TECHNIQUES OF INTEGRATION

Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad (\text{any number } k)$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

EXAMPLE Making a Simplifying Substitution

Evaluate $$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{du}{\sqrt{u}}$$ $u = x^2 - 9x + 1,$ $du = (2x - 9)\,dx.$

$$= \int u^{-1/2}\,du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C$$ Table 8.1 Formula 4, with $n = -1/2$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$ ■

EXAMPLE Completing the Square

Evaluate $$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16)$$
$$= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$ $a = 4, u = (x - 4),$ $du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$ Table 8.1, Formula 18

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

EXAMPLE : Expanding a Power and Using a Trigonometric Identity

Evaluate $$\int (\sec x + \tan x)^2\,dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\int (\sec x + \tan x)^2\,dx = \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\,dx$$

$$= 2\int \sec^2 x\,dx + 2\int \sec x \tan x\,dx - \int 1\,dx$$

$$= 2\tan x + 2\sec x - x + C.$$

EXAMPLE Eliminating a Square Root

Evaluate $$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$$

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2}\,\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx \qquad \sqrt{u^2} = |u|$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx \qquad \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x.$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \qquad \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \qquad = \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.$$

EXAMPLE 5 Reducing an Improper Fraction

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

$$\begin{array}{r} x - 3 \\ 3x + 2\overline{)3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\ 6 \end{array}$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2}\right)dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

EXAMPLE : Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate $\displaystyle\int \sec x\,dx.$

Solution

$$\int \sec x\,dx = \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C. \qquad u = \tan x + \sec x,\ du = (\sec^2 x + \sec x \tan x)\,dx$$

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u \, du = \ln|\sec u + \tan u| + C$
2. $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

Integration by Parts

Since

$$\int x \, dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2 \, dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx.$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du \qquad (2)$$

This formula expresses one integral, $\int u \, dv$, in terms of a second integral, $\int v \, du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

EXAMPLE Using Integration by Parts

Find $\int x \cos x \, dx$.

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

EXAMPLE Find $\int \ln x\,dx.$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

EXAMPLE Evaluate $\int x^2 e^x\,dx.$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Hence,

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$$
$$= x^2e^x - 2xe^x + 2e^x + C.$$

EXAMPLE

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

Evaluating Definite Integrals by Parts

EXAMPLE

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution $$\int_0^4 xe^{-x} \, dx.$$

Let $u = x$, $dv = e^{-x} \, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x} \, dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x}) \, dx$$

$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x} \, dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize

the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

which can be verified algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x + 1)(x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x - 3}{(x + 1)(x - 3)} dx = \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx$$

$$= 2 \ln |x + 1| + 3 \ln |x - 3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the above example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.$$

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE $\displaystyle\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

Coefficient of x^2: $A + B + C = 1$
Coefficient of x^1: $4A + 2B = 4$
Coefficient of x^0: $3A - 3B - C = 1$

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$
$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

EXAMPLE

Evaluate

$$\int \frac{6x + 7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$
$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$
$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$A = 6$ and $2A + B = 12 + B = 7$, or $A = 6$ and $B = -5$.

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$
$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$
$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

EXAMPLE

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r@{}l} & \quad 2x \\ x^2 - 2x - 3 \,\big) & \overline{2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \qquad\qquad\quad 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx \\ &= x^2 + 2\ln|x+1| + 3\ln|x-3| + C. \end{aligned}$$

■

EXAMPLE

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \qquad (2)$$

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$
$$= (A + C)x^3 + (-2A + B - C + D)x^2$$
$$+ (A - 2B + C)x + (B - C + D).$$

Equating coefficients of like terms gives

Coefficients of x^3: $0 = A + C$

Coefficients of x^2: $0 = -2A + B - C + D$

Coefficients of x^1: $-2 = A - 2B + C$

Coefficients of x^0: $4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$ Subtract fourth equation from second.

$C = -A = -2$ From the first equation

$B = 1$ $A = 2$ and $C = -2$ in third equation.

$D = 4 - B + C = 1.$ From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx = \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx$$

$$= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx$$

$$= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C.$$

EXAMPLE

Evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx + C)x(x^2+1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \qquad C = 0, \qquad 2A + B + D = 0, \qquad C + E = 0, \qquad A = 1.$$

Solving this system gives $A = 1,\quad B = -1,\quad C = 0,\quad D = -1$, and $E = 0$. Thus,

$$\begin{aligned} \int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \end{aligned}$$

$u = x^2 + 1,$
$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.$$

EXAMPLE Find A, B, and C in the partial-fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0,$$

$$A = 1.$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\boxed{(x-1)}\,(1-2)(1-3)} = \frac{2}{(-1)(-2)} = 1.$$

⇑ Cover

$$B = \frac{(2)^2 + 1}{(2-1)\,\boxed{(x-2)}\,(2-3)} = \frac{5}{(1)(-1)} = -5.$$

⇑ Cover

$$C = \frac{(3)^2 + 1}{(3-1)(3-2)\,\boxed{(x-3)}} = \frac{10}{(2)(1)} = 5.$$

⇑ Cover

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

EXAMPLE

Evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0+4}{\boxed{x}\,(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$\Uparrow$ Cover

$$A_2 = \frac{2+4}{2\,\boxed{(x-2)}\,(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$\Uparrow$ Cover

$$A_3 = \frac{-5+4}{(-5)(-5-2)\,\boxed{(x+5)}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

$\Uparrow$ Cover

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

Other Ways to Determine the Coefficients

EXAMPLE Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

EXAMPLE Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution $x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$

$$\begin{aligned}
x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\
2 &= 2A \\
A &= 1 \\
x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\
5 &= -B \\
B &= -5 \\
x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\
10 &= 2C \\
C &= 5.
\end{aligned}$$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$