Chapter four

Applications of derivatives

4-1- L'Hopital rule :

Suppose that $f(x_o) = g(x_o) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_o. Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_o. Then :

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists .}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_o$. Stop differentiating as soon as you get something else . L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit .

EX-1 – Evaluate the following limits :

$$1)\ \lim_{x \to o} \frac{\sin x}{x} \qquad 2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x \to 0} \frac{x-\sin x}{x^3} \qquad 4)\ \lim_{x \to \frac{\pi}{2}} -\left(x-\frac{\pi}{2}\right).\tan x$$

Sol. –

$$1)\ \lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 2} \frac{\dfrac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x \to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

$$3)\ \lim_{x \to 0} \frac{x-\sin x}{x^3} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{1-\cos x}{3x^2} \Rightarrow \frac{0}{0} \quad using\ L'Hopital's\ rule \Rightarrow$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6}$$

4) $\lim_{x\to\frac{\pi}{2}} -\left(x-\frac{\pi}{2}\right)\tan x \Rightarrow 0.\infty$ we can't using L'Hoptal's rule $\Rightarrow$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x} . \lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x} . \lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}} . \sin\frac{\pi}{2} = 1$$

4-2- <u>The slope of the curve</u> :

Secant to the curve is a line through two points on a curve.

Slopes and tangent lines :

1. we start with what we can calculate , namely the slope of secant through *P* and a point *Q* nearby on the curve .
2. we find the limiting value of the secant slope (if it exists) as *Q* approaches *p* along the curve .
3. we take this number to be the slope of the curve at *P* and define the tangent to the curve at *P* to be the line through *p* with this slope .

The derivative of the function f is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

<u>*EX-2*</u>- Write an equation for the tangent line at $x = 3$ of the curve :

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

<u>*Sol.*</u>-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$$

4-3- Velocity and acceleration and other rates of changes :

- The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

i.e. $$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v + at$.

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e. $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$average\ rate\ of\ change = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists .

EX-3- The position s (in meters) of a moving body as a function of time t (in second) is : $s = 2t^2 + 5t - 3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .

b) The body's velocity at $t = 2$ seconds .

Sol.-

a) 1) $\Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - \left[2t^2 + 5t - 3\right]$

$$= (4t+5)\Delta t + 2(\Delta t)^2$$

$$at\ t = 0\ and\ \Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2 + 2*2^2 = 18$$

2) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2.\Delta t$

$$at\ t = 0\ \ and\ \Delta t = 2 \Rightarrow v_{av} = 4*0 + 5 + 2*2 = 9$$

b) $v(t) = \dfrac{d}{dt} f(t) = 4t + 5$

$$v(2) = 4*2 + 5 = 13$$

EX-4- A particle moves along a straight line so that after t (seconds), its distance from O a fixed point on the line is s (meters), where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?

ii) what is its velocity and acceleration at these times ?

iii) what is its average velocity during the first second ?

iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. –

i) $at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$

$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ sec.$$

ii) $velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$

$$\Rightarrow v(1) = -1m/s$$

$$\Rightarrow v(2) = 2m/s$$

$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$

$$\Rightarrow a(1) = 0m/s^2$$

$$\Rightarrow a(2) = 6m/s^2$$

iii) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(1) - s(0)}{1 - 0} = \dfrac{1 - 3 + 2 - 0}{1} = 0m/s$

iv) $a_{av} = \dfrac{\Delta v}{\Delta t} = \dfrac{v(2) - v(0)}{2 - 0} = \dfrac{2-2}{2} = 0m/s^2$

4-4- Maxima and Minima :

Increasing and decreasing function **:** **Let f be defined on an interval and x_1 , x_2 denoted a number on that interval :**

- If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .

- If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .

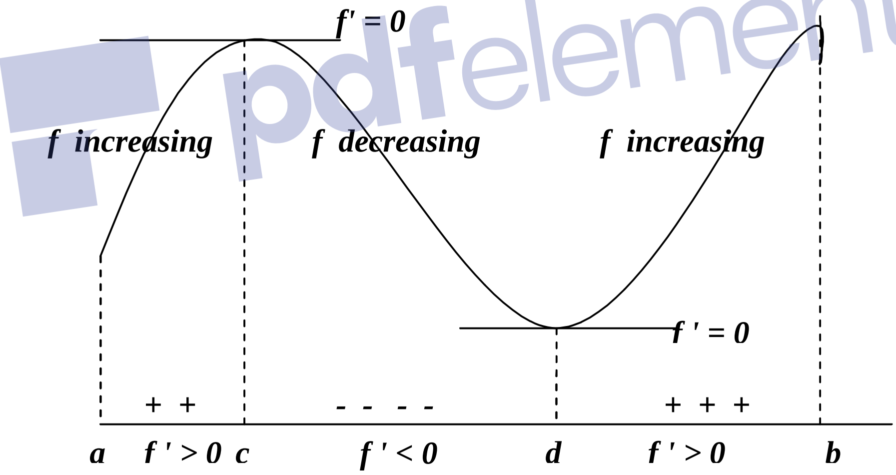
- If $f(x_1) = f(x_2)$ for all values of x_1 , x_2 then f is constant on that interval .

The first derivative test for rise and fall **: Suppose that a function f has a derivative at every point x of an interval I. Then :**

- f increases on I if $f'(x) > o, \quad \forall x \in I$

- f decreases on I if $f'(x) > o, \quad \forall x \in I$

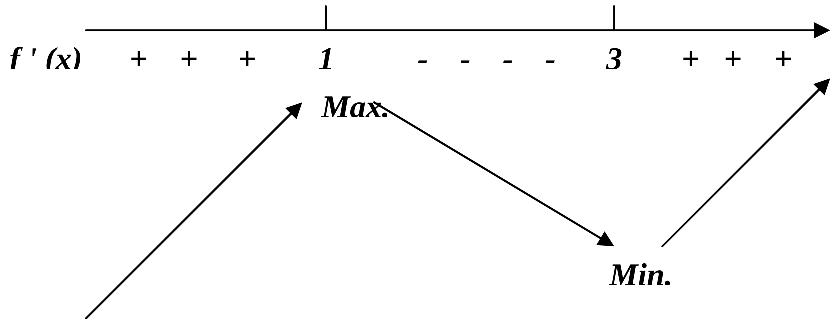
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

EX-5 **– Graph the function :** $y = f(x) = \dfrac{x^3}{3} - 2x^2 + 3x + 2$.

Sol. **-** $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

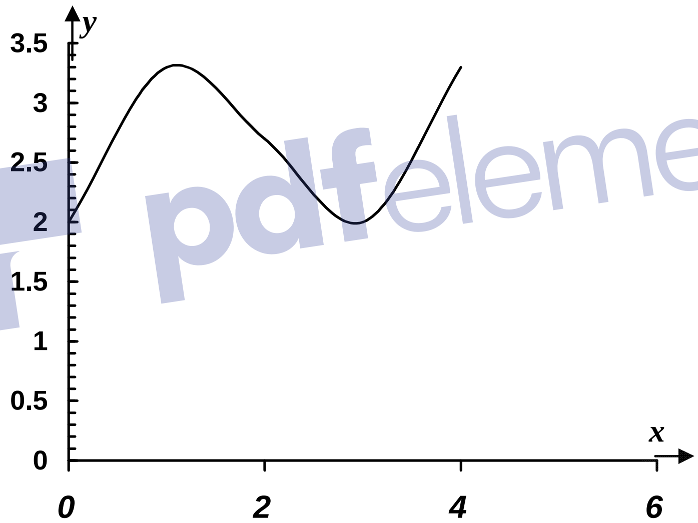


The function has a local maximum at $x = 1$ and a local minimum at $x = 3$.

To get a more accurate curve , we take :

x	*0*	*1*	*2*	*3*	*4*
f(x)	*2*	*3.3*	*2.7*	*2*	*3.3*

Then the graph of the function is :



<u>Concave down and concave up</u> : The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases , and concave up on an interval where f' increases.

<u>The second derivative test for concavity</u> : The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.

<u>Point of inflection</u> : A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where y'' is positive on one side and negative on other , i.e. $y'' = 0$.

<u>*EX-6*</u> – **Sketch the curve :** $y=\frac{1}{6}(x^3-6x^2+9x+6)$.

<u>***Sol.***</u> -

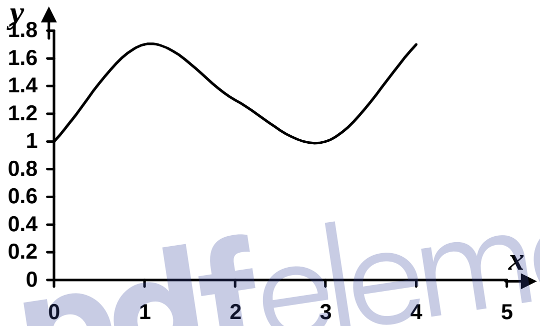
$$y'=\frac{1}{2}x^2-2x+\frac{3}{2}=0 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0 \Rightarrow x=1,3$$

$$y''=x-2 \Rightarrow at\ x=1 \Rightarrow y''=1-2=-1<0\ concave\ down\ .$$

$$\Rightarrow at\ x=3 \Rightarrow y''=3-2>0 \qquad concave\ up\ .$$

$$\Rightarrow at\ y''=0 \Rightarrow x-2=0 \Rightarrow x=2\ \ point\ of\ inflection\ .$$

x	*0*	*1*	*2*	*3*	*4*
y	*1*	*1.7*	*1.3*	*1*	*1.7*



<u>*EX-7*</u> – **What value of** a **makes the function :**

$$f(x)=x^2+\frac{a}{x}\ ,\ \text{have :}$$

i) a local minimum at $x=2$ **?**
ii) a local minimum at $x=-3$ **?**
iii) a point of inflection at $x=1$ **?**
iv) show that the function can't have a local maximum for any value of a **.**

<u>***Sol.***</u> –

$$f(x)=x^2+\frac{a}{x} \Rightarrow \frac{df}{dx}=2x-\frac{a}{x^2}=0 \Rightarrow a=2x^3\ and\ \frac{d^2y}{dx^2}=2+\frac{2a}{x^3}$$

$$i)\quad at\ x = 2 \Rightarrow a = 2*8 = 16\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2*16}{2^3} = 6 > 0\ Mini.$$

$$ii)\quad at\ x = -3 \Rightarrow a = 2(-3)^3 = -54\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0\ Mini.$$

$$iii)\quad at\ x = 1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$$

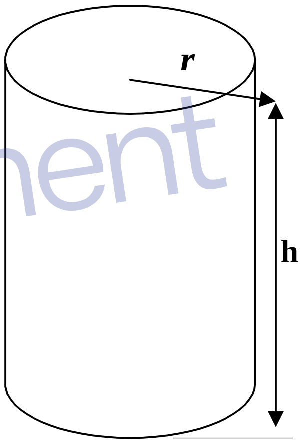
$$iv)\quad a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$$

Since $\frac{d^2 f}{dx^2} > 0$ *for all value of* x *in* $a = 2x^3$.

Hence the function don't have a local maximum .

<u>EX-8</u> **– What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold *1* gallon (*231* cubic inches) ?**

<u>Sol.</u> **– The volume of the can is :**



$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

where *r* is radius , *h* is height .

The total area of the outer surface (top, bottom , and side) is :

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252\ inches$$

$$\frac{d^2 A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow min.$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7}(3.3252)^2} = 6.6474\ inches$$

The dimensions of the can of volume *1* gallon have minimum surface area are :

r = 3.3252 **in. and *h = 6.6474*** **in.**

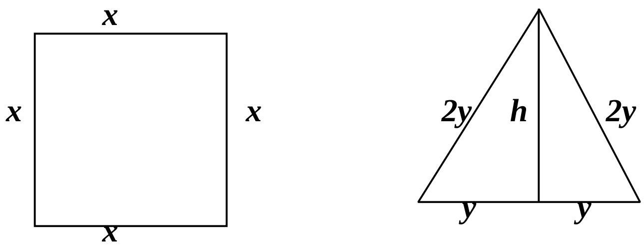
EX-9 **– A wire of length L is cut into two pieces , one being bent to form a square and the other to form an equilateral triangle . How should the wire be cut :**

a) if the sum of the two areas is minimum.

b) if the sum of the two areas is maximum.

Sol. **: Let x is a length of square.**

$2y$ is the edge of triangle .



The perimeter is $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}\,y$ *from triangle .*

The total area is $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}\,y$

$$\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}\,y^2$$

$$\frac{dA}{dy} = \frac{-3}{4}(L - 6y) + 2\sqrt{3}\,y = 0 \Rightarrow y = \frac{3L}{18 + 8\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9}{2} + 2\sqrt{3} > 0 \Rightarrow min.$$

a) To minimized total areas cut for triangle $6y = \frac{9L}{9 + 4\sqrt{3}}$

And for square $L - \frac{9L}{9 + 4\sqrt{3}} = \frac{4\sqrt{3}\,L}{9 + 4\sqrt{3}}$.

b) To maximized the value of A on endpoints of the interval

$$0 \le 4x \le L \Rightarrow 0 \le x \le \frac{L}{4}$$

$$at\ x = 0 \Rightarrow y = \frac{L}{6}\ and\ h = \frac{L}{2\sqrt{3}} \Rightarrow A_1 = \frac{L^2}{12\sqrt{3}}$$

$$at\ \ x = \frac{L}{4} \Rightarrow y = 0 \Rightarrow A_2 = \frac{L^2}{16}$$

$$Since\ \ A_2 = \frac{L^2}{16} > A_1 = \frac{L^2}{12\sqrt{3}}$$

Hence the wire should not be cut at all but should be bent into a square .

Problems – 4

1. Find the velocity v if a particle's position at time t is $s = 180t - 16t^2$
   When does the velocity vanish ? *(ans.: 5.625)*

2. If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ?
   (ans.: 1, 16)

3. A stone is thrown vertically upwards at 35 m./sec. . Its height is : $s = 35t - 4.9t^2$ in meter above the point of projection where t is time in second later :
   a) What is the distance moved, and the average velocity during the 3^{rd} sec. (from $t = 2$ to $t = 3$) ?
   b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; $t = 2$ to $t = 2 + h$.
   c) Deduce the actual velocity at the end of the 2^{nd} sec. .
   (ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)

4. A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t\ (5 - t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :
   i) when is the stone at the ledge level ?
   ii) find its height and velocity after 1 , 2 , 3 , and 6 sec. .
   iii) what meaning is attached to negative value of s ? a negative value of v ?
   iv) when is the stone momentarily at rest ? what is the greatest height reached ?
   v) find the total distance moved during the 3^{rd} sec. .
   (ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5. A stone is thrown vertically downwards with a velocity of 10 m./sec. , and gravity produces on it an acceleration of 9.8 m./sec.^{2} :
   a) what is the velocity after 1 , 2 , 3 , t sec. ?
   b) sketch the velocity –time graph . *(ans.: 19.8, 29.6, 39.4,10+9.8t)*

6. A car accelerates from 5 km./h. to 41 km./h. in 10 sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.^{2}, iii) km./h.^{2} .
   (ans.: i)3.6; ii)1; iii) 12960)

7. A car can accelerate at *4* m./sec.^{2} . How long will it take to reach *90* km./h. from rest ? *(ans.: 6.25)*

8. An express train reducing its velocity to *40* km./h. , has to apply the brakes for *50* sec. . If the retardation produced is *0.5* m./sec.^{2} , find its initial velocity in km./h. . *(ans.: 130)*

9. At the instant from which time is measured a particle is passing through O and traveling towards A , along the straight line OA. It is s m. from O after t sec. where $s = t\,(t-2)^2$:

i) when is it again at O ?

ii) when and where is it momentarily at rest ?

iii) what is the particle's greatest displacement from O , and how far does it moves , during the first *2* sec. ?

iv) what is the average velocity during the *3*rd sec. ?

v) at the end of the *1*st sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?

vi) repeat (v) for the instant when $t = -1$.

(ans.:i)2;ii)0,32/27;iii)64/27;iv)3;v)OA;inceasing; vi)AO;decreasing)

10. A particle moves in a straight line so that after t sec. it is s m. , from a fixed point O on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.

(ans.: i)18, 54,114; ii)58)

11. A particle moves along the x-axis in such away that its distance x cm. from the origin after t sec. is given by the formula $x = 27t - 2t^2$ what are its velocity and acceleration after *6.75* sec. ? How long does it take for the velocity to be reduced from *15* cm./sec. to *9* cm./sec., and how far does the particle travel mean while ? *(ans.: 0,-4,1.5 ;18)*

12. A point moves along a straight line OX so that its distance x cm. from the point O at time t sec. is given by the formula

$x = t^3 - 6t^2 + 9t$. Find :

i) at what times and in what positions the point will have zero velocity .

ii) its acceleration at these instants .

iii) its velocity when its acceleration is zero .

(ans.: i)1,3;4,0; ii)-6,6; iii)-3)

13. A particle moves in a straight line so that its distance *x* cm. from a fixed point *O* on the line is given by $x = 9t^2 - 2t^3$ where *t* is the time in seconds measured from *O* . Find the speed of the particle when $t= 3$. Also find the distance from *O* of the particle when $t = 4$, and show that it is then moving towards *O* . *(ans.: 0, 16)*

14. Find the limits for the following functions by using L'Hopital's rule :

1) $\lim\limits_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$ 2) $\lim\limits_{t\to 0}\dfrac{\sin t^2}{t}$

3) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{2x-\pi}{\cos x}$ 4) $\lim\limits_{t\to 0}\dfrac{\cos t-1}{t^2}$

5) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{1+\cos 2x}$ 6) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

7) $\lim\limits_{x\to 1}\dfrac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ 8) $\lim\limits_{x\to 0}\dfrac{x(\cos x-1)}{\sin x-x}$

9) $\lim\limits_{x\to 0}x.\csc^2\sqrt{2x}$ 10) $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x.\sin x}$

$$(ans.: 1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$$

15. Find any local maximum and local minimum values , then sketch each curve by using first derivative :

1) $f(x)=x^3-4x^2+4x+5$ $(ans.: max.(0.7,6.2);min.(2,5))$

2) $f(x)=\dfrac{x^2-1}{x^2+1}$ $(ans.: min.(0,-1))$

3) $f(x)=x^5-5x-6$ $(ans.: max.(-1,-2);min.(1,-10))$

4) $f(x)=x^{\frac{4}{3}}-x^{\frac{1}{3}}$ $(ans.: min.(0.25,-0.47))$

16. Find the interval of *x*-values on which the curve is concave up and concave down , then sketch the curve :

1) $f(x)=\dfrac{x^3}{3}+x^2-3x$ $(ans.: up(-1,\infty);down(-\infty,-1))$

2) $f(x)=x^2-5x+6$ $(ans.: up(-\infty,\infty))$

3) $f(x)=x^3-2x^2+1$ $(ans.: up(\frac{2}{3},\infty);down(-\infty,\frac{2}{3}))$

4) $f(x)=x^4-2x^2$ $(ans.: up(-\infty,-\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{3}},\infty);down(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}))$

17. Sketch the following curve by using second derivative :

1) $y = \dfrac{x}{1+x^2}$ (ans.: max.(1,0.5); min.(-1,-0.5))

2) $y = -x(x-7)^2$ (ans.: max.(7,0); min.(2.3,-50.8))

3) $y = (x+2)^2(x-3)$ (ans.: max.(-2,0); min.(1.3,-18.5))

4) $y = x^2(5-x)$ (ans.: max.(3.3,18.5); min.(0,0))

18. What is the smallest perimeter possible for a rectangle of area *16* in.^{2} ? *(ans.: 16)*

19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$. *(ans.:32)*

20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With *800* m. of fence at your disposal . What is the largest area you can enclose ? *(ans.:80000)*

21) Show that the rectangle that has maximum area for a given perimeter is a square .

22) A wire of length *L* is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?
(ans.: all bent into a circle)

23) A closed container is made from a right circular cylinder of radius *r* and height *h* with a hemispherical dome on top . Find the relationship between *r* and *h* that maximizes the volume for a given surface area *s* . (ans.: $r = h = \sqrt{\dfrac{s}{5\pi}}$)

24) An open rectangular box is to be made from a piece of cardboard *8* in. wide and *15* in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume . *(ans.: height=5/3; width=14/3; length=35/3)*