^9 + 2i^{-1} =$ _____
5. If $z = \dfrac{4i}{-3-4i}$, then $|z| =$ _____.
6. Describe the region defined by $1 \le |z+2| \le 3$. _____
7. $\text{Arg}(z+\bar{z}) = 0$ _____
8. If $z = \dfrac{5}{-\sqrt{3}+i}$, then $\text{Arg } z =$ _____.
9. If $e^z = 2i$, then $z =$ _____.
10. If $|e^z| = 1$, then z is a pure imaginary number. _____
11. The principal value of $(1+i)^{(2+i)}$ is _____.
12. If $f(z) = x^2 - 3xy - 5y^3 + i(4x^2y - 4x + 7y)$, then $f(-1+2i) =$ _____.
13. If the Cauchy–Riemann equations are satisfied at a point, then the function is necessarily analytic there. _____
14. $f(z) = e^z$ is periodic with period _____.
15. $\text{Ln}(-ie^3) =$ _____
16. $f(z) = \sin(x - iy)$ is nowhere analytic. _____

In Problems 17–20, write the given number in the form $a + ib$.

17. $i(2-3i)^2(4+2i)$
18. $\dfrac{3-i}{2+3i} + \dfrac{2-2i}{1+5i}$
19. $\dfrac{(1-i)^{10}}{(1+i)^3}$
20. $4e^{\pi i/3}e^{-\pi i/4}$

In Problems 21–24, sketch the set of points in the complex plane satisfying the given inequality.

21. $\text{Im}(z^2) \le 2$
22. $\text{Im}(z+5i) > 3$
23. $\dfrac{1}{|z|} \le 1$
24. $\text{Im}(z) < \text{Re}(z)$
25. Look up the definitions of conic sections in a calculus text. Now describe the set of points in the complex plane that satisfy the equation $|z-2i| + |z+2i| = 5$.