

Chapter seven

Application of integrals

7-1- Definite integrals:

If $f(x)$ is continuous in the interval $a \leq x \leq b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

$$1) \int_a^b c f(x) dx = c \int_a^b f(x) dx, \text{ where } c \text{ is constant.}$$

$$2) \int_a^b (f(x) \mp g(x)) dx = \int_a^b f(x) dx \mp \int_a^b g(x) dx$$

$$3) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$4) \text{ Let } a < c < b \text{ then } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$5) \int_a^a f(x) dx = 0$$

$$6) \text{ If } f(x) \geq 0 \text{ for } a \leq x \leq b \text{ then } \int_a^b f(x) dx \geq 0$$

$$7) \text{ If } f(x) \leq g(x) \text{ for } a \leq x \leq b \text{ then } \int_a^b f(x) dx \leq \int_a^b g(x) dx$$

EX-1 – Evaluate the following definite integrals:

$$1) \int_2^6 \frac{dx}{x+2}$$

$$2) \int_{\pi/2}^{3\pi/2} \cos x \, dx$$

$$3) \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

$$4) \int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$$

$$5) \int_{-2}^4 e^{-\frac{x}{2}} \, dx$$

$$6) \int_0^{\pi} (\pi - x) \cdot \cos x \, dx$$

Sol. –

$$1) \int_2^6 \frac{dx}{x+2} = \ln(x+2) \Big|_2^6 = \ln(6+2) - \ln(2+2) = \ln 8 - \ln 4 = 3\ln 2 - 2\ln 2 = \ln 2$$

$$2) \int_{\pi/2}^{3\pi/2} \cos x \, dx = \sin x \Big|_{\pi/2}^{3\pi/2} = \sin\left(\frac{3}{2}\pi\right) - \sin\left(\frac{\pi}{2}\right) = -1 - 1 = -2$$

$$3) \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-\sqrt{3}}^{\sqrt{3}} = \tan^{-1} \sqrt{3} - \tan^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2}{3}\pi$$

$$4) \int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{\sqrt{3}/2} = \sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$5) \int_{-2}^4 e^{-\frac{x}{2}} \, dx = -2e^{-\frac{x}{2}} \Big|_{-2}^4 = -2(e^{-2} - e) = 2(e - e^{-2})$$

$$6) \text{ Let } u = \pi - x \Rightarrow du = -dx \text{ \& } dv = \cos x \, dx \Rightarrow v = \sin x$$

$$\begin{aligned} \int_0^{\pi} (\pi - x) \cdot \cos x \, dx &= (\pi - x) \sin x \Big|_0^{\pi} + \int_0^{\pi} \sin x \, dx = (\pi - x) \sin x - \cos x \Big|_0^{\pi} \\ &= (\pi - \pi) \sin \pi - \cos \pi - ((\pi - 0) \sin 0 - \cos 0) = 0 - (-1) - (0 - 1) = 2 \end{aligned}$$

7-2- Area between two curves:

Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of x that are continuous for $a \leq x \leq b$ then the area bounded above by the y_1 curve, below by y_2 curve and on the sides by the vertical lines $x = a$ and $x = b$ is:-

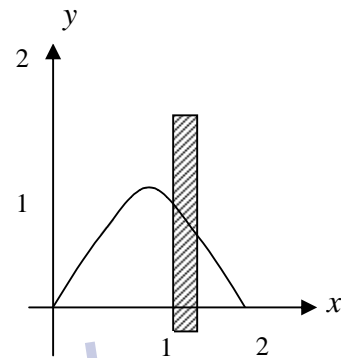
$$A = \int_a^b [f_1(x) - f_2(x)] dx$$

EX-2- Find the area bounded by the x -axis and the curve:

$$y = 2x - x^2$$

Sol.-

$$\left. \begin{array}{l} y = 0 \quad \dots\dots\dots(1) \\ y = 2x - x^2 \quad \dots\dots(2) \end{array} \right\} \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, 2$$



The points of the intersection of the curve and the x -axis are $(0,0)$ and $(2,0)$ then the area bounded by x -axis and the curve is:-

$$\int_0^2 (2x - x^2) dx = x^2 - \frac{x^3}{3} \Big|_0^2 = 4 - \frac{8}{3} - (0 - 0) = \frac{4}{3}$$

EX-3- Find the area bounded by the y -axis and the curve:

$$x = y^2 - y^3$$

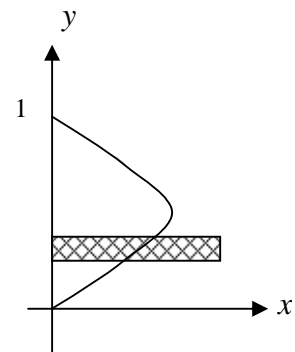
Sol.-

$$\left. \begin{array}{l} x = 0 \quad \dots\dots\dots(1) \\ x = y^2 - y^3 \quad \dots\dots(2) \end{array} \right\} \Rightarrow y^2(1 - y) = 0 \Rightarrow y = 0, 1$$

\Rightarrow intersection points $(0,0), (0,1)$

The area =

$$A = \int_0^1 (y^2 - y^3) dy = \frac{y^3}{3} - \frac{y^4}{4} \Big|_0^1 = \frac{1}{3} - \frac{1}{4} - (0 - 0) = \frac{1}{12}$$



EX-4- Find the area bounded by the curve $y = x^2$ and the line:
 $y = x$

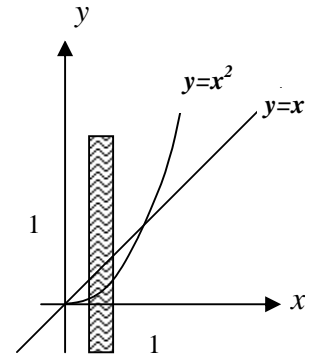
Sol.-

$$\left. \begin{array}{l} y = x^2 \dots\dots(1) \\ y = x \dots\dots(2) \end{array} \right\} \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$$

\Rightarrow intersection points $(0,0), (1,1)$

The area =

$$A = \int_0^1 (x - x^2) dx = \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$$



EX-5- Find the area bounded by the curves $y = x^4 - 2x^2$ and
 $y = 2x^2$

Sol.-

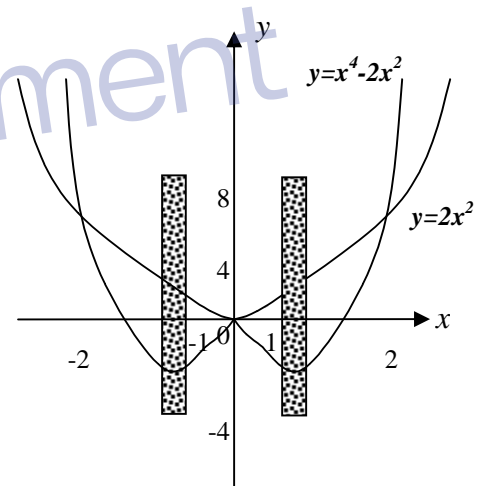
$$\left. \begin{array}{l} y = x^4 - 2x^2 \dots\dots(1) \\ y = 2x^2 \dots\dots\dots(2) \end{array} \right\} \Rightarrow x^2(x^2 - 4) = 0$$

$$\Rightarrow x = 0, 2, -2$$

\Rightarrow intersection points are $(0,0), (2,8), (-2,8)$

The area =

$$\begin{aligned} A &= \int_{-2}^0 (2x^2 - (x^4 - 2x^2)) dx + \int_0^2 (2x^2 - (x^4 - 2x^2)) dx \\ &= 2 \int_0^2 (4x^2 - x^4) dx = 2 \left[\frac{4}{3} x^3 - \frac{x^5}{5} \right]_0^2 = 2 \left[\frac{4}{3} \cdot 8 - \frac{32}{5} - 0 \right] \\ &= \frac{128}{15} \end{aligned}$$



Notice:- We can use the double integration to calculate the area between two curves which bounded above by the curve $y = f_2(x)$ below by $y = f_1(x)$ on the left by the line $x = a$ and on the right by $x = b$, then:-

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy dx$$

To evaluate above integrals we follow:-

- (a) integrating $\int dy$ with respect to y and evaluating the resulting integral the limits $y = f_1(x)$ and $y = f_2(x)$, then:
 (b) integrating the result of (a) with respect to x between the limits $x = a$ and $x = b$.

If the area is bounded on the left by the curve $x = g_1(y)$, on the right by $x = g_2(y)$, below by the line $y = c$, and above by the line $y = d$, then it is better to integrate first with respect to x and then with respect to y . That is:-

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx dy$$

EX-6- Find the area of the triangular region in the first quadrant bounded by the y -axis and the curve $y = \sin x$, $y = \cos x$.

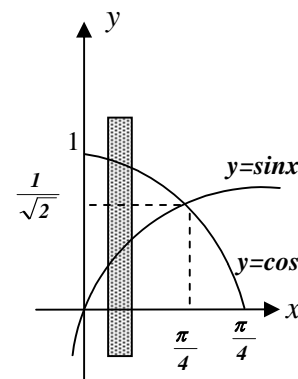
Sol.-

$$\left. \begin{array}{l} y = \sin x \dots(1) \\ y = \cos x \dots(2) \end{array} \right\} \Rightarrow \sin x = \cos x \quad \therefore x = \frac{\pi}{4}$$

The area =

$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy dx = \int_0^{\frac{\pi}{4}} y \Big|_{\sin x}^{\cos x} dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) = \sqrt{2} - 1 = 0.414$$

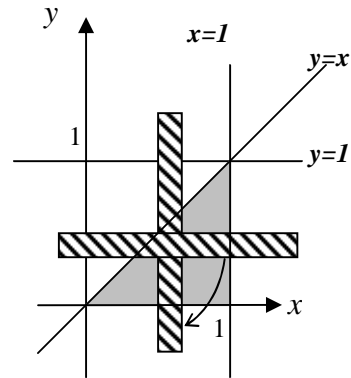


EX-7- Calculate: $\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$

Sol.- We cannot solve the integration

$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$, hence we reverse the order of integration as follow:-

$$\begin{aligned} x &= 1 \quad \text{and} \quad y = 1 \\ x &= y \quad \quad \quad y = 0 \end{aligned}$$



$$\begin{aligned} A &= \int_0^1 \int_0^x \frac{\sin x}{x} dy dx = \int_0^1 \frac{\sin x}{x} y \Big|_0^x dx = \int_0^1 \frac{\sin x}{x} (x - 0) dx \\ &= \int_0^1 \sin x dx = -\cos x \Big|_0^1 = -(\cos 1 - \cos 0) = 1 - \cos 1 \end{aligned}$$

EX-8- Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.

$$1) \int_{-2}^1 \int_{x^2+4x}^{3x+2} dy dx$$

$$2) \int_{-1}^0 \int_{-2x}^{1-x} dy dx + \int_0^2 \int_{-\frac{x}{2}}^{1-x} dy dx$$

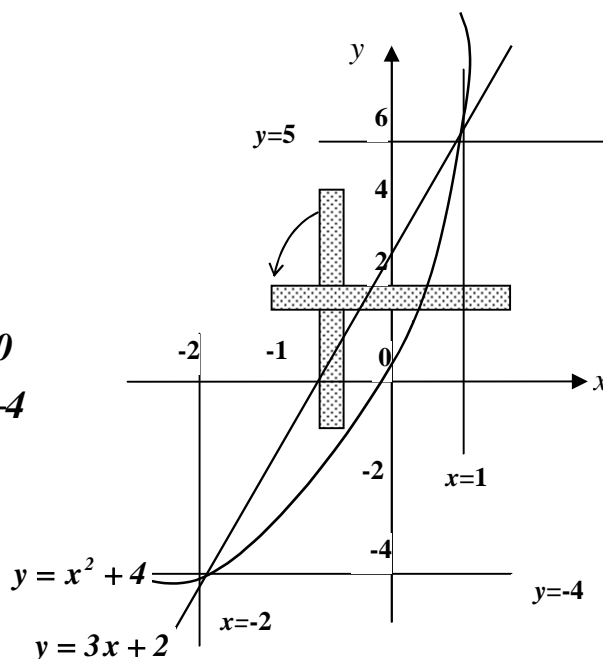
Sol.-

$$1) \left. \begin{aligned} y &= 3x + 2 \quad \dots (1) \\ y &= x^2 + 4x \quad \dots (2) \end{aligned} \right\} \Rightarrow$$

$$(x+2)(x-1) = 0$$

$$\text{either } x = -2 \Rightarrow y = -4$$

$$\text{or } x = 1 \Rightarrow y = 5$$



$$\begin{aligned}
 (a) \int_{-2}^1 \int_{x^2+4x}^{3x+2} dy dx &= \int_{-2}^1 y \Big|_{x^2+4x}^{3x+2} dx = \int_{-2}^1 (2-x-x^2) dx \\
 &= 2x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-2}^1 = 2 - \frac{1}{2} - \frac{1}{3} - (-4 - 2 + \frac{8}{3}) = \frac{9}{2}
 \end{aligned}$$

(b) *The reversed integral is :-*

$$y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$$

$$y = x^2 + 4x \Rightarrow (x+2)^2 = y+4 \Rightarrow x = -2 \mp \sqrt{y+4}$$

$$\text{Since } -2 \leq x \leq 1 \Rightarrow x = -2 + \sqrt{y+4}$$

$$\int_{-4}^5 \int_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dx dy = \int_{-4}^5 x \Big|_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dy = \int_{-4}^5 \left(-2 + \sqrt{y+4} - \frac{y-2}{3} \right) dy$$

$$\begin{aligned}
 &= -2y + \frac{2}{3}(y+4)^{3/2} - \frac{(y-2)^2}{6} \Big|_{-4}^5 \\
 &= -10 + \frac{2}{3}(27) - \frac{9}{6} - (8+0 - \frac{36}{6}) = \frac{9}{2}
 \end{aligned}$$

= *The same result as in (a).*

$$\begin{aligned}
 2) (a) \int_{-1}^0 \int_{-2x}^{1-x} dy dx + \int_0^2 \int_{-\frac{x}{2}}^{1-x} dy dx &= \int_{-1}^0 y \Big|_{-2x}^{1-x} dx + \int_0^2 y \Big|_{-\frac{x}{2}}^{1-x} dx \\
 &= \int_{-1}^0 (1+x) dx + \int_0^2 (1 - \frac{x}{2}) dx = x + \frac{x^2}{2} \Big|_{-1}^0 + x - \frac{x^2}{4} \Big|_0^2 \\
 &= 0 - (-1 + \frac{1}{2}) + 2 - 1 - 0 = \frac{3}{2}
 \end{aligned}$$

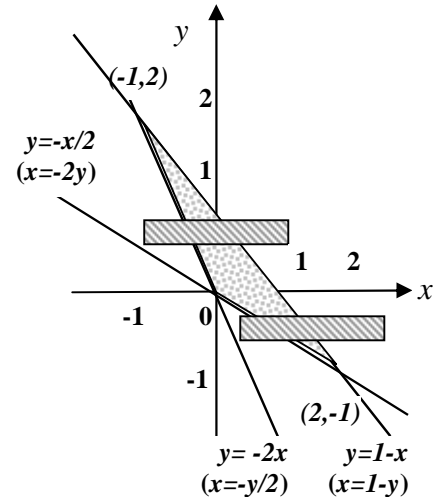
(b) *1st region*

$$\left. \begin{aligned} y &= 1-x \dots\dots(1) \\ y &= -2x \dots\dots(2) \end{aligned} \right\} \Rightarrow x = -1 \Rightarrow y = 2 \quad x \text{ from } -1 \text{ to } 0$$

2nd region

$$\left. \begin{array}{l} y = 1 - x \dots(1) \\ y = -\frac{x}{2} \dots(2) \end{array} \right\} \Rightarrow x = 2 \Rightarrow y = -1 \quad y \text{ from } 0 \text{ to } 2$$

$$\begin{aligned} \int_0^2 \int_{-\frac{y}{2}}^{1-y} dx dy + \int_{-1}^0 \int_{-2y}^{1-y} dx dy &= \int_0^2 x \Big|_{-\frac{y}{2}}^{1-y} dy + \int_{-1}^0 x \Big|_{-2y}^{1-y} dy \\ &= \int_0^2 \left(1 - \frac{y}{2}\right) dy + \int_{-1}^0 (1 + y) dy = y - \frac{y^2}{4} \Big|_0^2 + y + \frac{y^2}{2} \Big|_{-1}^0 \\ &= 2 - 1 - 0 + 0 - \left(-1 + \frac{1}{2}\right) = \frac{3}{2} \\ &= \text{The same result as in (a).} \end{aligned}$$

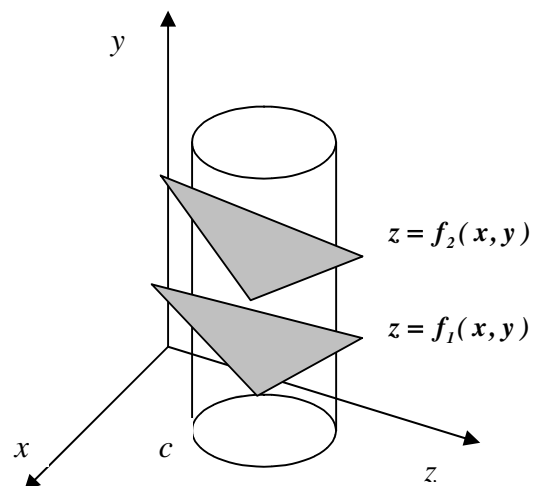


7-3- Triple integrals (Volume):

Consider a region N in xyz -space bounded below by a surface $z = f_1(x, y)$, above by the surface $z = f_2(x, y)$ and laterally by a cylinder c with elements parallel to the z -axis. Let A denote the region of the xy -plane enclosed by cylinder c (that is, A is the region covered by the orthogonal projection of the solid into xy -plane). Then the volume V of the region V can be found by evaluating the triply iterated integral:-

$$V = \iint_A \int_{f_1(x,y)}^{f_2(x,y)} dz dy dx$$

Let z -limits of integration indicate that for every (x, y) in the region A , Z may extend from the lower surface $z = f_1(x, y)$ to the surface $z = f_2(x, y)$. The y - and x -limits of integration have not been given explicitly in equation above, but are indicated as extending over the region A .



We can find the equation of the boundary of the region A by eliminating z between the two equations $z = f_1(x, y)$ and $z = f_2(x, y)$, thus obtaining an equation $f_1(x, y) = f_2(x, y)$ which contains no z , and interpret it as an equation in the xy -plane.

EX-9 The volume in the first octant bounded by the cylinder $x = 4 - y^2$, and the planes $z = y$, $x = 0$, $z = 0$.

Sol.-

$$x = 4 - y^2 \Rightarrow y = \mp \sqrt{4 - x} \quad \text{in first octant : -}$$

$$\begin{aligned} V &= \int_0^4 \int_0^{\sqrt{4-x}} \int_0^y dz dy dx = \int_0^4 \int_0^{\sqrt{4-x}} z \Big|_0^y dy dx = \int_0^4 \int_0^{\sqrt{4-x}} y dy dx = \int_0^4 \frac{y^2}{2} \Big|_0^{\sqrt{4-x}} dx \\ &= \frac{1}{2} \int_0^4 (4 - x - 0) dx = \frac{1}{2} \left[4x - \frac{x^2}{2} \right]_0^4 = \frac{1}{2} \left[16 - \frac{16}{2} - 0 \right] = 4 \end{aligned}$$

EX-10 The volume enclosed by the cylinders $z = 5 - x^2$, $z = 4x^2$ and the planes $y = 0$, $x + y = 1$.

Sol.-

$$\left. \begin{aligned} z &= 5 - x^2 \dots (1) \\ z &= 4x^2 \dots (2) \end{aligned} \right\} \Rightarrow x = \mp 1$$

$$\begin{aligned} V &= \int_{-1}^1 \int_0^{1-x} \int_{4x^2}^{5-x^2} dz dy dx = \int_{-1}^1 \int_0^{1-x} z \Big|_{4x^2}^{5-x^2} dy dx = \int_{-1}^1 \int_0^{1-x} (5 - 5x^2) dy dx \\ &= 5 \int_{-1}^1 (1 - x^2) y \Big|_0^{1-x} dx = 5 \int_{-1}^1 (1 - x^2)(1 - x) dx \\ &= 5 \int_{-1}^1 (1 - x - x^2 + x^3) dx = 5 \left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \right]_{-1}^1 \\ &= 5 \left[(1+1) - \frac{1}{2}(1-1) - \frac{1}{3}(1+1) + \frac{1}{4}(1-1) \right] = \frac{20}{3} \end{aligned}$$

EX-11 The volume enclosed by the cylinders $y^2 + 4z^2 = 16$ and the planes $x = 0$, $x + y = 4$.

Sol.-

$$y^2 + 4z^2 = 16 \Rightarrow y = \mp 2\sqrt{4 - z^2}$$

$$\begin{aligned} V &= \int_{-2}^2 \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} \int_0^{4-y} dx dy dz \\ &= \int_{-2}^2 \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} (4 - y) dy dz = \int_{-2}^2 4y - \frac{y^2}{2} \Big|_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} dz = 16 \int_{-2}^2 (4 - z^2)^{1/2} dz \end{aligned}$$

$$\text{let } z = 2 \sin \theta \Rightarrow dz = 2 \cos \theta d\theta, \quad \theta = \sin^{-1} \frac{z}{2} \quad \begin{array}{l} \text{at } z=2 \Rightarrow \theta = \frac{\pi}{2} \\ \text{at } z=-2 \Rightarrow \theta = \frac{\pi}{2} \end{array}$$

$$\begin{aligned} V &= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 - 4 \sin^2 \theta)^{1/2} 2 \cos \theta d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} d\theta \\ &= 32 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 32 \left[\left(\frac{\pi}{2} + \frac{\pi}{2} \right) + \frac{1}{2} (0 - 0) \right] = 32\pi \end{aligned}$$

EX-12 The volume bounded by the ellipse paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

Sol.-

$$\left. \begin{array}{l} z = 18 - x^2 - 9y^2 \quad \dots(1) \\ z = x^2 + 9y^2 \quad \dots\dots\dots(2) \end{array} \right\} \Rightarrow 9 - x^2 - 9y^2 = 0 \Rightarrow y = \mp \frac{1}{3} \sqrt{9 - x^2}$$

$$V = \int_{-3}^3 \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz dy dx = \int_{-3}^3 \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} [18 - x^2 - 9y^2 - (x^2 + 9y^2)] dy dx$$

$$\begin{aligned}
 V &= 2 \int_{-3}^3 (9-x^2)y - 3y^3 \Big|_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} dx \\
 &= 2 \int_{-3}^3 \left[(9-x^2) \left(\frac{\sqrt{9-x^2}}{3} + \frac{\sqrt{9-x^2}}{3} \right) - 3 \left(\frac{(9-x^2)^{3/2}}{27} + \frac{(9-x^2)^{3/2}}{27} \right) \right] dx \\
 &= \frac{8}{9} \int_{-3}^3 (9-x^2)^{3/2} dx
 \end{aligned}$$

let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta$, $\theta = \sin^{-1} \frac{x}{3}$ $\begin{matrix} \text{at } x=3 \Rightarrow \theta = \frac{\pi}{2} \\ \Rightarrow \Rightarrow \Rightarrow \\ \text{at } x=-3 \Rightarrow \theta = -\frac{\pi}{2} \end{matrix}$

$$\begin{aligned}
 &= \frac{8}{9} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (9-9\sin^2\theta)^{3/2} 3\cos\theta d\theta = 72 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta = 72 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta \\
 &= 18 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2\cos 2\theta + \cos^2 2\theta) d\theta = 18 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+2\cos 2\theta + \frac{\cos 4\theta}{2} \right) d\theta \\
 &= 9 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+4\cos 2\theta + \cos 4\theta) d\theta = 9 \left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= 9 \left[3\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + 2(\sin \pi - \sin(-\pi)) + \frac{1}{4}(\sin 2\pi - \sin(-2\pi)) \right] = 27\pi
 \end{aligned}$$

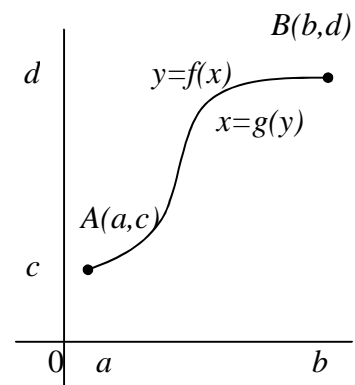
7-4- The length of a plane curve:-

The length of the curve $y = f(x)$ from point $A(a,c)$ to $B(b,d)$ is:-

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If x can be expressed as a function of y then the length is:-

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$



Let the equation of motion be $x = g(t)$ and $y = h(t)$ continuously differentiable for t between t_a (at A) and t_b (at B), then the length of the curve is:-

$$L = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

EX-13 – Find the length of the curve:

1) $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$

2) $9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3},3)$

3) $y = x^{\frac{2}{3}}$ from $x = -1$ to $x = 8$

Sol.

1) $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = x(x^2 + 2)^{\frac{1}{2}}$

$$L = \int_0^3 \sqrt{1 + x^2(x^2 + 2)} dx = \int_0^3 (x^2 + 1) dx = \left. \frac{x^3}{3} + x \right|_0^3 = 9 + 3 - 0 = 12$$

2) $9x^2 = 4y^3 \Rightarrow x = \pm \frac{2}{3}y^{\frac{3}{2}}$ Since x from 0 to $2\sqrt{3}$

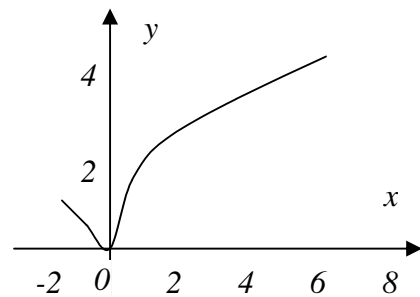
then $x = \frac{2}{3}y^{\frac{3}{2}} \Rightarrow \frac{dx}{dy} = y^{\frac{1}{2}}$

$$L = \int_0^3 \sqrt{1 + y} dy = \left. \frac{2}{3}(1 + y)^{\frac{3}{2}} \right|_0^3 = \frac{2}{3}[8 - 1] = \frac{14}{3}$$

3) $y = x^{\frac{2}{3}} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

Since $\frac{dy}{dx} = \infty$ at $x = 0$

then $x = \pm y^{\frac{3}{2}} \Rightarrow \frac{dx}{dy} = \pm \frac{3}{2}y^{\frac{1}{2}}$



$$L = \int_0^1 \sqrt{1 + \frac{9}{4}y} dy + \int_0^4 \sqrt{1 + \frac{9}{4}y} dy = \frac{1}{18} \left[\left. \frac{(4 + 9y)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^1 + \left. \frac{(4 + 9y)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^4 \right]$$

$$= \frac{1}{27} [(13\sqrt{13} - 4\sqrt{4}) + (40\sqrt{40} - 4\sqrt{4})] = 10.51$$

EX-14 – Find the distance traveled between $t=0$ and $t=\frac{\pi}{2}$ a particle $P(x,y)$ whose position at time t is given by:-
 $x = a \cos t + a \cdot t \sin t$ and $y = a \sin t - a \cdot t \cos t$ where a is a positive constant.

Sol.

$$x = a \cos t + a \cdot t \sin t \Rightarrow \frac{dx}{dt} = a \cdot t \cos t$$

$$y = a \sin t - a \cdot t \cos t \Rightarrow \frac{dy}{dt} = a \cdot t \sin t$$

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{a^2 \cdot t^2 \cos^2 t + a^2 \cdot t^2 \sin^2 t} dt \\ &= a \int_0^{\frac{\pi}{2}} t dt = \frac{a}{2} t^2 \Big|_0^{\frac{\pi}{2}} = \frac{a}{2} \left[\frac{\pi^2}{4} - 0 \right] = \frac{a}{8} \pi^2 \end{aligned}$$

EX-15 – Find the length of the curve:-

$$x = t - \sin t \text{ and } y = 1 - \cos t ; 0 \leq t \leq 2\pi$$

Sol.

$$x = t - \sin t \Rightarrow \frac{dx}{dt} = 1 - \cos t$$

$$y = 1 - \cos t \Rightarrow \frac{dy}{dt} = \sin t$$

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt \\ &= \int_0^{2\pi} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt = \int_0^{2\pi} \sqrt{1 - 2 \cos t + 1} dt \\ &= 2 \int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}} dt = 2 \int_0^{2\pi} \sin \frac{t}{2} dt = -4 \cos \frac{t}{2} \Big|_0^{2\pi} \\ &= -4 [\cos \pi - \cos 0] = -4 [-1 - 1] = 8 \end{aligned}$$

7-5- The surface area:

Suppose that the curve $y = f(x)$ is rotated about the x -axis. It will generate a surface in space. Then the surface area of the shape is:-

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If the curve rotated about the y -axis, then the surface area is:-

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

If the curve sweeps out the surface is given in parametric form with x and y as functions of a third variable t that varies from t_a to t_b then we may compute the surface area from the formula:-

$$S = \int_{t_a}^{t_b} 2\pi \rho \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

where ρ is the distance from the axis of revolution to the element of arc length and is expressed as a function of t .

EX-16 – The circle $x^2 + y^2 = r^2$ is revolved about the x -axis. Find the area of the sphere generated.

Sol.-

$$y = \sqrt{r^2 - x^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-r}^r 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 2\pi r \int_{-r}^r dx \\ &= 2\pi r x \Big|_{-r}^r = 2\pi r(r - (-r)) = 4\pi r^2 \end{aligned}$$

EX-17 – Find the area of the surface generated by rotating the portion of the curve $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$ between $x=0$ and $x=3$ about the y -axis.

Sol.-

$$y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}} \Rightarrow x = ((3y)^{\frac{2}{3}} - 2)^{\frac{1}{2}} \Rightarrow \frac{dx}{dy} = \frac{1}{(3y)^{\frac{1}{3}} \cdot ((3y)^{\frac{2}{3}} - 2)^{\frac{1}{2}}}$$

$$y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}} \quad \Rightarrow \Rightarrow \Rightarrow \quad \text{at } x=0 \quad y = \frac{2\sqrt{2}}{3} \quad \text{and} \quad \Rightarrow \Rightarrow \Rightarrow \quad \text{at } x=3 \quad y = \frac{11\sqrt{11}}{3}$$

$$\begin{aligned} S &= \int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} 2\pi \sqrt{(3y)^{\frac{2}{3}} - 2} \cdot \sqrt{1 + \frac{1}{((3y)^{\frac{2}{3}} - 2)(3y)^{\frac{2}{3}}}} dy \\ &= 2\pi \int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} \sqrt{\frac{(3y)^{\frac{4}{3}} - 2(3y)^{\frac{2}{3}} + 1}{(3y)^{\frac{2}{3}}}} dy = 2\pi \int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} \sqrt{\frac{((3y)^{\frac{2}{3}} - 1)^2}{(3y)^{\frac{2}{3}}}} dy \\ &= 2\pi \int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} \left[(3y)^{\frac{1}{3}} - (3y)^{-\frac{1}{3}} \right] dy = 2\pi \left[\frac{1}{3} \frac{(3y)^{\frac{4}{3}}}{\frac{4}{3}} - \frac{1}{3} \frac{(3y)^{\frac{2}{3}}}{\frac{2}{3}} \right]_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} \\ &= \pi \left[\frac{(3 \cdot \frac{11\sqrt{11}}{3})^{\frac{4}{3}}}{2} - (3 \cdot \frac{11\sqrt{11}}{3})^{\frac{2}{3}} - \frac{(3 \cdot \frac{2\sqrt{2}}{3})^{\frac{4}{3}}}{2} - (3 \cdot \frac{2\sqrt{2}}{3})^{\frac{2}{3}} \right] = \frac{99}{2} \pi \end{aligned}$$

EX-18 – The arc of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x=1$ to $x=3$ is rotated about the line $y=-1$. Find the surface area generated.

Sol.-

$$y = \frac{x^3}{3} + \frac{1}{4x} \Rightarrow \frac{dy}{dx} = x^2 - \frac{1}{4x^2} = \frac{4x^4 - 1}{4x^2}$$

$$\begin{aligned} S &= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1 \right) \sqrt{1 + \frac{(4x^4 - 1)^2}{16x^4}} dx \\ &= 2\pi \int_1^3 \frac{4x^4 + 12x + 3}{12x} \sqrt{\frac{(4x^4 + 1)^2}{16x^4}} dx \\ &= \frac{\pi}{24} \int_1^3 (16x^5 + 48x^2 + 16x + 12x^{-2} + 3x^{-3}) dx \\ &= \frac{\pi}{24} \left[\frac{8}{3}x^6 + 16x^3 + 8x^2 - \frac{12}{x} - \frac{3}{2x^2} \right]_1^3 \\ &= \frac{\pi}{24} \left[\frac{8}{3}(729 - 1) + 16(27 - 1) + 8(9 - 1) - 12\left(\frac{1}{3} - 1\right) - \frac{3}{2}\left(\frac{1}{9} - 1\right) \right] \\ &= \frac{1823}{18} \pi \end{aligned}$$

EX-19 – Find the area of the surface generated by rotating the curve $x = t^2$, $y = t$, $0 \leq t \leq 1$ about the x -axis.

Sol.-

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t \quad \text{and} \quad y = t \Rightarrow \frac{dy}{dt} = 1$$

$$\begin{aligned} S &= \int_{t_a}^{t_b} 2\pi \rho \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2\pi \int_0^1 t \sqrt{4t^2 + 1} dt \\ &= \frac{\pi}{4} \left[\frac{(4t^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \frac{\pi}{6} [5\sqrt{5} - 1] \end{aligned}$$

Problems – 7

1) Find the area of the region bounded by the given curves and lines for the following problems:-

1. The coordinate axes and the line $x + y = a$
2. The x -axis and the curve $y = e^x$ and the lines $x = 0$, $x = 1$
3. The curve $y^2 + x = 0$ and the line $y = x + 2$
4. The curves $x = y^2$ and $x = 2y - y^2$
5. The parabola $x = y - y^2$ and the line $x + y = 0$

$$(ans. : 1. \frac{a^2}{2} ; 2. e - 1 ; 3. \frac{9}{2} ; 4. \frac{1}{3} ; 5. \frac{4}{3})$$

2) Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.

$$1. \int_0^2 \int_1^{e^x} dy dx$$

$$2. \int_0^1 \int_{\sqrt{y}}^1 dx dy$$

$$3. \int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y dx dy$$

$$(ans. : \int_1^{e^2} \int_{\ln y}^2 dx dy ; e^2 - 3)$$

$$(ans. : \int_0^1 \int_0^{x^2} dy dx ; \frac{1}{3})$$

$$(ans. : \int_{-2}^2 \int_0^{\sqrt{\frac{4-x^2}{2}}} y dy dx ; \frac{8}{3})$$

3) Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.

$$(ans. : \frac{1}{6} |abc|)$$

4) Find the volume bounded by the plane $z = 0$ laterally by the elliptic cylinder $x^2 + 4y^2 = 4$ and above by the plane $z = x + 2$.

$$(ans. : 4\pi)$$

5) Find the lengths of the following curves:-

1. $y = x^{\frac{3}{2}}$ from $(0,0)$ to $(4,8)$ (ans.: $\frac{8}{27}(10\sqrt{10} - 1)$)

2. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x=1$ to $x=3$ (ans.: $\frac{53}{6}$)

3. $x = \frac{y^4}{4} + \frac{1}{8y^2}$ from $y=1$ to $y=2$ (ans.: $\frac{123}{32}$)

4. $(y+1)^2 = 4x^3$ from $x=0$ to $x=1$ (ans.: $\frac{4}{27}(10\sqrt{10} - 1)$)

6) Find the distance traveled by the particle $P(x,y)$ between $t=0$ and

$t=4$ if the position at time t is given by: $x = \frac{t^2}{2}$; $y = \frac{1}{3}(2t+1)^{\frac{3}{2}}$
(ans.: 12)

7) The position of a particle $P(x,y)$ at time t is given by:

$x = \frac{1}{3}(2t+3)^{\frac{3}{2}}$; $y = \frac{t^2}{2} + t$. Find the distance it travel between $t=0$

and $t=3$. (ans.: $\frac{21}{2}$)

8) Find the area of the surface generated by rotating about the x -axis the arc of the curve $y = x^3$ between $x=0$ and $x=1$.

(ans.: $\frac{\pi}{27}(10\sqrt{10} - 1)$)

9) Find the area of the surface generated by rotating about the y -axis the arc of the curve $y = x^2$ between $(0,0)$ and $(2,4)$.

(ans.: $\frac{\pi}{6}(17\sqrt{17} - 1)$)

10) Find the area of the surface generated by rotating about the y -

axis the curve $y = \frac{x^2}{2} + \frac{1}{2}$; $0 \leq x \leq 1$. (ans.: $\frac{2}{3}\pi(2\sqrt{2} - 1)$)

11) The curve described by the particle $P(x,y)$ $x = t+1$, $y = \frac{t^2}{2} + t$ from $t=0$ to $t=4$ is rotated about the y -axis. Find the surface area that is generated.

(ans.: $\frac{2\sqrt{2}}{3}\pi(13\sqrt{13} - 1)$)