University Of Anbar College Of Engineering

Electrical Engineering Department

MATHEMATICS-1

1 st class students

Dr. Falah Sh. Khalifa

INTEGRATION

The Definite Integral

DEFINITION **The Definite Integral as a Limit of Riemann Sums**

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

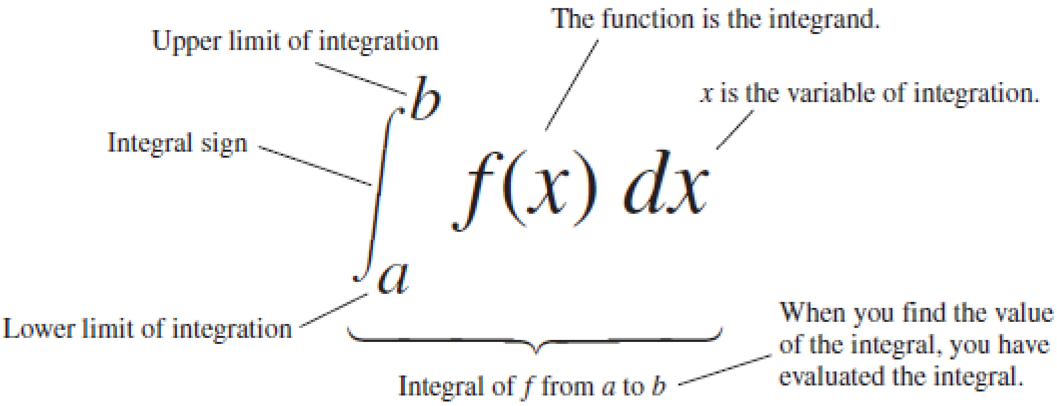
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(u)\,du \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

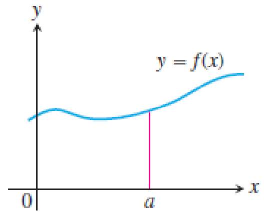
1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

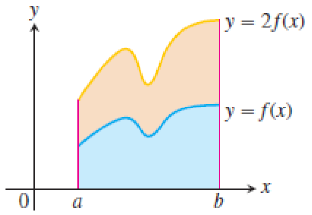
   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

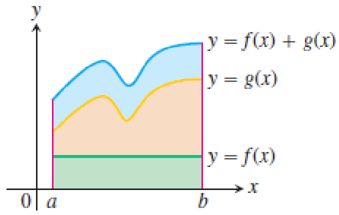
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

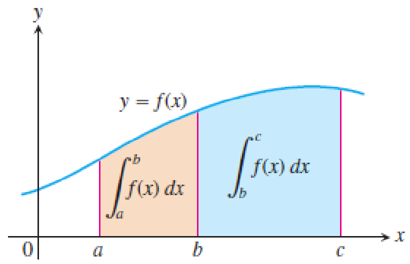
(Shown for $k = 2$.)



(c) *Sum:*

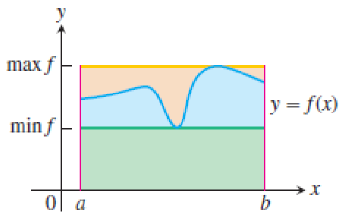
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



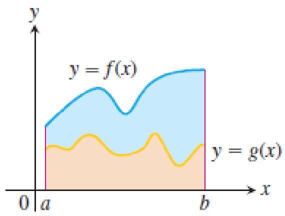
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11

DEFINITION Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

The Fundamental Theorem of Calculus

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

EXAMPLE Use the Fundamental Theorem to find

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\,dt$

(c) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t\sin t\,dt$

(d) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_1^{x^2} \cos t\,dt$

Solution

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt = \cos x$ Eq. 2 with $f(t) = \cos t$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$ Eq. 2 with $f(t) = \dfrac{1}{1+t^2}$

(c) Rule 1 for integrals in Table 5.3 of Section 5.3 sets this up for the Fundamental Theorem.

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right) \quad \text{Rule 1}$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x$$

(d) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\,dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx} = \cos u\cdot\frac{du}{dx} = \cos(x^2)\cdot 2x$$

$$= 2x\cos x^2$$

EXAMPLE Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

Solution The Fundamental Theorem makes it easy to construct a function with derivative $\tan x$ that equals 0 at $x = 3$:

$$y = \int_3^x \tan t\, dt.$$

Since $y(3) = \int_3^3 \tan t\, dt = 0$, we have only to add 5 to this function to construct one with derivative $\tan x$ whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t\, dt + 5.$$ ■

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The theorem says that to calculate the definite integral of f over $[a, b]$ all we need to do is:

1. Find an antiderivative F of f, and
2. Calculate the number $\int_a^b f(x)\, dx = F(b) - F(a)$.

The usual notation for $F(b) - F(a)$ is

(a) $$\int_0^\pi \cos x\, dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^0 \sec x \tan x\, dx = \sec x\Big]_{-\pi/4}^0 = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$
$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$
$$= [8 + 1] - [5] = 4.$$

Total Area

EXAMPLE Calculate the area bounded by the x-axis and the parabola $y = 6 - x - x^2$.

Solution We find where the curve crosses the x-axis by setting

$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x),$$

which gives $\quad x = -3 \quad$ or $\quad x = 2.$

The curve is sketched in Figure 5.21, and is nonnegative on $[-3, 2]$.

The area is

$$\int_{-3}^{2} (6 - x - x^2)\, dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = 20\tfrac{5}{6}.$$

EXAMPLE ' the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$.

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution The definite integral for $f(x) = \sin x$ is given by

$$\int_{0}^{2\pi} \sin x\, dx = -\cos x\Big]_{0}^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_{0}^{\pi} \sin x\, dx = -\cos x\Big]_{0}^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2.$$

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

EXAMPLE

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals,

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them.

The Power Rule in Integral Form

If u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n \, du,$$

If u is any differentiable function, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \qquad (1)$$

EXAMPLE

$$\int \sqrt{1+y^2}\cdot 2y\,dy = \int \sqrt{u}\cdot\left(\frac{du}{dy}\right)dy$$

Let $u = 1 + y^2$, $du/dy = 2y$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C$$

Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{2}{3}u^{3/2} + C$$

Simpler form

$$= \frac{2}{3}(1+y^2)^{3/2} + C$$

Replace u by $1 + y^2$. ■

EXAMPLE

$$\int \sqrt{4t-1}\,dt = \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt$$

Let $u = 4t - 1$, $du/dt = 4$.

$$= \frac{1}{4}\int u^{1/2}\,du$$

With the 1/4 out front, the integral is now in standard form.

$$= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C$$

Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{1}{6}u^{3/2} + C$$

Simpler form

$$= \frac{1}{6}(4t-1)^{3/2} + C$$

Replace u by $4t - 1$. ■

Substitution: Running the Chain Rule Backwards

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof The rule is true because, by the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) \qquad \text{Chain Rule}$$

$$= f(g(x))\cdot g'(x). \qquad \text{Because } F' = f$$

If we make the substitution $u = g(x)$ then

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx$$

$$= F(g(x)) + C \qquad \text{Fundamental Theorem} \qquad = F(u) + C \qquad u = g(x)$$

$$= \int F'(u)\,du \qquad \text{Fundamental Theorem} \qquad = \int f(u)\,du \qquad F' = f$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx, \qquad \int f(u)\,du.$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE Using Substitution

$$\int \cos(7\theta + 5)\,d\theta = \int \cos u \cdot \frac{1}{7}\,du \qquad \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta.$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{With the (1/7) out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u, \text{ Table 4.2.}$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$. ■

EXAMPLE Using Substitution

$$\int x^2 \sin(x^3)\, dx = \int \sin(x^3) \cdot x^2\, dx$$

$$= \int \sin u \cdot \frac{1}{3}\, du \qquad \text{Let } u = x^3,\ du = 3x^2\, dx,\ (1/3)\, du = x^2\, dx.$$

$$= \frac{1}{3}\int \sin u\, du \qquad = \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

EXAMPLE Using Identities and Substitution

$$\int \frac{1}{\cos^2 2x}\, dx = \int \sec^2 2x\, dx \qquad \frac{1}{\cos 2x} = \sec 2x$$

$$= \int \sec^2 u \cdot \frac{1}{2}\, du \qquad u = 2x,\ du = 2\, dx,\ dx = (1/2)\, du$$

$$= \frac{1}{2}\int \sec^2 u\, du \qquad = \frac{1}{2}\tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \frac{1}{2}\tan 2x + C \qquad u = 2x$$

EXAMPLE Using Different Substitutions $\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}}$.

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\, dz.$$

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C \qquad = \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$ Let $u = \sqrt[3]{z^2+1}$,
$u^3 = z^2 + 1$,
$3u^2\,du = 2z\,dz$.

$$= 3\int u\,du \qquad = 3\cdot\frac{u^2}{2} + C$$ Integrate with respect to u.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$.

The Integrals of $\sin^2 x$ and $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can using the substitution rule. Here is an example giving the integral formulas for $\sin^2 x$ and $\cos^2 x$ which arise frequently in applications.

EXAMPLE

(a)
$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b)
$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

As in part (a), but with a sign change

EXAMPLE Area Beneath the Curve $y = \sin^2 x$

(a) the definite integral of $g(x)$ over $[0, 2\pi]$.

(b) the area between the graph of the function and the x-axis over $[0, 2\pi]$.

Solution

$$\int_0^{2\pi} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \left[\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]$$
$$= [\pi - 0] - [0 - 0] = \pi.$$

(b) The function $\sin^2 x$ is nonnegative, so the area is equal to the definite integral, or π.

Substitution and Area Between Curves

Substitution Formula

In the following formula, the limits of integration change when the variable of integration is changed by substitution.

THEOREM 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE Substitution by Two Methods

$$\text{Evaluate } \int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$$

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_0^2 \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C$$ Integrate with respect to u.

$$= \frac{2}{3}(x^3 + 1)^{3/2} + C$$ Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx = \frac{2}{3}(x^3 + 1)^{3/2}\Big]_{-1}^{1}$$

Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3 + 1)^{3/2} - ((-1)^3 + 1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

EXAMPLE Using the Substitution Formula

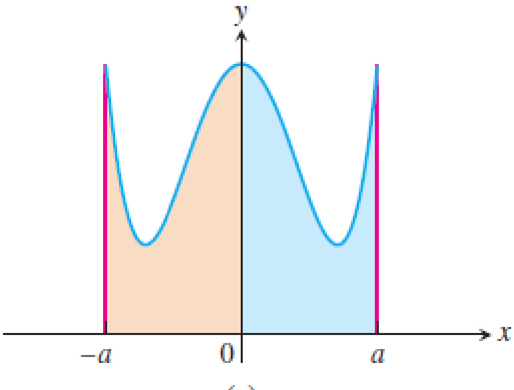
$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\, d\theta = \int_{1}^{0} u \cdot (-du)$$

Let $u = \cot\theta$, $du = -\csc^2\theta\, d\theta$, $-du = \csc^2\theta\, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

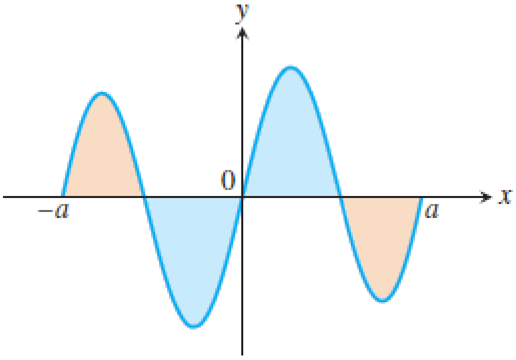
$$= -\int_{1}^{0} u\, du$$

$$= -\left[\frac{u^2}{2}\right]_{1}^{0} = -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

Definite Integrals of Symmetric Functions



(a)



(b)

(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

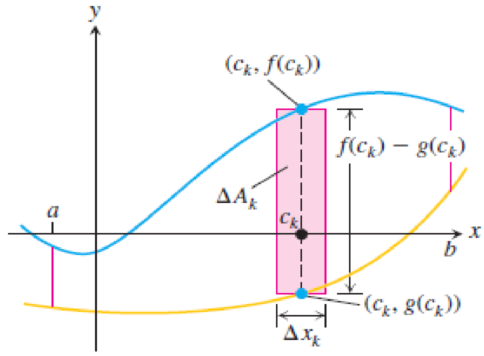
EXAMPLE Evaluate $\displaystyle\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 = 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

Areas Between Curves



DEFINITION **Area Between Curves**

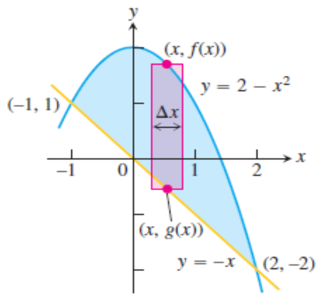
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.



$2 - x^2 = -x$	Equate $f(x)$ and $g(x)$.
$x^2 - x - 2 = 0$	Rewrite.
$(x + 1)(x - 2) = 0$	Factor.
$x = -1, \quad x = 2.$	Solve.

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^2 [(2 - x^2) - (-x)]\, dx$$

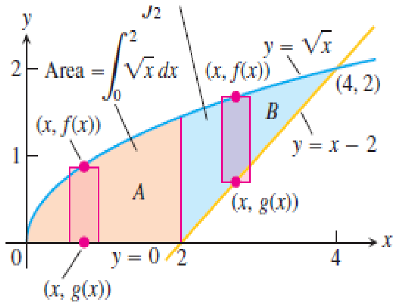
$$= \int_{-1}^2 (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:



$$\sqrt{x} = x - 2$$
$$x = (x-2)^2 = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1, \qquad x = 4.$$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

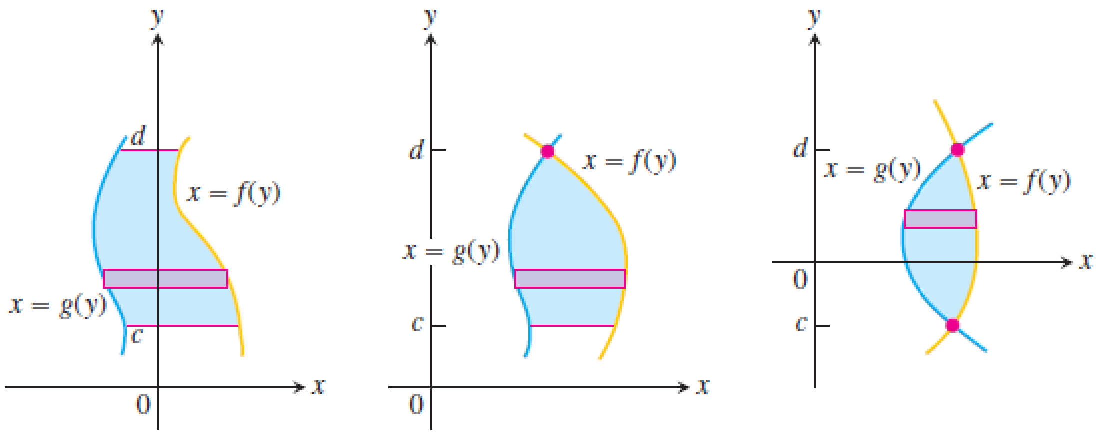
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

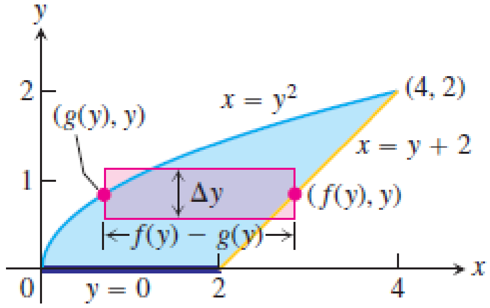
Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.



$$A = \int_c^d [f(y) - g(y)]\,dy.$$

EXAMPLE



$$y + 2 = y^2$$

$$y^2 - y - 2 = 0$$

$$(y + 1)(y - 2) = 0$$

$$y = -1, \qquad y = 2$$

$$A = \int_a^b [f(y) - g(y)]\, dy = \int_0^2 [y + 2 - y^2]\, dy$$

$$= \int_0^2 [2 + y - y^2]\, dy \quad = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \quad = \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

EXAMPLE find the area

$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3}x^{3/2}\Big]_0^4 - 2$$

$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.$$