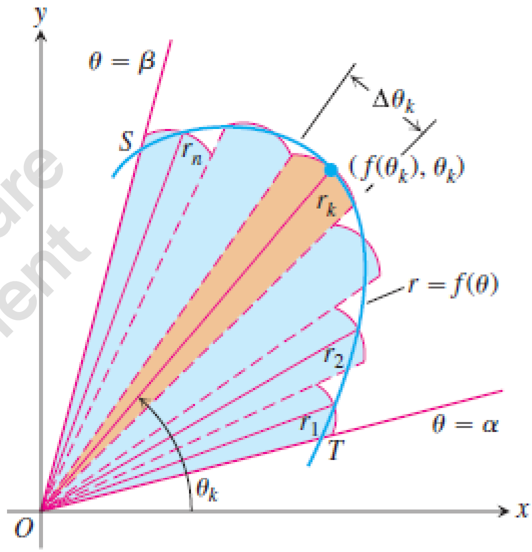
Areas and Lengths in Polar Coordinates

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

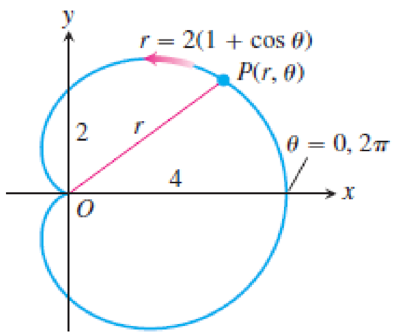
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta .$$



EXAMPLE Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.



$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

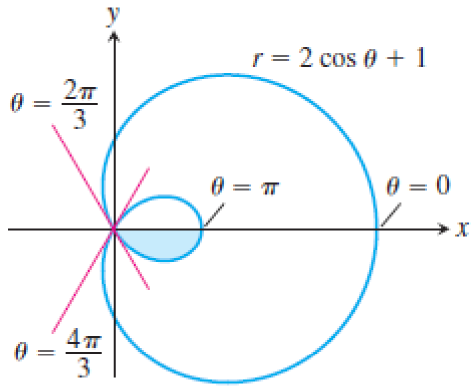
$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \quad = \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta = \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .$$

EXAMPLE

Find the area inside the smaller loop of the limaçon $r = 2\cos\theta + 1$.

$$r^2 = (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1$$
$$= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1$$
$$= 2 + 2\cos 2\theta + 4\cos\theta + 1$$
$$= 3 + 2\cos 2\theta + 4\cos\theta,$$



$$A = \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta$$
$$= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi}$$
$$= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right)$$
$$= \pi - \frac{3\sqrt{3}}{2}.$$

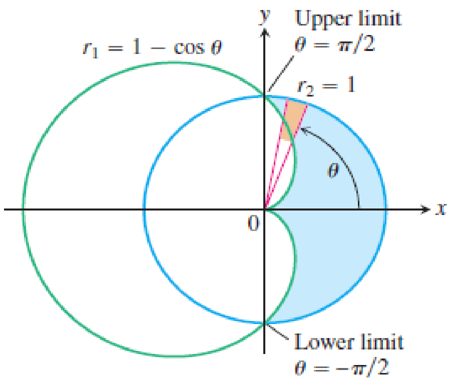
Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$
$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$
$$= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta$$
$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2}\left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}.$$



Length of a Polar Curve

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad (3)$$

EXAMPLE Find the length of the cardioid $r = 1 - \cos\theta$.

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$

$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

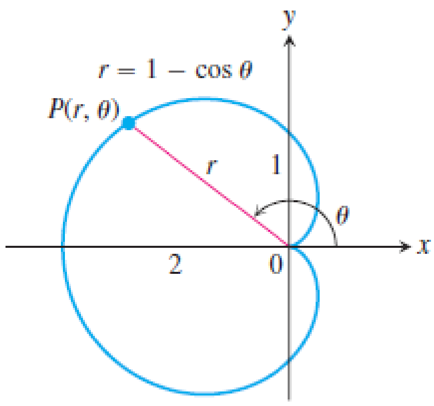
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$



Sequences and Series

Sequences of Numbers

A ***sequence*** of numbers is a function whose domain is the set of positive integers.

Example

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = n-1$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = \frac{1}{n}$

The index n is the ***domain*** of the sequence. While the numbers in the ***range*** of the sequence are called the ***terms*** of the sequence, and the number a_n being called the ***n^{th}-term***, or ***the term with index*** n.

Example $\quad a_n = \frac{n+1}{n}$ then the terms are

$$\begin{array}{ccccc} 1^{st}\ term & 2^{nd}\ term & 3^{rd}\ term & & n^{th}\ term \\ a_1 = 2, & a_2 = \frac{3}{2}, & a_3 = \frac{4}{3}, & . \quad . \quad . & a_n = \frac{n+1}{n}, \quad . \quad . \quad . \end{array}$$

and we use the notation $\{a_n\}$ as the sequence a_n.

Example

Find the first five terms of the following:

(a) $\left\{\frac{2n-1}{3n+2}\right\}$, $\quad$ (b) $\left\{\frac{1-(-1)^n}{n^3}\right\}$, $\quad$ (c) $\left\{(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\frac{1}{5}, \quad \frac{3}{8}, \quad \frac{5}{11}, \quad \frac{7}{14}, \quad \frac{9}{17}$ $\qquad$ **(b)** $2, \quad 0, \quad \frac{2}{27}, \quad 0, \quad \frac{2}{125}$

(c) $x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \quad \frac{x^9}{9!}$

Example

Find the n[th]-term of the following:

(a) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4},$ (b) $0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4},$ (c) $0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16},$

(d) $2, 1, \frac{2^3}{3^2}, \frac{2^4}{4^2}, \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n}$, **(b)** $a_n = \frac{\ln n}{n}$, **(c)** $a_n = \frac{n-1}{n^2}$, **(d)** $a_n = \frac{2^n}{n^2}$

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

From that we can say that

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty}\{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty}\{ka_n\} = kA$

iii) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B},$ provided $B \neq 0$ and b_n is never 0

Example

Test the convergence of the following:

(a) $\left\{\frac{1}{n}\right\},$ (b) $\{1+(-1)^n\},$ (c) $\{n^2\},$ (d) $\{\sqrt{n+1}-\sqrt{n}\},$

(e) $\left\{\frac{3n^2-5n}{5n^2+2n+6}\right\},$ (f) $\left\{\frac{3n^2-4n}{2n-1}\right\},$ (g) $\left\{\left(\frac{2n-3}{3n-7}\right)^4\right\},$ (h) $\left\{\frac{2n^5-4n^2}{3n^7+n^2-10}\right\},$

(i) $\left\{\frac{2^n}{5n}\right\},$ (j) $\left\{\frac{\ln n}{e^n}\right\}$

Solution

(a) $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ ***(Conv.)***

(b) $\lim_{n\to\infty}\left(1+(-1)^n\right) = 1 + \lim_{n\to\infty}(-1)^n = \begin{cases} 0 & n \quad odd \\ 2 & n \quad even \end{cases}$ ***(Div.)***

(c) $\lim_{n\to\infty}\left(n^2\right) = \infty$ ***(Div.)***

(d) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\left(\sqrt{n+1}-\sqrt{n}\right)\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{1}{\infty+\infty} = 0$ ***(Conv.)***

(e) $\lim_{n\to\infty}\left(\dfrac{3n^2-5n}{5n^2+2n+6}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{5n}{n^2}}{\dfrac{5n^2}{n^2}+\dfrac{2n}{n^2}+\dfrac{6}{n^2}}\right)=\dfrac{3}{5}$ ***(Conv.)***

(f) $\lim_{n\to\infty}\left(\dfrac{3n^2-4n}{2n-1}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{4n}{n^2}}{\dfrac{2n}{n^2}-\dfrac{1}{n^2}}\right)=\dfrac{3}{0}=\infty$ ***(Div.)***

(g) $\lim_{n\to\infty}\left(\dfrac{2n-3}{3n-7}\right)^4=\left(\dfrac{2}{3}\right)^4=\dfrac{16}{81}$ ***(Conv.)***

(h) $\lim_{n\to\infty}\left(\dfrac{2n^5-4n^2}{3n^7+n^2-10}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{2}{n^2}-\dfrac{4}{n^5}}{3+\dfrac{1}{n^5}-\dfrac{10}{n^7}}\right)=0$ ***(Conv.)***

(i) $\lim_{n\to\infty}\left(\dfrac{2^n}{5n}\right)=\lim_{n\to\infty}\left(\dfrac{2^n.\ln 2}{5}\right)=\infty$ ***(Div.)***

(j) $\lim_{n\to\infty}\left(\dfrac{\ln n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1/n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1}{n.e^n}\right)=\dfrac{1}{\infty}=0$ ***(Conv.)***

Example

Prove the following limits

(a) $\lim_{n\to\infty}\left(\dfrac{\ln n}{n}\right)=0$, (b) $\lim_{n\to\infty}\left(\sqrt[n]{n}\right)=1$, (c) $\lim_{n\to\infty}\left(x^{1/n}\right)=1$ $(x>0)$,

(d) $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x$ (any x), (e) $\lim_{n\to\infty}\left(\dfrac{x^n}{n!}\right)=0$ (any x)

Solution

(a) $\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)=\lim_{n\to\infty}\left(\frac{1/n}{1}\right)=\frac{0}{1}=0$

(b) Let $a_n = n^{1/n}$, then $\ln a_n = \ln n^{1/n} = \frac{1}{n}\ln n \to 0$,

So, $\lim_{n\to\infty} n^{1/n} = e^{\ln a_n} \to e^0 = 1$

(c) Let $a_n = x^{1/n}$, then $\ln a_n = \ln x^{1/n} = \frac{1}{n}\ln x \to 0$,

So, $\lim_{n\to\infty} x^{1/n} = e^{\ln a_n} \to e^0 = 1$

(d) Let $a_n = \left(1+\frac{x}{n}\right)^n$, then

$$\ln a_n = \ln\left(1+\frac{x}{n}\right)^n = n.\ln\left(1+\frac{x}{n}\right)$$

So, $$\lim_{n\to\infty} n.\ln\left(1+\frac{x}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+x/n)}{1/n}=\lim_{n\to\infty}\frac{\left(\frac{1}{1+x/n}\right)\cdot\left(-\frac{x}{n^2}\right)}{-1/n^2}$$

$$=\lim_{n\to\infty}\frac{x}{1+x/n}=x,$$

Thus, $\left(1+\frac{x}{n}\right)^n = a_n = e^{\ln a_n} \to e^x$

(e) $\lim_{n\to\infty}\left(\frac{x^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{x}{1}\right)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right)\ldots\left(\frac{x}{n}\right)=0$

Exercises on Sequences

Find the values of a_1, a_2, a_3 and a_4 for the following sequences

1) $a_n = \dfrac{1-n}{n^2}$ **2)** $a_n = \dfrac{1}{n!}$ **3)** $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4) $a_n = 2 + (-1)^n$ **5)** $a_n = \dfrac{2^n}{2^{n+1}}$ **6)** $a_n = \dfrac{2^n - 1}{2^n}$

Find a formula for the n^{th} term of the following sequences

1) $1, -1, 1, -1, 1, \dots$ **2)** $-1, 1, -1, 1, -1, \dots$ **3)** $1, -4, 9, -16, 25, \dots$

4) $1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \dfrac{1}{25}, \dots$ **5)** $0, 3, 8, 15, 24, \dots$ **6)** $-3, -2, -1, 0, 1, \dots$

7) $1, 5, 9, 13, 17, \dots$ **8)** $2, 6, 10, 14, 18, \dots$ **9)** $1, 0, 1, 0, 1, \dots$

Which of the following sequences converge and which diverge?

1) $a_n = 2 + (0.1)^n$ *Ans. Converges, 2*

2) $a_n = \dfrac{1-2n}{1+2n}$ *Ans. Converges, -1*

3) $a_n = \dfrac{1-5n^4}{n^4+8n^3}$ *Ans. Converges, -5*

4) $a_n = \dfrac{n^2-2n+1}{n-1}$ *Ans. Diverges*

5) $a_n = 1 + (-1)^n$ *Ans. Diverges*

6) $a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$ *Ans.* *Converges,* $\frac{1}{2}$

7) $a_n = \frac{(-1)^{n+1}}{2n-1}$ *Ans.* *Converges,* 0

8) $a_n = \sqrt{\frac{2n}{n+1}}$ *Ans.* *Converges,* $\sqrt{2}$

9) $a_n = \sin\left(\frac{\pi}{2}+\frac{1}{n}\right)$ *Ans.* *Converges,* 1

10) $a_n = \frac{\sin n}{n}$ *Ans.* *Converges,* 0

11) $a_n = \frac{n}{2^n}$ *Ans.* *Converges,* 0

12) $a_n = \frac{\ln(n+1)}{n}$ *Ans.* *Converges,* 0

13) $a_n = 8^{1/n}$ *Ans.* *Converges,* 1

14) $a_n = \left(1+\frac{7}{n}\right)^n$ *Ans.* *Converges,* e^7

15) $a_n = \sqrt[n]{10n}$ *Ans.* *Converges,* 1

16) $a_n = \left(\frac{3}{n}\right)^{1/n}$ *Ans.* *Converges,* 1

17) $a_n = \frac{\ln n}{n^{1/n}}$ *Ans.* *Diverges*

18) $a_n = \sqrt[n]{4^n n}$ *Ans.* *Converges,* 4

20) $a_n = \dfrac{n!}{10^{6n}}$ *Ans.* *Diverges*

21) $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$ *Ans.* *Converges,* e^{-1}

22) $a_n = \left(\dfrac{3n+1}{3n-1}\right)^{n}$ *Ans.* *Converges,* $e^{2/3}$

23) $a_n = \left(\dfrac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$ *Ans.* *Converges,* $x \ (x > 0)$

24) $a_n = \dfrac{3^n \times 6^n}{2^{-n} \times n!}$ *Ans.* *Converges,* 0

25) $a_n = \tanh(n)$ *Ans.* *Converges,* 1

26) $a_n = \dfrac{n^2}{2n-1} \sin \dfrac{1}{n}$ *Ans.* *Converges,* $\dfrac{1}{2}$

27) $a_n = \tan^{-1}(n)$ *Ans.* *Converges,* $\dfrac{\pi}{2}$

28) $a_n = \left(\dfrac{1}{3}\right)^{n} + \dfrac{1}{\sqrt{2^n}}$ *Ans.* *Converges,* 0

29) $a_n = \dfrac{(\ln n)^{200}}{n}$ *Ans.* *Converges,* 0

30) $a_n = n - \sqrt{n^2 - n}$ *Ans.* *Converges,* $\dfrac{1}{2}$

31) $a_n = \dfrac{1}{n} \int_1^n \dfrac{1}{x}\, dx$ *Ans.* *Converges,* 0

Infinite Series

Infinite series are sequences of a special kind: those in which the n^{th}-term is the sum of the first n terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

If we denote the above sequence as a_n, and the resultant sequence of the series as s_n, then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by the rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- ❖ The number $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} ***partial sum*** of the series.
- ❖ Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- ❖ The series $\sum a_n$ is said to ***converge*** to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + +ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r. If $|r| < 1$, the geometric series converges to $a/(1-r)$. If $|r| \geq 1$, the series diverges unless $a = 0$. If $a = 0$, the series converges to 0.

Example

Geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$.

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

Geometric series with $a = 4$ and $r = -\frac{1}{2}$.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \ldots = 4\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots\right)$$

$$= \frac{4}{1+(1/2)} = \frac{8}{3}$$

Example

Determine whether each series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$, (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

Solution

(a) Since the series is a geometric series with $r=\frac{2}{3}<1$, so the series is convergent with a sum of $\frac{1}{1-(2/3)}=3$

(b) Since the series is a geometric series with $r=\frac{3}{2}>1$, so the series is divergent.

(c) $\cos\pi/3=1/2$. This is a geometric series with first term $a_1=1$ and the ratio $r=1/2$; so the series converges and its sum is $1/(1-\frac{1}{2})=2$.

(d) $\tan\pi/4=1$. This is a geometric series with $r=1$, so the series diverges.

(e) This is a geometric series with first term $a_1=-5/4$ and ratio $r=-1/4$. So the series converges and its sum is $\frac{-5/4}{1+(1/4)}=-1$.

Test Convergence of Series with Non-negative Terms

1) The n^th^- Term Test

- If $\lim_{n\to\infty}a_n\neq 0$, or if $\lim_{n\to\infty}a_n$ fails to exist, then $\sum_{n=1}^{\infty}a_n$ diverges.
- If $\sum_{n=1}^{\infty}a_n$ converges, then $a_n\to 0$.
- If $\lim_{n\to\infty}a_n=0$, then the test fails.

From the above, it can not be concluded that if $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $a_n \to 0$. Thus $\lim_{n\to\infty} a_n = 0$ is a necessary but not a sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

Examples

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$,

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$,

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist,

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ can not be tested by the nth-term test for divergence because $\frac{1}{n} \to 0$.

2) The Integral Test

Let the function $y = f(x)$, obtained by introducing the continuous variable x in place of the discrete variable n in the nth-term of the positive series $\sum_{n=1}^{\infty} a_n$, then

$$\int_1^{\infty} f(x)dx = \begin{cases} +\infty & Div. \\ -\infty & Div. \\ -\infty < c < \infty & Conv. \end{cases}$$

Example

Prove that, for the p-series, if p is a real constant, the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

converges if $p>1$ and diverges if $p\le 1$.

Solution

To prove this, let

$$f(x)=\frac{1}{x^p}$$

Then, if $p>1$, we have

$$\int_1^{\infty}x^{-p}dx=\lim_{b\to\infty}\frac{x^{-p+1}}{-p+1}\Bigg|_1^b=\frac{1}{p-1}$$

which is finite. Hence, the p-series converges if $p>1$.

If $p=1$, which is called a harmonic series, we have

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

and the integral test is

$$\int_1^{\infty}x^{-1}dx=\lim_{b\to\infty}\ \ln x\Big|_1^b=+\infty$$

which diverges.

Finally, for $p<1$, then the terms of the series are greater than the corresponding terms of the divergent harmonic series. Hence the p-series diverges for $p<1$.

Thus, we have a convergence for $p>1$, but divergence for $p\le 1$.

Example

Test the convergence of

(a) $\sum_{n=1}^{\infty} \frac{1}{e^n}$,   (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution

(a) $\int_1^{\infty} e^{-x} dx = -e^{-x}\Big|_1^{\infty} = -(e^{-\infty} - e^{-1}) = \frac{1}{e}$ ***(Conv.)***

(b) $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_2^{\infty} \frac{1/x}{(\ln x)^2} dx = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{-1}{\infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ ***(Conv.)***

3) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- The series converges if $\rho < 1$,
- The series diverges if $\rho > 1$,
- The series may converge or it may diverge if $\rho = 1$. (Test fails)

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$, (b) $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$, (c) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$, (d) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$, (e) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

(a) If $a_n = \frac{n!n!}{(2n)!}$, then $a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{4n+2} \to \frac{1}{4} < 1 \qquad \textit{(Conv.)}$$

(b) If $a_n = \frac{4^n n!n!}{(2n)!}$, then $a_{n+1} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \times \frac{(2n)!}{4^n n!n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{2(n+1)}{2n+1} \to 1 \qquad \textit{(Test fails)}$$

(c) If $a_n = \frac{2^n+5}{3^n}$, then $a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3} \times \frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3} \times \left(\frac{2+5\times 2^{-n}}{1+5\times 2^{-n}}\right) \to \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} < 1 \qquad \textit{(Conv.)}$$

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \to \infty > 1 \qquad \textit{(Div.)}$$

(e) If $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n (n+1) n!}{(n+1) n! n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e^1 = 2.7 > 1 \qquad \textit{(Div.)}$$

4) The n^th Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > n_0$ and suppose that

$$\sqrt[n]{a_n} \to \rho$$

Then

- ❖ The series converges if $\rho < 1$.
- ❖ The series diverges if $\rho > 1$.
- ❖ The test is not conclusive if $\rho = 1$.

<u>Example</u>

Test the convergence of the following series using the n^th Root Test.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$, (d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

(a) $\sqrt[n]{\dfrac{1}{n^n}} = \dfrac{1}{n} \to 0 < 1$ *(Conv.)*

(b) $\sqrt[n]{\dfrac{2^n}{n^2}} = \dfrac{2}{\sqrt[n]{n^2}} = \dfrac{2}{\left(\sqrt[n]{n}\right)^2} \to \dfrac{2}{1^2} = 2 > 1$ *(Div.)*

(c) $\sqrt[n]{\left(1-\dfrac{1}{n}\right)^n} = \left(1-\dfrac{1}{n}\right) \to 1$ *(Test fails)*

(d) $\sqrt[n]{\left(\dfrac{n}{n+1}\right)^{n^2}} = \left(\dfrac{n}{n+1}\right)^{\frac{n^2}{n}} = \left(\dfrac{n}{n+1}\right)^n = \left(\dfrac{1}{1+1/n}\right)^n \to \dfrac{1}{e} = \dfrac{1}{2.7} < 1$ *(Conv.)*

(e) $\sqrt[n]{\left(\dfrac{2n}{n+1}\right)^n} = \dfrac{2n}{n+1} \to 2 > 1$ *(Div.)*

Exercises on Series

Find the sum of the following series

1) $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{4^n}$ *Ans.* $\dfrac{4}{5}$

2) $\sum_{n=1}^{\infty} \dfrac{7}{4^n}$ *Ans.* $\dfrac{7}{3}$

3) $\sum_{n=0}^{\infty} \left(\dfrac{5}{2^n} + \dfrac{1}{3^n}\right)$ *Ans.* $\dfrac{23}{2}$

4) $\sum_{n=0}^{\infty} \left(\dfrac{1}{2^n} + \dfrac{(-1)^n}{5^n}\right)$ *Ans.* $\dfrac{17}{6}$

5) $\sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}$ *Ans.* 1

6) $\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}$ *Ans.* 5

7) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ *Ans.* 1

8) $\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$ *Ans.* $-\frac{1}{\ln 2}$

Which of the following series converges and which diverges? Find the sum of the convergent series.

1) $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ *Ans.* *Converges,* $2+\sqrt{2}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ *Ans.* *Converges,* 1

3) $\sum_{n=0}^{\infty}\cos(n\pi)$ *Ans.* *Diverges*

4) $\sum_{n=0}^{\infty}e^{-2n}$ *Ans.* *Converges,* $\frac{e^2}{e^2-1}$

5) $\sum_{n=1}^{\infty}\frac{2}{10^n}$ *Ans.* *Converges,* $\frac{2}{9}$

6) $\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$ *Ans.* *Converges,* $\frac{3}{2}$

7) $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$ *Ans.* *Diverges*

8) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ *Ans.* *Diverges*

9) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$ *Ans.* *Converges,* $\frac{\pi}{\pi - e}$

Which of the following series converges and which diverges?

1) $\sum_{n=1}^{\infty} \frac{1}{10^n}$ *Ans.* ***Converges (Geometric)***

2) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *Ans.* ***Diverges (n^{th}-term test)***

3) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ *Ans.* ***Diverges (p-series)***

4) $\sum_{n=1}^{\infty} \frac{-1}{8^n}$ *Ans.* ***Converges (Geometric)***

5) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ *Ans.* ***Diverges (Integral Test)***

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ *Ans.* ***Converges (Geometric)***

7) $\sum_{n=0}^{\infty} \frac{-2}{n+1}$ *Ans.* ***Diverges (Integral Test)***

8) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ *Ans. Diverges (n^{th}-term test)*

9) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ *Ans. Diverges (n^{th}-term test)*

10) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ *Ans. Diverges (Geometric)*

11) $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$ *Ans. Converges (Integral Test)*

12) $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ *Ans. Diverges (n^{th}-term test)*

13) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ *Ans. Converges (Integral Test)*

14) $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$ *Ans. Converges (Integral Test)*

15) $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ *Ans. Diverges (n^{th}-term test)*

16) $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$ *Ans. Converges (Ratio Test)*

17) $\sum_{n=1}^{\infty} n! e^{-n}$ *Ans. Diverges (Ratio Test)*

18) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ *Ans. Converges (Ratio Test)*

19) $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$ *Ans. Diverges (n^{th}-term test)*

20) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ *Ans. Converges (Ratio Test)*

21) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ *Ans. Converges (Ratio Test)*

22) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ *Ans. Converges (Ratio Test)*

23) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ *Ans. Converges (Root Test)*

24) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ *Ans. Diverges (Root Test)*

25) $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$ *Ans. Converges (Root Test)*

Alternating Series

A series in which the terms are alternately positive and negative.

Example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{n}+\ldots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n}4}{2^{n}}+\ldots$$

$$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$$

The Convergence Test of Alternating Series

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\ldots$$

converges if all three of the following conditions are satisfied:

1) The u_n's are all positive.
2) $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3) $u_n \to 0$.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

satisfies the three requirements of convergence; it therefore converges.

Absolute Convergence

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges, i.e.,

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Example

The geometric series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ converges absolutely because the corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ converges .

Conditional Convergence

A series that converges but does not converge absolutely ***converges conditionally***.

Example

The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ does not converge absolutely. The corresponding series of absolute values $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is the divergent harmonic series.

Power Series

- A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

- A power series about $x=a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

in which the center a and the coefficients $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

Example

The series $\sum_{n=0}^{\infty} x^n$ is a geometric series with first term 1 and ratio x. It converges to

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \qquad \text{for } |x| < 1$$

Convergence of Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

The test of power series is done using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \rho \begin{cases} <1 & \text{Conv.} \\ >1 & \text{Div.} \\ =1 & \text{Fails} \end{cases}$$

Notes:

- Use the Ratio Test to find the interval where the series converges absolutely.
- If the interval of absolute convergence is finite, test the convergence or divergence at each endpoint. Use the integral test or the Alternating Series Test for endpoints.

- If the interval of absolute convergence is $|x-a|<R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the n^th^-term does not approach zero for those values of x.

Example

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$ (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,$ (d) $\sum_{n=0}^{\infty}n!x^n=1+x+2!x^2+3!x^3+\ldots$

Solution

(a) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right|=\frac{n}{n+1}|x|\to|x|.$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$ because the n^th^-term does not converge to zero. At $x=1$, we get the alternating harmonic series $1-1/2+1/3-1/4+\ldots$, which converges. At $x=-1$, we get $-1-1/2-1/3-1/4-\ldots$, the negative of the harmonic series; it diverges. So, the series converges for $-1<x\le 1$ and diverges elsewhere.

(b) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\times\frac{2n-1}{x^{2n-1}}\right|=\frac{2n-1}{2n+1}x^2\to x^2.$

The series converges absolutely for $x^2<1$. It diverges for $x^2>1$ because the n^th^-term does not converge to zero. At $x=1$, the series becomes $1-1/3+1/5-1/7+\ldots$, which converges because it satisfies the three conditions of convergence of alternating series. It also converges at $x=-1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x=-1$ is the negative of the value at $x=1$. So, the series converges for $-1 \le x \le 1$ and diverges elsewhere.

(c) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \dfrac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.

(d) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$ unless $x=0$.

The series diverges for all values of x except $x=0$.

Exercises on Alternating & Power Series

Which of the following series converges and which diverges?

1) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ *Ans.* *Converges*

2) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$ *Ans.* *Diverges,* $a_n \to \infty$

3) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$ *Ans.* *Converges*

4) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln(n^2)}$ *Ans.* *Diverges,* $a_n \to \dfrac{1}{2}$

5) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ *Ans.* *Converges*

Which of the following series converges absolutely, conditionally, and which diverges?

1) $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$ ***Ans.*** *Converges absolutely*

2) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ ***Ans.*** *Converges conditionally*

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$ ***Ans.*** *Diverges,* $a_n \to 1$

4) $\sum_{n=1}^{\infty}(-1)^n n^2\left(\frac{2}{3}\right)^n$ ***Ans.*** *Converges absolutely*

5) $\sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1}n}{n^2+1}$ ***Ans.*** *Converges absolutely*

6) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$ ***Ans.*** *Diverges,* $a_n \to 1$

7) $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$ ***Ans.*** *Converges absolutely*

8) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n\sqrt{n}}$ ***Ans.*** *Converges absolutely*

9) $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$ ***Ans.*** *Converges absolutely*

10) $\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{2^n n! n}$ ***Ans.*** *Diverges,* $a_n \to \infty$

Find the interval of convergence for the following series

1) $\sum_{n=0}^{\infty} x^n$ *Ans.* $-1 < x < 1$

2) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ *Ans.* $-\frac{1}{2} < x < 0$

3) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ *Ans.* $-8 < x < 12$

4) $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$ *Ans.* $-1 < x < 1$

5) $\sum_{n=1}^{\infty} \frac{x^n}{3^n n\sqrt{n}}$ *Ans.* $-3 \le x \le 3$

6) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ *Ans.* For all x

7) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ *Ans.* For all x

8) $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$ *Ans.* $-1 \le x < 1$

9) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ *Ans.* $-8 < x < 2$

10) $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ *Ans.* $-3 < x < 3$

11) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ *Ans.* $-1 < x < 1$

12) $\sum_{n=1}^{\infty} n^n x^n$ *Ans.* $x = 0$

13) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{2^n n}$ *Ans.* $-4 < x \le 0$

Find the interval of convergence and the sum within this interval for the following series

1) $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4n}$ *Ans.* $-1 < x < 3, \frac{4}{3+2x-x^2}$

2) $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1 \right)^n$ *Ans.* $0 < x < 16, \frac{2}{4-\sqrt{x}}$

3) $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3} \right)^n$ *Ans.* $-\sqrt{2} < x < \sqrt{2}, \frac{3}{2-x^2}$

Taylor Series & Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the ***Taylor Series*** generated by f at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots$$

The ***Maclaurin Series*** generated by f is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

which is a Taylor series generated by f at $x=0$.

Example

Find the Taylor series and the interval of convergence for the following functions (a) $f(x)=1/x$ at $x=2$, (b) $f(x)=\ln(x)$ at $x=1$.

Solution

(a) We need to find $f(2), f'(2), f''(2),\ldots$. Taking derivatives we get

$$f(x)=x^{-1}, \qquad f(2)=\frac{1}{2},$$

$$f'(x)=-x^{-2}, \qquad f'(2)=-\frac{1}{2^2},$$

$$f''(x)=2!x^{-3}, \qquad \frac{f''(2)}{2!}=\frac{1}{2^3},$$

$$f'''(x)=-3!x^{-4}, \qquad \frac{f'''(2)}{3!}=-\frac{1}{2^4},$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}, \qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \ldots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \ldots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \ldots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \ldots$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ or $0 < x < 4$.

(b) $f(x) = \ln(x)$, $\qquad f(1) = 0$,

$f'(x) = \frac{1}{x}$, $\qquad f'(1) = 1$,

$f''(x) = -\frac{1}{x^2}$, $\qquad f''(1) = -1$,

$f'''(x) = \frac{2}{x^3}$, $\qquad f'''(1) = 2$,

$f^{(4)}(x) = \frac{-6}{x^4}$, $\qquad f^{(4)}(1) = -6$,

$$\vdots \qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}, \qquad f^{(n)}(1) = (-1)^{n+1}(n-1)!,$$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \ldots$$

$$\ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \ldots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n + \ldots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + \frac{(-1)^{n+1}}{n}(x-1)^n + \ldots$$

$$\ln(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1} \times \frac{n}{(x-1)^n}\right| = \lim_{n\to\infty}\left|(x-1)\frac{n}{n+1}\right| = |x-1| < 1$$

So, $-1 < x-1 < 1$ or $0 < x < 2$. For $x = 0$, we get $-1-1/2-1/3-1/4-\ldots$ which diverges because it is the negative of the harmonic series. While, for $x = 2$, we get $1-1/2+1/3-1/4+\ldots$ which converges because it is an alternating series that satisfies the three conditions of convergence of alternating series. So, the region of convergence will be $0 < x \le 2$.

Example

Find the Maclaurin series generated by the following functions

(a) e^x, (b) $\cosh(x)$, (c) $\sinh(x)$

Solution

(a) $f(x)=e^x \quad\Rightarrow\quad f(0)=1,$

$f'(x)=e^x \quad\Rightarrow\quad f'(0)=1,$

$f''(x)=e^x \quad\Rightarrow\quad f''(0)=1,$

$f'''(x)=e^x \quad\Rightarrow\quad f'''(0)=1,$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x)=e^x \quad\Rightarrow\quad f^{(n)}(0)=1$

$$e^x = 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1.$$

So, the series is convergent for all values of x.

(b) $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$

$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right]$$

$$=\frac{1}{2}\left[(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)+(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots)\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\right]$$

$$= \frac{1}{2} \times 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!} \times \frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(n+1)}\right| = 0 < 1 .$$

So, the series is convergent for all values of x.

(c) $\sinh(x) = f'(\cosh(x))$

$$= f'\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}\right)$$

$$= 0 + \frac{2x}{2!} + \frac{4x^3}{4!} + \frac{6x^5}{6!} + \ldots + \frac{2nx^{2n-1}}{(2n)!}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n-1}}{(2n-1)!}$$

$$\sinh(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}, \quad \text{or} \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

To find the interval of convergence, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2n+1)!} \times \frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n)}\right| = 0 < 1 .$$

So, the series is convergent for all values of x.

Exercises on Taylor Series

Find Maclaurin series for the following functions

1) e^{-x} *Ans.* $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

2) $\frac{1}{1+x}$ *Ans.* $\sum_{n=0}^{\infty} (-1)^n x^n$

3) $\sin(3x)$ *Ans.* $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$

4) $7\cos(-x)$ *Ans.* $7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

5) $x^4 - 2x^3 - 5x + 4$ *Ans.* $x^4 - 2x^3 - 5x + 4$

Find the Taylor series generated by f at $x = a$ for the following functions

1) $f(x) = x^3 - 2x + 4$, $a = 2$ *Ans.* $8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

2) $f(x) = x^4 + x^2 + 1$, $a = -2$ *Ans.* $21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

3) $f(x) = \frac{1}{x^2}$, $a = 1$ *Ans.* $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

4) $f(x) = e^x$, $a = 2$ *Ans.* $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

Find Maclaurin series for the following functions

1) e^{-5x} *Ans.* $\sum_{n=0}^{\infty}\frac{(-5x)^n}{n!}$

2) $5\sin(-x)$ *Ans.* $\sum_{n=0}^{\infty}\frac{5(-1)^n(-x)^{2n+1}}{(2n+1)!}$

3) $\cos\sqrt{x+1}$ *Ans.* $\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{(2n)!}$

4) xe^x *Ans.* $\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$

5) $\frac{x^2}{2}-1+\cos x$ *Ans.* $\sum_{n=2}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

6) $x\cos(\pi\cdot x)$ *Ans.* $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}x^{2n+1}}{(2n)!}$

7) $\cos^2(x)$ *Ans.* $1+\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{2\cdot(2n)!}$

8) $\frac{x^2}{1-2x}$ *Ans.* $x^2\sum_{n=0}^{\infty}(2x)^n$

9) $\frac{1}{(1-x)^2}$ *Ans.* $\sum_{n=1}^{\infty}nx^{n-1}$