13.1 Separable Partial Differential Equations

Introduction Partial differential equations (PDEs), like ordinary differential equations (ODEs), are classified as *linear* or *nonlinear*. Analogous to a linear ODE (see (6) of Section 1.1), the dependent variable and its partial derivatives appear only to the first power in a linear PDE. In this and the chapters that follow, we are concerned only with linear partial differential equations.

Linear Partial Differential Equation If we let u denote the dependent variable and x and y the independent variables, then the general form of a **linear second-order partial differential equation** is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{1}$$

where the coefficients $A, B, C, \ldots, G$ are constants or functions of x and y. When $G(x, y) = 0$, equation (1) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

EXAMPLE 1 Linear Second-Order PDEs

The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$$

are examples of linear second-order PDEs. The first equation is homogeneous and the second is nonhomogeneous. ≡

Solution of a PDE A **solution** of a linear partial differential equation (1) is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

It is not our intention to examine procedures for finding *general solutions* of linear partial differential equations. Not only is it often difficult to obtain a general solution of a linear second-order PDE, but a general solution is usually not all that useful in applications. Thus our focus throughout will be on finding *particular solutions* of some of the important linear PDEs, that is, equations that appear in many applications.

◀ We are interested only in particular solutions of PDEs.

Separation of Variables Although there are several methods that can be tried to find particular solutions of a linear PDE, in the **method of separation of variables** we seek to find a particular solution of the form of a *product* of a function of x and a function of y,

$$u(x, y) = X(x)Y(y).$$

With this assumption, it is *sometimes* possible to reduce a linear PDE in two variables to two ODEs. To this end we observe that

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

where the primes denote ordinary differentiation.

EXAMPLE 2 Using Separation of Variables

Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$.

SOLUTION Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields

$$X''Y = 4XY'.$$

After dividing both sides by $4XY$, we have separated the variables:

$$\frac{X''}{4X} = \frac{Y'}{Y}.$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x *and* y. In other words, each side of the equation must be a constant. As a practical matter it is convenient to write this real **separation constant** as $-\lambda$. From the two equalities,

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

we obtain the two linear ordinary differential equations

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0. \tag{2}$$

See Example 2, Section 3.9 and Example 1, Section 12.5.

For the three cases for λ: zero, negative, or positive; that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$, the ODEs in (2) are, in turn,

$$X'' = 0 \quad \text{and} \quad Y' = 0, \tag{3}$$

$$X'' - 4\alpha^2 X = 0 \quad \text{and} \quad Y' - \alpha^2 Y = 0, \tag{4}$$

$$X'' + 4\alpha^2 X = 0 \quad \text{and} \quad Y' + \alpha^2 Y = 0. \tag{5}$$

Case I ($\lambda = 0$): The DEs in (3) can be solved by integration. The solutions are $X = c_1 + c_2 x$ and $Y = c_3$. Thus a particular product solution of the given PDE is

$$u = XY = (c_1 + c_2 x)c_3 = A_1 + B_1 x, \tag{6}$$

where we have replaced c_1c_3 and c_2c_3 by A_1 and B_1, respectively.

Case II ($\lambda = -\alpha^2$): The general solutions of the DEs in (4) are

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \quad \text{and} \quad Y = c_6 e^{\alpha^2 y},$$

respectively. Thus, another particular product solution of the PDE is

$$u = XY = (c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x)c_6 e^{\alpha^2 y}$$

or

$$u = A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x, \tag{7}$$

where $A_2 = c_4c_6$ and $B_2 = c_5c_6$.

Case III ($\lambda = \alpha^2$): Finally, the general solutions of the DEs in (5) are

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \quad \text{and} \quad Y = c_9 e^{-\alpha^2 y},$$

respectively. These results give yet another particular solution

$$u = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x, \tag{8}$$

where $A_3 = c_7c_9$ and $B_3 = c_8c_9$. ≡

It is left as an exercise to verify that (6), (7), and (8) satisfy the given partial differential equation $u_{xx} = 4u_y$. See Problem 29 in Exercises 13.1.

Separation of variables is not a general method for finding particular solutions; some linear partial differential equations are simply not separable. You should verify that the assumption $u = XY$ does not lead to a solution for $\partial^2 u/\partial x^2 - \partial u/\partial y = x$.

Superposition Principle

The following theorem is analogous to Theorem 3.1.2 and is known as the **superposition principle**.

Theorem 13.1.1 Superposition Principle

If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k,$$

where the c_i, $i = 1, 2, \ldots, k$ are constants, is also a solution.

Throughout the remainder of the chapter we shall assume that whenever we have an infinite set $u_1, u_2, u_3, \ldots$ of solutions of a homogeneous linear equation, we can construct yet another solution u by forming the infinite series

$$u = \sum_{k=1}^{\infty} c_k u_k$$

where the c_k, $k = 1, 2, \ldots$, are constants.

Classification of Equations A linear second-order partial differential equation in two independent variables with constant coefficients can be classified as one of three types. This classification depends only on the coefficients of the second-order derivatives. Of course, we assume that at least one of the coefficients A, B, and C is not zero.

Definition 13.1.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F, and G are real constants, is said to be

hyperbolic if $\quad B^2 - 4AC > 0,$

parabolic if $\quad B^2 - 4AC = 0,$

elliptic if $\quad B^2 - 4AC < 0.$

EXAMPLE 3 Classifying Linear Second-Order PDEs

Classify the following equations:

(a) $3\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$ **(b)** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$ **(c)** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

SOLUTION **(a)** By rewriting the given equation as

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$. Since $B^2 - 4AC = 0$, the equation is parabolic.

(b) By rewriting the equation as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

we see that $A = 1$, $B = 0$, $C = -1$, and $B^2 - 4AC = -4(1)(-1) > 0$. The equation is hyperbolic.

(c) With $A = 1$, $B = 0$, $C = 1$, and $B^2 - 4AC = -4(1)(1) < 0$, the equation is elliptic. ≡

A detailed explanation of why we would want to classify a second-order partial differential equation is beyond the scope of this text. But the answer lies in the fact that we wish to solve partial differential equations subject to certain side conditions known as boundary and initial conditions. The kinds of side conditions appropriate for a given equation depend on whether the equation is hyperbolic, parabolic, or elliptic.

13.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1–16, use separation of variables to find, if possible, product solutions for the given partial differential equation.

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$

2. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = 0$

3. $u_x + u_y = u$

4. $u_x = u_y + u$

5. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$

6. $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$

7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

8. $y\dfrac{\partial^2 u}{\partial x \partial y} + u = 0$

9. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}, \quad k > 0$

10. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

11. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

12. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

13. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

14. $x^2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

15. $u_{xx} + u_{yy} = u$

16. $a^2 u_{xx} - g = u_{tt}, \quad g$ a constant

In Problems 17–26, classify the given partial differential equation as hyperbolic, parabolic, or elliptic.

17. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

18. $3\dfrac{\partial^2 u}{\partial x^2} + 5\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

19. $\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x \partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0$

20. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$

21. $\dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial^2 u}{\partial x \partial y}$

22. $\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0$

23. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - 6\dfrac{\partial u}{\partial y} = 0$

24. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

25. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

26. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

In Problems 27 and 28, show that the given partial differential equation possesses the indicated product solution.

27. $k\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = \dfrac{\partial u}{\partial t};$

$u = e^{-k\alpha^2 t}(c_1 J_0(\alpha r) + c_2 Y_0(\alpha r))$

28. $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0;$

$u = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha})$

29. Verify that each of the products $u = X(x)Y(y)$ in (6), (7), and (8) satisfies the second-order PDE in Example 2.

30. Definition 13.1.1 generalizes to linear PDEs with coefficients that are functions of x and y. Determine the regions in the xy-plane for which the equation

$$(xy + 1)\frac{\partial^2 u}{\partial x^2} + (x + 2y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, parabolic, or elliptic.

Discussion Problems

In Problems 31 and 32, discuss whether product solutions $u = X(x)Y(y)$ can be found for the given partial differential equation. [*Hint*: Use the superposition principle.]

31. $\dfrac{\partial^2 u}{\partial x^2} - u = 0$

32. $\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial x} = 0$

13.2 Classical PDEs and Boundary-Value Problems

Introduction For the remainder of this and the next chapter we shall be concerned with finding product solutions of the second-order partial differential equations

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0 \tag{1}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

or slight variations of these equations. These classical equations of mathematical physics are known, respectively, as the **one-dimensional heat equation**, the **one-dimensional wave equation**, and **Laplace's equation in two dimensions**. "One-dimensional" refers to the fact that x denotes a spatial dimension whereas t represents time; "two dimensional" in (3) means that x and y are both spatial dimensions. Laplace's equation is abbreviated $\nabla^2 u = 0$, where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is called the **two-dimensional Laplacian** of the function u. In three dimensions the **Laplacian** of u is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

By comparing equations (1)–(3) with the linear second-order PDE given in Definition 13.1.1, with t playing the part of y, we see that the heat equation (1) is parabolic, the wave equation (2) is hyperbolic, and Laplace's equation (3) is elliptic. This classification is important in Chapter 16.

Heat Equation Equation (1) occurs in the theory of heat flow—that is, heat transferred by conduction in a rod or thin wire. The function $u(x, t)$ is temperature. Problems in mechanical vibrations often lead to the wave equation (2). For purposes of discussion, a solution $u(x, t)$ of (2) will represent the displacement of an idealized string. Finally, a solution $u(x, y)$ of Laplace's equation (3) can be interpreted as the steady-state (that is, time-independent) temperature distribution throughout a thin, two-dimensional plate.

Even though we have to make many simplifying assumptions, it is worthwhile to see how equations such as (1) and (2) arise.

Suppose a thin circular rod of length L has a cross-sectional area A and coincides with the x-axis on the interval $[0, L]$. See **FIGURE 13.2.1**. Let us suppose:

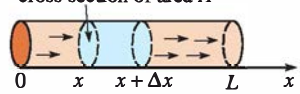

FIGURE 13.2.1 One-dimensional flow of heat

- The flow of heat within the rod takes place only in the x-direction.
- The lateral, or curved, surface of the rod is insulated; that is, no heat escapes from this surface.
- No heat is being generated within the rod.
- The rod is homogeneous; that is, its mass per unit volume ρ is a constant.
- The specific heat γ and thermal conductivity K of the material of the rod are constants.

To derive the partial differential equation satisfied by the temperature $u(x, t)$, we need two empirical laws of heat conduction:

(*i*) *The quantity of heat Q in an element of mass m is*

$$Q = \gamma m u, \tag{4}$$

where u is the temperature of the element.

(*ii*) *The rate of heat flow Q_t through the cross section indicated in Figure* 13.2.1 *is proportional to the area A of the cross section and the partial derivative with respect to x of the temperature*:

$$Q_t = -KAu_x. \tag{5}$$

Since heat flows in the direction of decreasing temperature, the minus sign in (5) is used to ensure that Q_t is positive for $u_x < 0$ (heat flow to the right) and negative for $u_x > 0$ (heat flow to the left). If the circular slice of the rod shown in Figure 13.2.1 between x and $x + \Delta x$ is very thin, then $u(x, t)$ can be taken as the approximate temperature at each point in the interval. Now the mass of the slice is $m = \rho(A\,\Delta x)$, and so it follows from (4) that the quantity of heat in it is

$$Q = \gamma\rho A\,\Delta x\, u. \tag{6}$$

Furthermore, when heat flows in the positive x-direction, we see from (5) that heat builds up in the slice at the net rate

$$-KAu_x(x, t) - [-KAu_x(x + \Delta x, t)] = KA[u_x(x + \Delta x, t) - u_x(x, t)]. \tag{7}$$

By differentiating (6) with respect to t we see that this net rate is also given by

$$Q_t = \gamma\rho A\,\Delta x\,u_t. \tag{8}$$

Equating (7) and (8) gives

$$\frac{K}{\gamma\rho}\frac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x} = u_t. \tag{9}$$

Taking the limit of (9) as $\Delta x \to 0$ finally yields (1) in the form*

$$\frac{K}{\gamma\rho}u_{xx} = u_t.$$

It is customary to let $k = K/\gamma\rho$ and call this positive constant the **thermal diffusivity**.

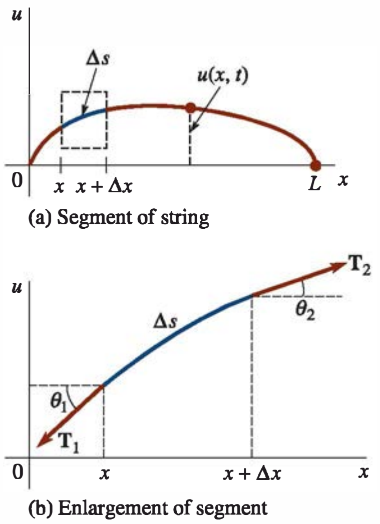

FIGURE 13.2.2 Taut string anchored at two points on the x-axis

Wave Equation

Consider a string of length L, such as a guitar string, stretched taut between two points on the x-axis—say, $x = 0$ and $x = L$. When the string starts to vibrate, assume that the motion takes place in the xy-plane in such a manner that each point on the string moves in a direction perpendicular to the x-axis (transverse vibrations). As shown in FIGURE 13.2.2(a), let $u(x, t)$ denote the vertical displacement of any point on the string measured from the x-axis for $t > 0$. We further assume:

- The string is perfectly flexible.
- The string is homogeneous; that is, its mass per unit length ρ is a constant.
- The displacements u are small compared to the length of the string.
- The slope of the curve is small at all points.
- The tension $\mathbf{T}$ acts tangent to the string, and its magnitude T is the same at all points.
- The tension is large compared with the force of gravity.
- No other external forces act on the string.

Now in Figure 13.2.2(b) the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent to the ends of the curve on the interval $[x, x + \Delta x]$. For small values of θ_1 and θ_2 the net vertical force acting on the corresponding element Δs of the string is then

$$\begin{aligned} T\sin\theta_2 - T\sin\theta_1 &\approx T\tan\theta_2 - T\tan\theta_1 \\ &= T[u_x(x+\Delta x, t) - u_x(x, t)],^{\dagger} \end{aligned}$$

where $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Now $\rho\,\Delta s \approx \rho\,\Delta x$ is the mass of the string on $[x, x + \Delta x]$, and so Newton's second law gives

$$T[u_x(x+\Delta x, t) - u_x(x, t)] = \rho\,\Delta x\,u_{tt}$$

or

$$\frac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x} = \frac{\rho}{T}u_{tt}.$$

If the limit is taken as $\Delta x \to 0$, the last equation becomes $u_{xx} = (\rho/T)u_{tt}$. This of course is (2) with $a^2 = T/\rho$.

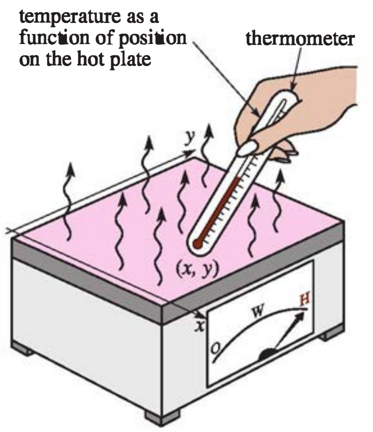

FIGURE 13.2.3 Steady-state temperatures in a rectangular plate

Laplace's Equation

Although we shall not present its derivation, Laplace's equation in two and three dimensions occurs in time-independent problems involving potentials such as electrostatic, gravitational, and velocity in fluid mechanics. Moreover, a solution of Laplace's equation can also be interpreted as a steady-state temperature distribution. As illustrated in FIGURE 13.2.3, a solution $u(x, y)$ of (3) could represent the temperature that varies from point to point—but not with time—of a rectangular plate.

We often wish to find solutions of equations (1), (2), and (3) that satisfy certain side conditions.

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*Recall from calculus that $u_{xx} = \lim_{\Delta x \to 0} \dfrac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x}$.

†$\tan\theta_2 = u_x(x + \Delta x, t)$ and $\tan\theta_1 = u_x(x, t)$ are equivalent expressions for slope.

Initial Conditions Since solutions of (1) and (2) depend on time t, we can prescribe what happens at $t = 0$; that is, we can give **initial conditions (IC)**. If $f(x)$ denotes the initial temperature distribution throughout the rod in Figure 13.2.1, then a solution $u(x, t)$ of (1) must satisfy the single initial condition $u(x, 0) = f(x)$, $0 < x < L$. On the other hand, for a vibrating string, we can specify its initial displacement (or shape) $f(x)$ as well as its initial velocity $g(x)$. In mathematical terms we seek a function $u(x, t)$ satisfying (2) and the two initial conditions:

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{10}$$

For example, the string could be plucked, as shown in **FIGURE 13.2.4**, and released from rest ($g(x) = 0$).

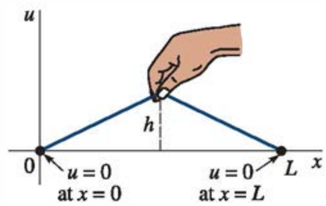


FIGURE 13.2.4 Plucked string

Boundary Conditions The string in Figure 13.2.4 is secured to the x-axis at $x = 0$ and $x = L$ for all time. We interpret this by the two **boundary conditions (BC)**:

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0.$$

Note that in this context the function f in (10) is continuous, and consequently $f(0) = 0$ and $f(L) = 0$. In general, there are three types of boundary conditions associated with equations (1), (2), and (3). On a boundary we can specify the values of *one* of the following:

$$(i)\ u, \quad (ii)\ \frac{\partial u}{\partial n}, \quad \text{or} \quad (iii)\ \frac{\partial u}{\partial n} + hu, \quad h \text{ a constant.}$$

Here $\partial u/\partial n$ denotes the normal derivative of u (the directional derivative of u in the direction perpendicular to the boundary). A boundary condition of the first type (i) is called a **Dirichlet condition**; a boundary condition of the second type (ii) is called a **Neumann condition**; and a boundary condition of the third type (iii) is known as a **Robin condition**. For example, for $t > 0$ a typical condition at the right-hand end of the rod in Figure 13.2.1 can be

$$(i)' \quad u(L, t) = u_0, \quad u_0 \text{ a constant,}$$

$$(ii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad \text{or}$$

$$(iii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -h(u(L, t) - u_m), \quad h > 0 \text{ and } u_m \text{ constants.}$$

Condition $(i)'$ simply states that the boundary $x = L$ is held by some means at a constant *temperature* u_0 for all time $t > 0$. Condition $(ii)'$ indicates that the boundary $x = L$ is *insulated*. From the empirical law of heat transfer, the flux of heat across a boundary (that is, the amount of heat per unit area per unit time conducted across the boundary) is proportional to the value of the normal derivative $\partial u/\partial n$ of the temperature u. Thus when the boundary $x = L$ is thermally insulated, no heat flows into or out of the rod and so

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

We can interpret $(iii)'$ to mean that *heat is lost* from the right-hand end of the rod by being in contact with a medium, such as air or water, that is held at a constant temperature. From Newton's law of cooling, the outward flux of heat from the rod is proportional to the difference between the temperature $u(L, t)$ at the boundary and the temperature u_m of the surrounding medium. We note that if heat is lost from the left-hand end of the rod, the boundary condition is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = h(u(0, t) - u_m).$$

The change in algebraic sign is consistent with the assumption that the rod is at a higher temperature than the medium surrounding the ends so that $u(0, t) > u_m$ and $u(L, t) > u_m$. At $x = 0$ and $x = L$, the slopes $u_x(0, t)$ and $u_x(L, t)$ must be positive and negative, respectively.

Of course, at the ends of the rod we can specify different conditions at the same time. For example, we could have

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad u(L, t) = u_0, \; t > 0.$$

We note that the boundary condition in (*i*)′ is homogeneous if $u_0 = 0$; if $u_0 \neq 0$, the boundary condition is nonhomogeneous. The boundary condition (*ii*)′ is homogeneous; (*iii*)′ is homogeneous if $u_m = 0$ and nonhomogeneous if $u_m \neq 0$.

Boundary-Value Problems

Problems such as

$$\textit{Solve:} \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \; 0 < x < L, \; t > 0$$

$$\textit{Subject to:} \quad \text{(BC)} \; u(0, t) = 0, \; u(L, t) = 0, \; t > 0 \tag{11}$$

$$\text{(IC)} \quad u(x, 0) = f(x), \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \; 0 < x < L$$

and

$$\textit{Solve:} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < a, \; 0 < y < b$$

$$\textit{Subject to:} \quad \text{(BC)} \begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \; \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0, \; 0 < y < b \\ u(x, 0) = 0, \; u(x, b) = f(x), \; 0 < x < a \end{cases} \tag{12}$$

are called **boundary-value problems**. The problems in (11) and (12) are classified as **homogeneous BVPs** since the partial differential equation and the boundary conditions are homogeneous.

Variations

The partial differential equations (1), (2), and (3) must be modified to take into consideration internal or external influences acting on the physical system. More general forms of the one-dimensional heat and wave equations are, respectively,

$$k \frac{\partial^2 u}{\partial x^2} + F(x, t, u, u_x) = \frac{\partial u}{\partial t} \tag{13}$$

and

$$a^2 \frac{\partial^2 u}{\partial x^2} + F(x, t, u, u_t) = \frac{\partial^2 u}{\partial t^2}. \tag{14}$$

For example, if there is heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m, then the heat equation (13) is

$$k \frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t},$$

where h is a constant. In (14) the function F could represent the various forces acting on the string. For example, when external, damping, and elastic restoring forces are taken into account, (14) assumes the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{\overset{\text{external force}}{f(x, t)} - \overset{\text{damping}}{c \frac{\partial u}{\partial t}} - \overset{\text{restoring force}}{ku}}_{F(x, t, u, u_t)} = \frac{\partial^2 u}{\partial t^2}. \tag{15}$$

Remarks

The analysis of a wide variety of diverse phenomena yields the mathematical models (1), (2), or (3) or their generalizations involving a greater number of spatial variables. For example, (1) is sometimes called the **diffusion equation** since the diffusion of dissolved substances in solution is analogous to the flow of heat in a solid. The function $c(x, t)$ satisfying the partial differential equation in this case represents the concentration of the dissolved substance. Similarly, equation (2) and its generalization (15) arise in the analysis of the flow of electricity in a long cable or transmission line. In this setting (2) is known as the **telegraph equation**. It can be shown that under certain assumptions the current $i(x, t)$ and the voltage $v(x, t)$ in the line satisfy two partial differential equations identical to (2) (or (15)). The wave equation (2) also appears in fluid mechanics, acoustics, and elasticity. Laplace's equation (3) is encountered in determining the static displacement of membranes.

13.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1–6, a rod of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the temperature $u(x, t)$.

1. The left end is held at temperature zero, and the right end is insulated. The initial temperature is $f(x)$ throughout.
2. The left end is held at temperature u_0, and the right end is held at temperature u_1. The initial temperature is zero throughout.
3. The left end is held at temperature 100°, and there is heat transfer from the right end into the surrounding medium at temperature zero. The initial temperature is $f(x)$ throughout.
4. There is heat transfer from the left end into a surrounding medium at temperature 20°, and the right end is insulated. The initial temperature is $f(x)$ throughout.
5. The left end is at temperature $\sin(\pi t/L)$, the right end is held at zero, and there is heat transfer from the lateral surface of the rod into the surrounding medium held at temperature zero. The initial temperature is $f(x)$ throughout.
6. The ends are insulated, and there is heat transfer from the lateral surface of the rod into the surrounding medium held at temperature 50°. The initial temperature is 100° throughout.

In Problems 7–10, a string of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the displacement $u(x, t)$.

7. The ends are secured to the x-axis. The string is released from rest from the initial displacement $x(L - x)$.
8. The ends are secured to the x-axis. Initially the string is undisplaced but has the initial velocity $\sin(\pi x/L)$.
9. The left end is secured to the x-axis, but the right end moves in a transverse manner according to $\sin \pi t$. The string is released from rest from the initial displacement $f(x)$. For $t > 0$ the transverse vibrations are damped with a force proportional to the instantaneous velocity.
10. The ends are secured to the x-axis, and the string is initially at rest on that axis. An external vertical force proportional to the horizontal distance from the left end acts on the string for $t > 0$.

In Problems 11 and 12, set up the boundary-value problem for the steady-state temperature $u(x, y)$.

11. A thin rectangular plate coincides with the region in the xy-plane defined by $0 \le x \le 4$, $0 \le y \le 2$. The left end and the bottom of the plate are insulated. The top of the plate is held at temperature zero, and the right end of the plate is held at temperature $f(y)$.
12. A semi-infinite plate coincides with the region defined by $0 \le x \le \pi$, $y \ge 0$. The left end is held at temperature e^{-y}, and the right end is held at temperature 100° for $0 < y \le 1$ and temperature zero for $y > 1$. The bottom of the plate is held at temperature $f(x)$.

13.3 Heat Equation

Introduction Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. If the rod shown in **FIGURE 13.3.1** satisfies the assumptions given on page 693, then the temperature $u(x, t)$ in the rod is determined from the boundary-value problem

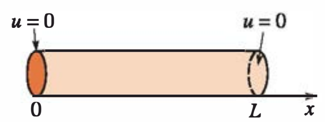


FIGURE 13.3.1 Find the temperature u in a finite rod

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad 0 < x < L. \tag{3}$$

In the discussion that follows next we show how to solve this BVP using the method of separation of variables introduced in Section 13.1.

Solution of the BVP Using the product $u(x, t) = X(x)T(t)$, and $-\lambda$ as the separation constant, leads to

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \tag{4}$$

and

$$X'' + \lambda X = 0 \tag{5}$$

$$T' + k\lambda T = 0. \tag{6}$$

Now the boundary conditions in (2) become $u(0, t) = X(0)T(t) = 0$ and $u(L, t) = X(L)T(t) = 0$. Since the last equalities must hold for all time t, we must have $X(0) = 0$ and $X(L) = 0$. These homogeneous boundary conditions together with the homogeneous ODE (5) constitute a regular Sturm–Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{7}$$

The solution of this BVP was discussed in detail in Example 2 of Section 3.9 and on page 675 of Section 12.5. In that example, we considered three possible cases for the parameter λ: zero, negative, and positive. The corresponding general solutions of the DEs are

$$X(x) = c_1 + c_2 x, \qquad \lambda = 0 \tag{8}$$

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \qquad \lambda = -\alpha^2 < 0 \tag{9}$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \qquad \lambda = \alpha^2 > 0. \tag{10}$$

Recall, when the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to (8) and (9) these solutions yield only $X(x) = 0$ and so we are left with the unusable result $u = 0$. Applying the first boundary condition $X(0) = 0$ to the solution in (10) gives $c_1 = 0$. Therefore $X(x) = c_2 \sin \alpha x$. The second boundary condition $X(L) = 0$ now implies

$$X(L) = c_2 \sin \alpha L = 0. \tag{11}$$

If $c_2 = 0$, then $X = 0$ so that $u = 0$. But (11) can be satisfied for $c_2 \neq 0$ when $\sin \alpha L = 0$. This last equation implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$, where $n = 1, 2, 3, \ldots$. Hence (7) possesses nontrivial solutions when $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$. The values λ_n and the corresponding solutions

$$X(x) = c_2 \sin \frac{n\pi}{L} x, \quad n = 1, 2, 3, \ldots \tag{12}$$

are the **eigenvalues** and **eigenfunctions**, respectively, of the problem in (7).

The general solution of (6) is $T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$, and so

$$u_n = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x, \tag{13}$$

where we have replaced the constant $c_2 c_3$ by A_n. The products $u_n(x, t)$ given in (13) satisfy the partial differential equation (1) as well as the boundary conditions (2) for each value of the positive integer n. However, in order for the functions in (13) to satisfy the initial condition (3), we would have to choose the coefficient A_n in such a manner that

$$u_n(x, 0) = f(x) = A_n \sin \frac{n\pi}{L} x. \tag{14}$$

In general, we would not expect condition (14) to be satisfied for an arbitrary, but reasonable, choice of f. Therefore we are forced to admit that $u_n(x, t)$ *is not a solution of the problem given in* (1)–(3). Now by the superposition principle the function

$$u(x, t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x \tag{15}$$

must also, although formally, satisfy equation (1) and the conditions in (2). If we substitute $t = 0$ into (15), then

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x.$$

This last expression is recognized as the half-range expansion of f in a sine series. If we make the identification $A_n = b_n$, $n = 1, 2, 3, \ldots$, it follows from (5) of Section 12.3 that

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx. \tag{16}$$

We conclude that a solution of the boundary-value problem described in (1), (2), and (3) is given by the infinite series

$$u(x, t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx\right) e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x. \tag{17}$$

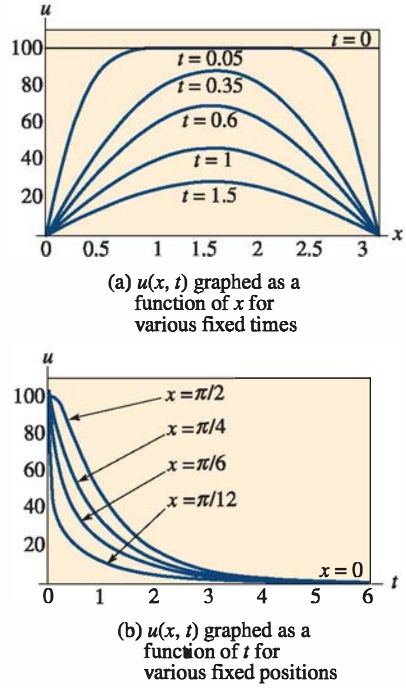
In the special case when the initial temperature is $u(x, 0) = 100$, $L = \pi$, and $k = 1$, you should verify that the coefficients (16) are given by

$$A_n = \frac{200}{\pi}\left[\frac{1 - (-1)^n}{n}\right],$$

and that the series (17) is

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n}\right] e^{-n^2 t} \sin nx. \tag{18}$$

Use of Computers The solution u in (18) is a function of two variables and as such its graph is a surface in 3-space. We could use the 3D-plot application of a computer algebra system to approximate this surface by graphing partial sums $S_n(x, t)$ over a rectangular region defined by $0 \le x \le \pi$, $0 \le t \le T$. Alternatively, with the aid of the 2D-plot application of a CAS we plot the solution $u(x, t)$ on the x-interval $[0, \pi]$ for increasing values of time t. See **FIGURE 13.3.2(a)**. In Figure 13.3.2(b) the solution $u(x, t)$ is graphed on the t-interval [0, 6] for increasing values of x ($x = 0$ is the left end and $x = \pi/2$ is the midpoint of the rod of length $L = \pi$). Both sets of graphs verify that which is apparent in (18)—namely, $u(x, t) \to 0$ as $t \to \infty$.



(a) $u(x, t)$ graphed as a function of x for various fixed times

(b) $u(x, t)$ graphed as a function of t for various fixed positions

FIGURE 13.3.2 Graphs obtained using partial sums of (18)

13.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1 and 2, solve the heat equation (1) subject to the given conditions. Assume a rod of length L.

1. $u(0, t) = 0, \quad u(L, t) = 0$

$$u(x, 0) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

2. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0) = x(L - x)$

3. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.

4. Solve Problem 3 if $L = 2$ and

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

5. Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \quad 0 < x < L, \ t > 0,$$

h a constant. Find the temperature $u(x, t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated. See **FIGURE 13.3.3**.

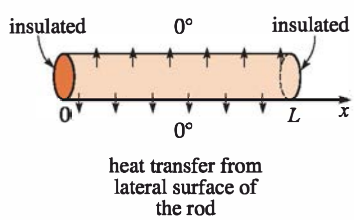


FIGURE 13.3.3 Rod in Problem 5

6. Solve Problem 5 if the ends $x = 0$ and $x = L$ are held at temperature zero.

7. A thin wire coinciding with the x-axis on the interval $[-L, L]$ is bent into the shape of a circle so that the ends $x = -L$ and $x = L$ are joined. Under certain conditions the temperature $u(x, t)$ in the wire satisfies the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -L < x < L, \ t > 0,$$

$$u(-L, t) = u(L, t), \ t > 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}, \ t > 0$$

$$u(x, 0) = f(x), \ -L < x < L,$$

Find the temperature $u(x, t)$.

8. Find the temperature $u(x, t)$ for the boundary-value problem (1)–(3) when $L = 1$ and $f(x) = 100 \sin 6\pi x$. [*Hint*: Look closely at (13) and (14).]

Computer Lab Assignments

9. **(a)** Solve the heat equation (1) subject to

$$u(0, t) = 0, \ u(100, t) = 0, \ t > 0$$

$$u(x, 0) = \begin{cases} 0.8x, & 0 \le x \le 50 \\ 0.8(100 - x), & 50 < x \le 100. \end{cases}$$

(b) Use the 3D-plot application of your CAS to graph the partial sum $S_5(x, t)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le 100$, $0 \le t \le 200$. Assume that $k = 1.6352$. Experiment with various three-dimensional viewing perspectives of the surface (called the **ViewPoint** option in *Mathematica*).

Discussion Problems

10. In Figure 13.3.2(b) we have the graphs of $u(x, t)$ on the interval $[0, 6]$ for $x = 0$, $x = \pi/12$, $x = \pi/6$, $x = \pi/4$, and $x = \pi/2$. Describe or sketch the graphs of $u(x, t)$ on the same time interval but for the fixed values $x = 3\pi/4$, $x = 5\pi/6$, $x = 11\pi/12$, and $x = \pi$.

13.4 Wave Equation

Introduction We are now in a position to solve the boundary-value problem (11) discussed in Section 13.2. The vertical displacement $u(x, t)$ of a string of length L that is freely vibrating in the vertical plane shown in Figure 13.2.2(a) is determined from

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \ t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \ t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ 0 < x < L. \tag{3}$$

Solution of the BVP With the usual assumption that $u(x, t) = X(x)T(t)$, separating variables in (1) gives

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

so that

$$X'' + \lambda X = 0 \tag{4}$$

$$T'' + a^2\lambda T = 0. \tag{5}$$

As in Section 13.3, the boundary conditions (2) translate into $X(0) = 0$ and $X(L) = 0$. The ODE in (4) along with these boundary-conditions is the regular Sturm–Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{6}$$

Of the usual three possibilities for the parameter λ: $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, only the last choice leads to nontrivial solutions. Corresponding to $\lambda = \alpha^2$, $\alpha > 0$, the general solution of (4) is

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$X(0) = 0$ and $X(L) = 0$ indicate that $c_1 = 0$ and $c_2 \sin \alpha L = 0$. The last equation again implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$. The eigenvalues and corresponding eigenfunctions of (6) are $\lambda_n = n^2\pi^2/L^2$ and $X(x) = c_2 \sin \frac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$. The general solution of the second-order equation (5) is then

$$T(t) = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t.$$

By rewriting c_2c_3 as A_n and c_2c_4 as B_n, solutions that satisfy both the wave equation (1) and boundary conditions (2) are

$$u_n = \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x \tag{7}$$

and

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x. \tag{8}$$

Setting $t = 0$ in (8) and using the initial condition $u(x, 0) = f(x)$ gives

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x.$$

Since the last series is a half-range expansion for f in a sine series, we can write $A_n = b_n$:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx. \tag{9}$$

To determine B_n we differentiate (8) with respect to t and then set $t = 0$:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi a}{L} \sin \frac{n\pi a}{L} t + B_n \frac{n\pi a}{L} \cos \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left(B_n \frac{n\pi a}{L}\right) \sin \frac{n\pi}{L} x.$$

In order for this last series to be the half-range sine expansion of the initial velocity g on the interval, the *total* coefficient $B_n n\pi a/L$ must be given by the form b_n in (5) of Section 12.3—that is,

$$B_n \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi}{L} x\, dx$$

from which we obtain

$$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin \frac{n\pi}{L} x\, dx. \tag{10}$$

The solution of the boundary-value problem (1)–(3) consists of the series (8) with coefficients A_n and B_n defined by (9) and (10), respectively.

We note that when the string is released from *rest*, then $g(x) = 0$ for every x in the interval $[0, L]$ and consequently $B_n = 0$.

Plucked String A special case of the boundary-value problem in (1)–(3) when $g(x) = 0$ is a model of a **plucked string**. We can see the motion of the string by plotting the solution or displacement $u(x, t)$ for increasing values of time t and using the animation feature of a CAS. Some frames of a movie generated in this manner are given in **FIGURE 13.4.1**. You are asked to emulate the results given in the figure by plotting a sequence of partial sums of (8). See Problems 7, 8, and 27 in Exercises 13.4.

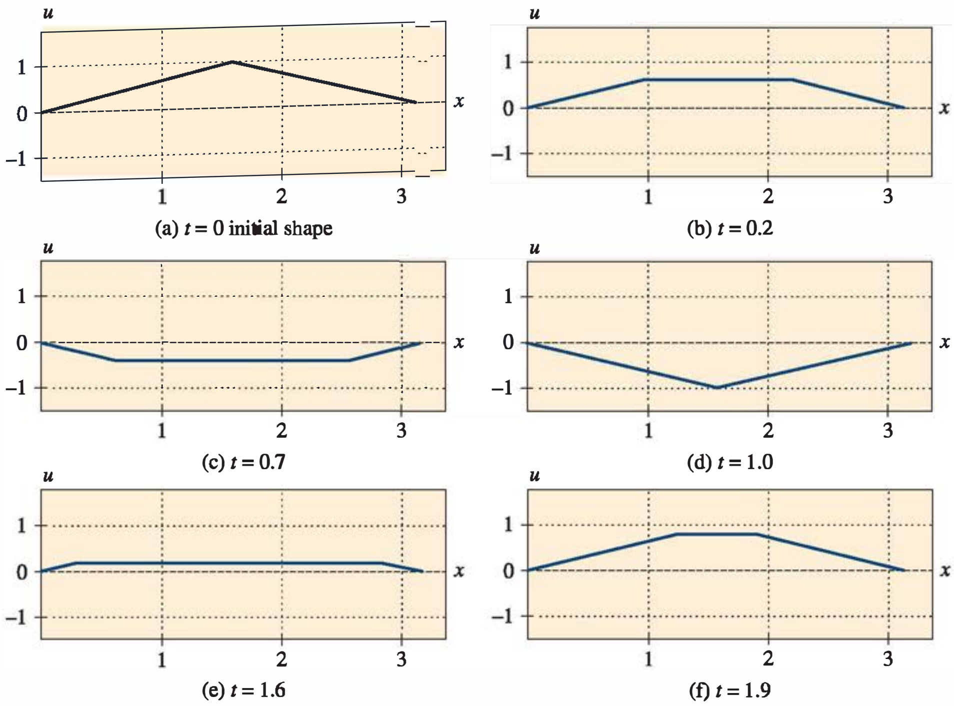


FIGURE 13.4.1 Frames of plucked-string movie

Standing Waves

Recall from the derivation of the wave equation in Section 13.2 that the constant a appearing in the solution of the boundary-value problem in (1)–(3) is given by $\sqrt{T/\rho}$, where ρ is mass per unit length and T is the magnitude of the tension in the string. When T is large enough, the vibrating string produces a musical sound. This sound is the result of standing waves. The solution (8) is a superposition of product solutions called **standing waves** or **normal modes**:

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots.$$

In view of (6) and (7) of Section 3.8, the product solutions (7) can be written as

$$u_n(x, t) = C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right)\sin\frac{n\pi}{L}x, \tag{11}$$

where $C_n = \sqrt{A_n^2 + B_n^2}$ and ϕ_n is defined by $\sin\phi_n = A_n/C_n$ and $\cos\phi_n = B_n/C_n$. For $n = 1, 2, 3, \ldots$ the standing waves are essentially the graphs of $\sin(n\pi x/L)$, with a time-varying amplitude given by

$$C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right).$$

Alternatively, we see from (11) that at a fixed value of x each product function $u_n(x, t)$ represents simple harmonic motion with amplitude $C_n|\sin(n\pi x/L)|$ and frequency $f_n = na/2L$. In other words, each point on a standing wave vibrates with a different amplitude but with the same frequency. When $n = 1$,

$$u_1(x, t) = C_1 \sin\left(\frac{\pi a}{L}t + \phi_1\right)\sin\frac{\pi}{L}x$$

is called the **first standing wave**, the **first normal mode**, or the **fundamental mode of vibration**. The first three standing waves, or normal modes, are shown in **FIGURE 13.4.2**. The dashed graphs represent the standing waves at various values of time. The points in the interval $(0, L)$, for which $\sin(n\pi/L)x = 0$, correspond to points on a standing wave where there is no motion. These points are called **nodes**. For example, in Figures 13.4.2(b) and (c) we see that the second standing wave has one node at $L/2$ and the third standing wave has two nodes at $L/3$ and $2L/3$. In general, the nth normal mode of vibration has $n - 1$ nodes.

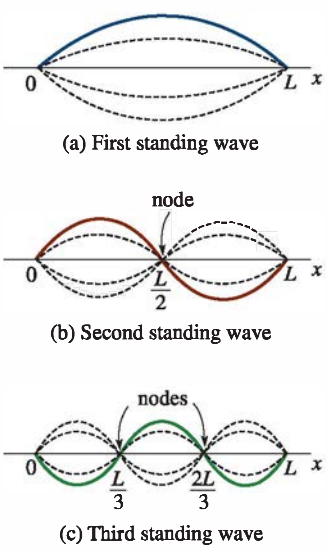


FIGURE 13.4.2 First three standing waves

The frequency

$$f_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

of the first normal mode is called the **fundamental frequency** or **first harmonic** and is directly related to the pitch produced by a stringed instrument. It is apparent that the greater the tension on the string, the higher the pitch of the sound. The frequencies f_n of the other normal modes, which are integer multiples of the fundamental frequency, are called **overtones**. The second harmonic is the first overtone, and so on.

Superposition Principle The superposition principle, Theorem 13.1.1, is the key in making the method of separation of variables an effective means of solving certain kinds of boundary-value problems involving linear partial differential equations. Sometimes a problem can also be solved by using a superposition of solutions of two easier problems. If we can solve each of the problems,

$$\overbrace{\begin{aligned} &a^2\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \\ &u_1(0, t) = 0, \quad u_1(L, t) = 0, \quad t > 0 \\ &u_1(x, 0) = f(x), \quad \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L \end{aligned}}^{\text{Problem 1}} \qquad \overbrace{\begin{aligned} &a^2\frac{\partial^2 u_2}{\partial x^2} = \frac{\partial^2 u_2}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \\ &u_2(0, t) = 0, \quad u_2(L, t) = 0, \quad t > 0 \\ &u_2(x, 0) = 0, \quad \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L \end{aligned}}^{\text{Problem 2}} \qquad (12)$$

then a solution of (1)–(3) is given by $u(x, t) = u_1(x, t) + u_2(x, t)$. To see this we know that $u(x, t) = u_1(x, t) + u_2(x, t)$ is a solution of the homogeneous equation in (1) because of Theorem 13.1.1. Moreover, $u(x, t)$ satisfies the boundary condition (2) and the initial conditions (3) because, in turn,

$$\text{BC}\begin{cases} u(0, t) = u_1(0, t) + u_2(0, t) = 0 + 0 = 0 \\ u(L, t) = u_1(L, t) + u_2(L, t) = 0 + 0 = 0, \end{cases}$$

and

$$\text{IC}\begin{cases} u(x, 0) = u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \left.\frac{\partial u_1}{\partial t}\right|_{t=0} + \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = 0 + g(x) = g(x). \end{cases}$$

You are encouraged to try this method to obtain (8), (9), and (10). See Problems 5 and 14 in Exercises 13.4.

13.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1–6, solve the wave equation (1) subject to the given conditions.

1. $u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$

   $u(x, 0) = \frac{1}{4}x(L - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

2. $u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$

   $u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(L - x), \quad 0 < x < L$

3. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x, \quad 0 < x < \pi$

4. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = \frac{1}{6}x(\pi^2 - x^2), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$

5. $u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$

   $u(x, 0) = x(1 - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(1 - x), \quad 0 < x < 1$

6. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = 0.01 \sin 3\pi x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$

In Problems 7–10, a string is tied to the x-axis at $x = 0$ and at $x = L$ and its initial displacement $u(x, 0) = f(x)$, $0 < x < L$, is shown in the figure. Find $u(x, t)$ if the string is released from rest.

7.

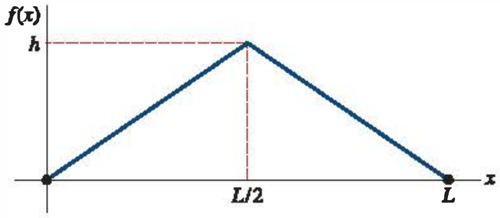


FIGURE 13.4.3 Initial displacement for Problem 7

8.

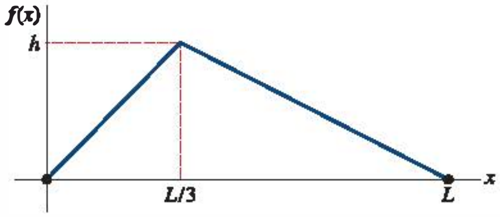


FIGURE 13.4.4 Initial displacement for Problem 8

9.

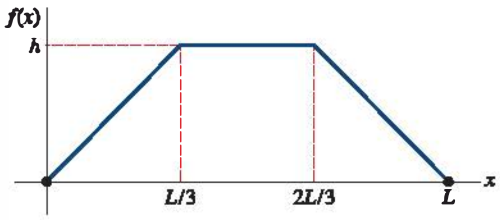


FIGURE 13.4.5 Initial displacement for Problem 9

10.

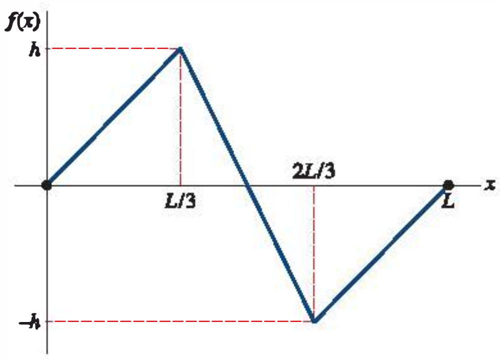


FIGURE 13.4.6 Initial displacement for Problem 10

11. The longitudinal displacement of a vibrating elastic bar shown in **FIGURE 13.4.7** satisfies the wave equation (1) and the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \ t > 0$$

$$u(x, 0) = x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ 0 < x < L.$$

The boundary conditions at $x = 0$ and $x = L$ are called **free-end conditions**. Find the displacement $u(x, t)$.

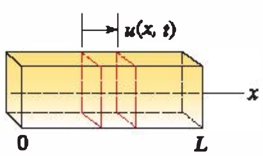


FIGURE 13.4.7 Elastic bar in Problem 11

12. A model for the motion of a vibrating string whose ends are allowed to slide on frictionless sleeves attached to the vertical axes $x = 0$ and $x = L$ is given by the wave equation (1) and the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \ t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ 0 < x < L.$$

See **FIGURE 13.4.8**. The boundary conditions indicate that the motion is such that the slope of the curve is zero at its ends for $t > 0$. Find the displacement $u(x, t)$.

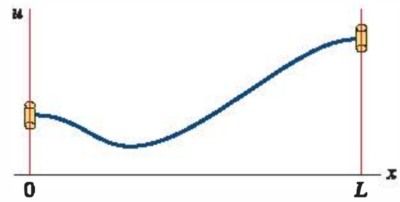


FIGURE 13.4.8 String whose ends are attached to frictionless sleeves in Problem 12

13. In Problem 10, determine the value of $u(L/2, t)$ for $t \geq 0$.

14. Rederive the results given in (8), (9), and (10), but this time use the superposition principle discussed on page 703.

15. A string is stretched and secured on the x-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the wave equation takes on the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t}, \quad 0 < \beta < 1, \quad t > 0.$$

Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.

16. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u, \quad 0 < x < \pi, \ t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \ t > 0$$

$$u(x, 0) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$$

is

$$u(x, t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x\cos\sqrt{(2k-1)^2 + 1}\,t.$$

17. Consider the boundary-value problem given in (1)–(3) of this section. If $g(x) = 0$ on $0 < x < L$, show that the solution of the problem can be written as

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)].$$

[*Hint*: Use the identity
$2\sin\theta_1\cos\theta_2 = \sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)$.]

18. The vertical displacement $u(x, t)$ of an infinitely long string is determined from the initial-value problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \ t > 0$$
$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \tag{13}$$

This problem can be solved without separating variables.

(a) Show that the wave equation can be put into the form $\partial^2 u/\partial\eta\partial\xi = 0$ by means of the substitutions $\xi = x + at$ and $\eta = x - at$.

(b) Integrate the partial differential equation in part (a), first with respect to η and then with respect to ξ, to show that $u(x, t) = F(x + at) + G(x - at)$, where F and G are arbitrary twice differentiable functions, is a solution of the wave equation. Use this solution and the given initial conditions to show that

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds + c$$

and $$G(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds - c,$$

where x_0 is arbitrary and c is a constant of integration.

(c) Use the results in part (b) to show that

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds. \tag{14}$$

Note that when the initial velocity $g(x) = 0$ we obtain

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)], \ -\infty < x < \infty.$$

The last solution can be interpreted as a superposition of two **traveling waves**, one moving to the right (that is, $\frac{1}{2}f(x - at)$) and one moving to the left ($\frac{1}{2}f(x + at)$). Both waves travel with speed a and have the same basic shape as the initial displacement $f(x)$. The form of $u(x, t)$ given in (14) is called **d'Alembert's solution**.

In Problems 19–21, use d'Alembert's solution (14) to solve the initial-value problem in Problem 18 subject to the given initial conditions.

19. $f(x) = \sin x$, $g(x) = 1$

20. $f(x) = \sin x$, $g(x) = \cos x$

21. $f(x) = 0$, $g(x) = \sin 2x$

22. Suppose $f(x) = 1/(1 + x^2)$, $g(x) = 0$, and $a = 1$ for the initial-value problem given in Problem 18. Graph d'Alembert's solution in this case at the time $t = 0$, $t = 1$, and $t = 3$.

23. The transverse displacement $u(x, t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \ 0 < x < L, \ t > 0.$$

If the beam is **simply supported**, as shown in **FIGURE 13.4.9**, the boundary and initial conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0, \ t > 0$$
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \ t > 0$$
$$u(x, 0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: For convenience use $\lambda = \alpha^4$ when separating variables.]

FIGURE 13.4.9 Simply supported beam in Problem 23

≡ Computer Lab Assignments

24. If the ends of the beam in Problem 23 are **embedded** at $x = 0$ and $x = L$, the boundary conditions become, for $t > 0$,

$$u(0, t) = 0, \qquad u(L, t) = 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

(a) Show that the eigenvalues of the problem are $\lambda = x_n^2/L^2$ where x_n, $n = 1, 2, 3, \ldots$, are the positive roots of the equation $\cosh x \cos x = 1$.

(b) Show graphically that the equation in part (a) has an infinite number of roots.

(c) Use a CAS to find approximations to the first four eigenvalues. Use four decimal places.

25. A model for an infinitely long string that is initially held at the three points $(-1, 0)$, $(1, 0)$, and $(0, 1)$ and then simultaneously released at all three points at time $t = 0$ is given by (13) with

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = 0.$$

(a) Plot the initial position of the string on the interval $[-6, 6]$.

(b) Use a CAS to plot d'Alembert's solution (14) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(c) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

26. An infinitely long string coinciding with the x-axis is struck at the origin with a hammer whose head is 0.2 inch in diameter. A model for the motion of the string is given by (13) with

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1, & |x| \le 0.1 \\ 0, & |x| > 0.1. \end{cases}$$