Applications of Definite Integrals

Volumes by Slicing and Rotation About an Axis

Volume = area × height = $A \cdot h$.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

Solids of Revolution: The Disk Method

$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

EXAMPLE

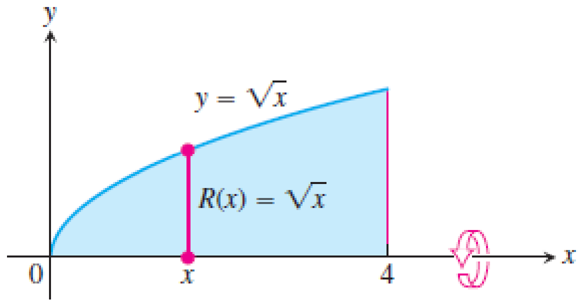
The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid

$$V = \int_a^b \pi[R(x)]^2\,dx$$

$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx$$

$$= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.$$



EXAMPLE Volume of a Sphere

The circle $x^2 + y^2 = a^2$

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

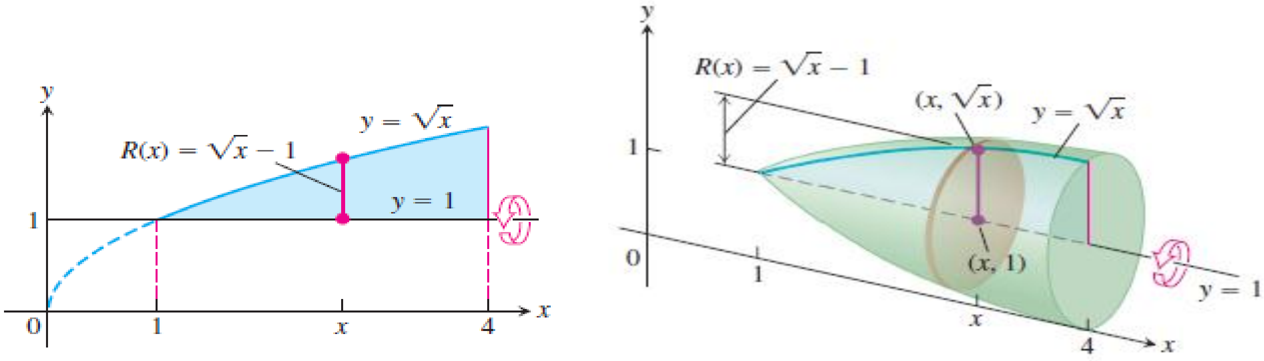
$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

EXAMPLE

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

$$V = \int_1^4 \pi[R(x)]^2\,dx \qquad = \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx$$

$$= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \quad = \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.$$

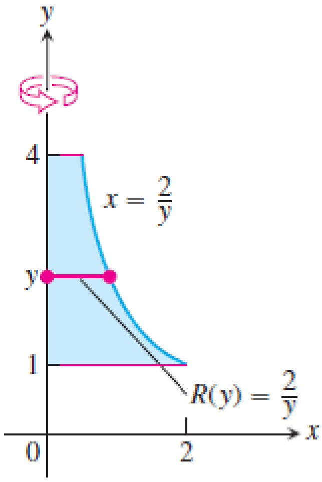


(a) (b)

EXAMPLE

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid



$$V = \int_1^4 \pi [R(y)]^2 \, dy$$

$$= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy$$

$$= \pi \int_1^4 \frac{4}{y^2} \, dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right]$$

$$= 3\pi .$$

EXAMPLE

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

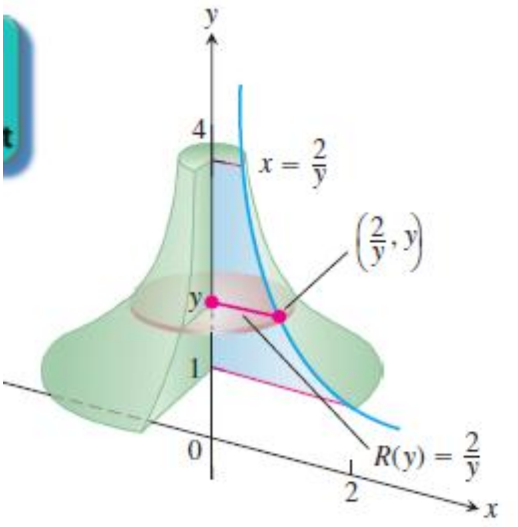
$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 \, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 \, dy$$

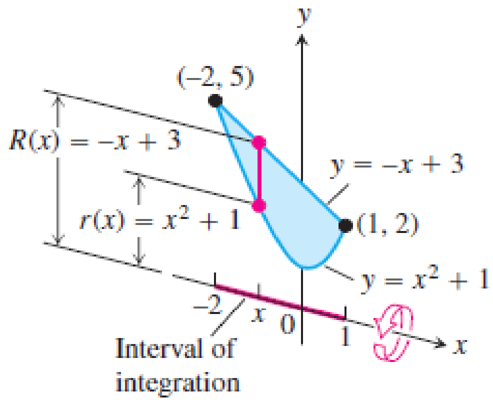
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] \, dy$$

$$= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$



Solids of Revolution: The Washer Method



Outer radius: $R(x)$

Inner radius: $r(x)$

The washer's area is $A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2)$.

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

EXAMPLE

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution

2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

Outer radius: $R(x) = -x + 3$

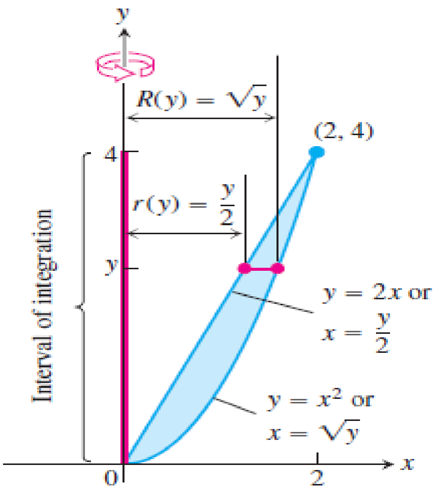
Inner radius: $r(x) = x^2 + 1$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

4. Evaluate the volume integral.

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx$$
$$= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx$$
$$= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx$$
$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}$$



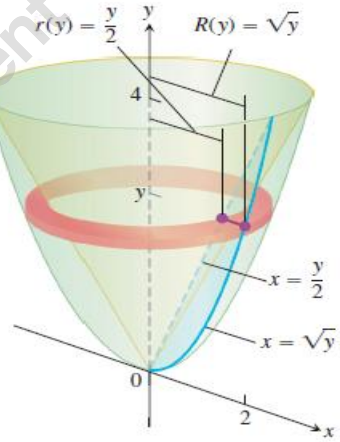
EXAMPLE

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$
$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right)dy$$
$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$



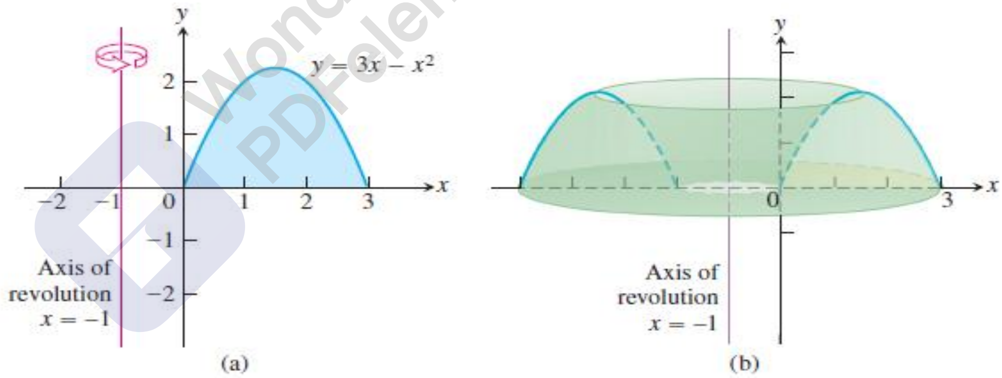
Volumes by Cylindrical Shells

we defined the volume of a solid S as the definite integral $V = \int_a^b A(x)\,dx,$

where $A(x)$ is an integrable cross-sectional area of S from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. In this section we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way. Now we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid perpendicular to the x-axis, with the axis of the cylinder parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid S is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This allows us to apply the same integral definition for volume as before. Before describing the method in general, let's look at an example to gain some insight.

EXAMPLE

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid Find the volume of the solid.



$$V = \int_0^3 2\pi(x+1)(3x - x^2)\,dx$$

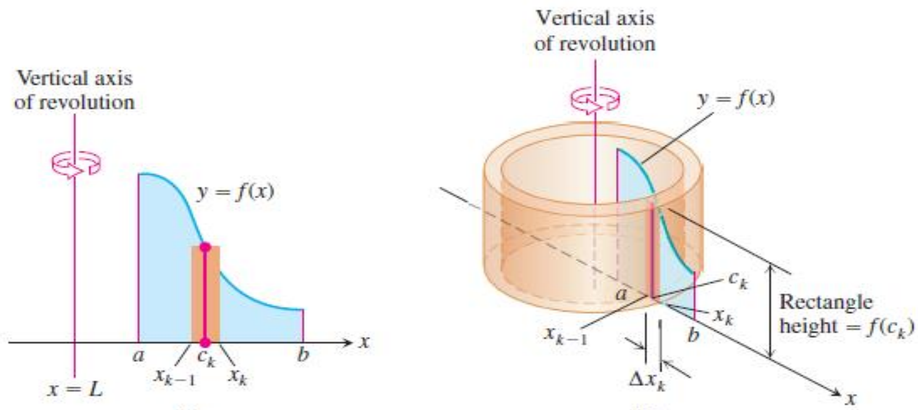
$$= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2)\,dx$$

$$= 2\pi \int_0^3 (2x^2 + 3x - x^3)\,dx$$

$$= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_0^3$$

$$= \frac{45\pi}{2}.$$

The Shell Method



(a) (b)

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx.$$

We refer to the variable of integration, here x, as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

> **Shell Formula for Revolution About a Vertical Line**
> The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is
>
> $$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx.$$

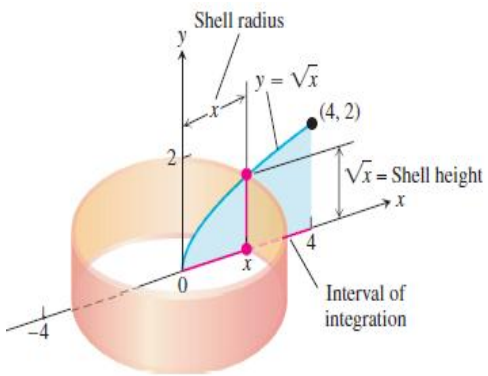
EXAMPLE

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$



EXAMPLE

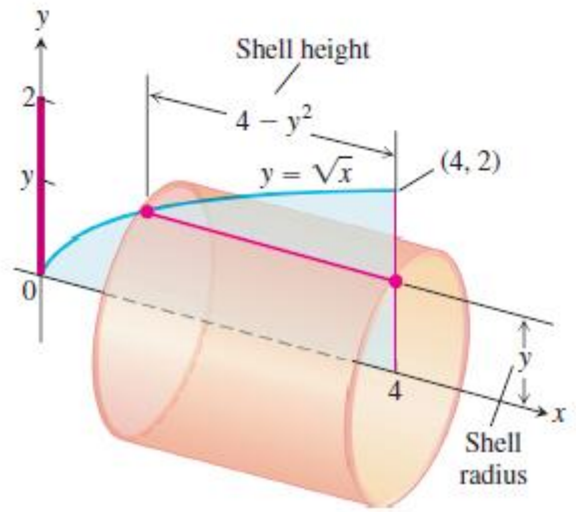
The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

$$= \int_0^2 2\pi(y)(4 - y^2)\, dy$$

$$= \int_0^2 2\pi(4y - y^3)\, dy$$

$$= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi .$$



Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find* the limits of integration for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

Lengths of Plane Curves

We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve. The idea of approximating the length of a curve running from point A to point B by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks. Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter

Length of a Parametrically Defined Curve

DEFINITION **Length of a Parametric Curve**

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \quad \text{and} \quad y = r\sin t, \quad 0 \le t \le 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

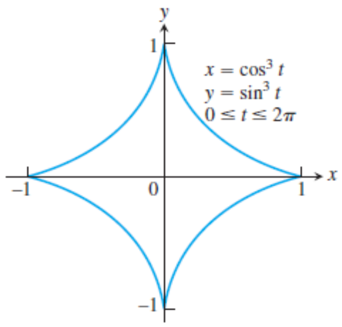
$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\big[t\big]_0^{2\pi} = 2\pi r.$$

EXAMPLE Find the length of the astroid $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have



$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t} \quad = 3|\cos t \sin t| \quad = 3\cos t \sin t.$$

$\cos t \sin t \ge 0$ for $0 \le t \le \pi/2$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt$$

$\cos t \sin t = (1/2)\sin 2t$

$$= -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}. \qquad = \frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt$$

Formula for the Length of $y = f(x), \quad a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2}$$

EXAMPLE Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1.$

Solution

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad \frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (2) with $a = 0, b = 1$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}.$$

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

Dealing with Discontinuities in dy/dx

Formula for the Length of $x = g(y)$, $\quad c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (3)$$

EXAMPLE Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$. The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

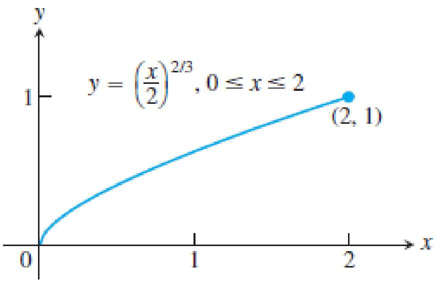
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1 + 9y}\,dy$$

Eq. (3) with $c = 0, d = 1$.

$$= \frac{1}{9}\cdot\frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1$$

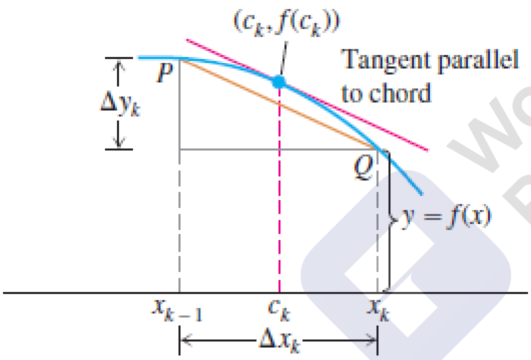
Let $u = 1 + 9y$, $du/9 = dy$, integrate, and substitute back.

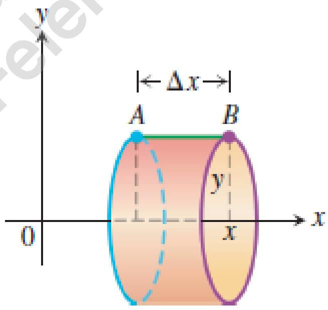
$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

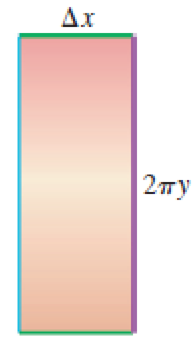


Areas of Surfaces of Revolution and the Theorems of Pappus

When you jump rope, the rope sweeps out a surface in the space around you called a *surface of revolution*. The "area" of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution.







DEFINITION **Surface Area for Revolution About the *x*-Axis**

If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

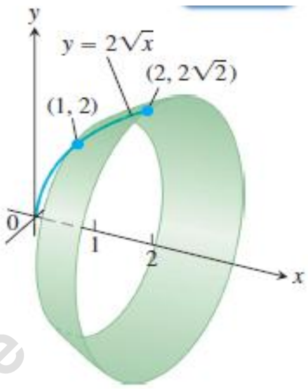
$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \qquad (3)$$

EXAMPLE

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

Revolution About the y-Axis

Surface Area for Revolution About the y-Axis

If $x = g(y) \ge 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

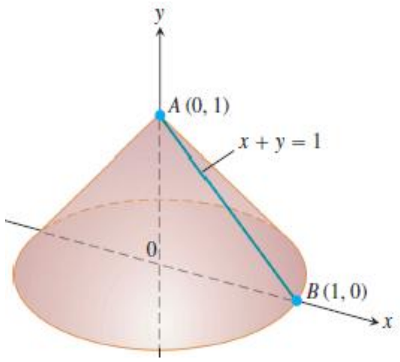
EXAMPLE

The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$



$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(-1)^2} = \sqrt{2}$$

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1-y)\sqrt{2}\,dy$$

$$= 2\pi\sqrt{2}\left[y-\frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1-\frac{1}{2}\right)$$

$$= \pi\sqrt{2}.$$

Parametrized Curves

$$\sqrt{[f'(t)]^2+[g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.$$

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad (5)$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad (6)$$

EXAMPLE

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1+\sin t, \qquad 0 \le t \le 2\pi.$$

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1}}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi\Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$

Inverse Functions and Their Derivatives

One-to-One Functions

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

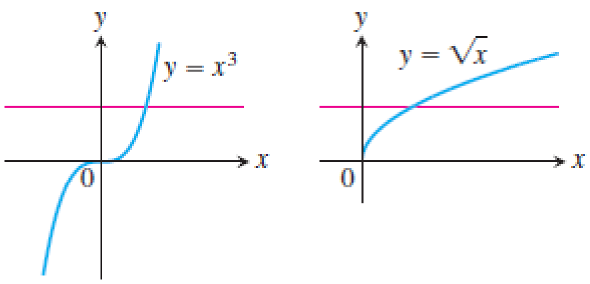
EXAMPLE Domains of One-to-One Functions

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

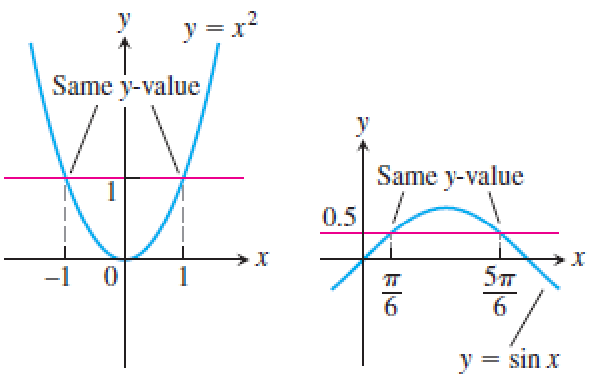
(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION **Inverse Function**

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$(f^{-1} \circ f)(x) = x,$ for all x in the domain of f

$(f \circ f^{-1})(y) = y,$ for all y in the domain of f^{-1} (or range of f)

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

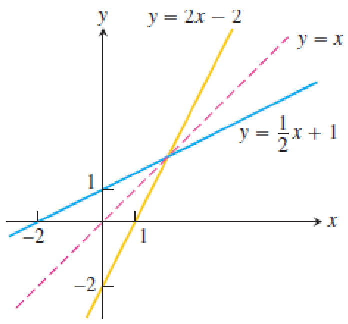
The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE :

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution



1. *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

2. *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

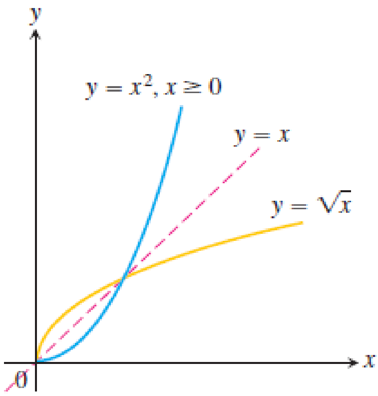
$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

EXAMPLE Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$



We then interchange x and y, obtaining $y = \sqrt{x}$.

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

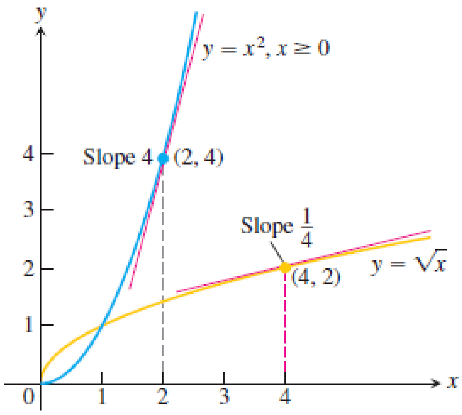
$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \qquad (1)$$

EXAMPLE

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))} = \frac{1}{2(\sqrt{x})}.$$



Theorem 1 gives a derivative that agrees with our calculation using the Power Rule for the derivative of the square root function.

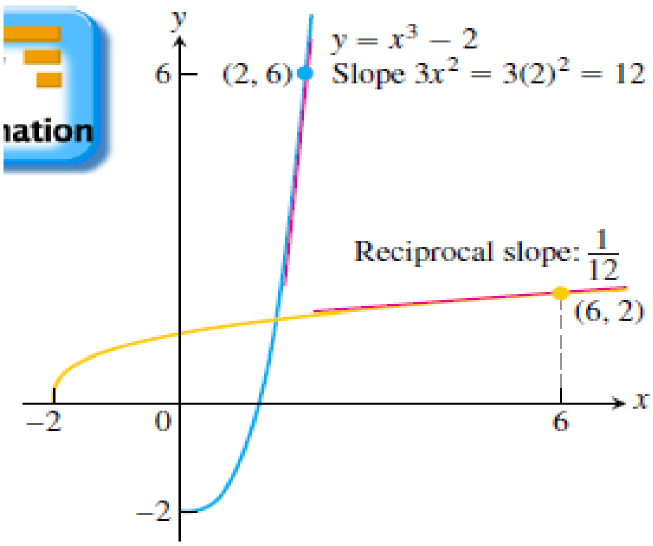
Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

EXAMPLE Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution



$$\frac{df}{dx}\bigg|_{x=2} = 3x^2\bigg|_{x=2} = 12$$

$$\frac{df^{-1}}{dx}\bigg|_{x=f(2)} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=2}} = \frac{1}{12}$$