Chapter One: Vectors and the Geometry of Space

1.1 3-D Coordinate Systems

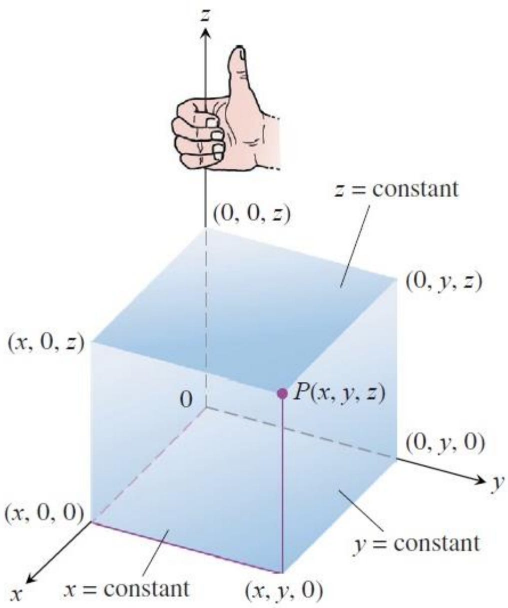


Figure 1.1 The Cartesian coordinate system is right-handed.

The Cartesian coordinates (x, y, z) of a point P in space are the numbers at which the planes through *P* perpendicular to the axes cut the axes (Figure 1.1). Cartesian coordinates for space are also called **rectangular coordinates** because the axes that define them meet at right angles. Points on the *x*-axis have *y*- and *z*-coordinates equal to zero. That is, they have coordinates of the form (*x*, 0, 0). Similarly, points on the *y*-axis have coordinates of the form (0, *y*, 0), and points on the *z*-axis have coordinates of the form (0, 0, *z*).

The planes determined by the coordinates axes are the *xy*-plane, whose standard equation is $z = 0$; the *yz*-plane, whose standard equation is $x = 0$; and the *xz*-plane, whose standard equation is $y = 0$. They meet at the origin (0, 0, 0) (Figure 1.2). The origin is also identified by simply 0 or sometimes the letter *O*. The three coordinate planes $x= 0$, $y = 0$, and $z = 0$ divide space into eight cells called octants. The octant in which the point coordinates are all positive is called the first octant; no conventional numbering for the other seven octants.

The points in a plane perpendicular to the *x*-axis all have the same *x*-coordinate, this being the number at which that plane cuts the *x*-axis. The *y*- and *z*-coordinates can be any numbers. Similarly, the points in a plane perpendicular to the *y*-axis have a common *y*-coordinate and the points in a plane perpendicular to the *z*-axis have a common *z*-coordinate. To write equations for these planes, we name the common coordinate's value. The plane $x = 2$ is the plane perpendicular to the *x*-axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the *y*-axis at $y = 3$. The plane $z = 5$ is the plane perpendicular to the *z*-axis at z = 5. Figure 1.3 shows the planes x = 2, y = 3, and z = 5 together with their intersection point (2, 3, 5).

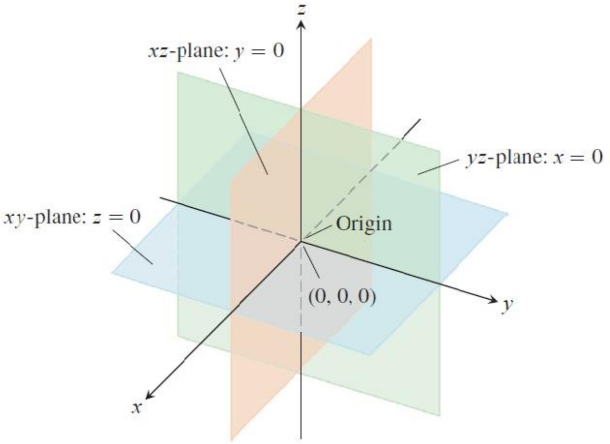


Figure 1.2 The planes x = 0, y = 0, and z = 0 divide space into eight octants.

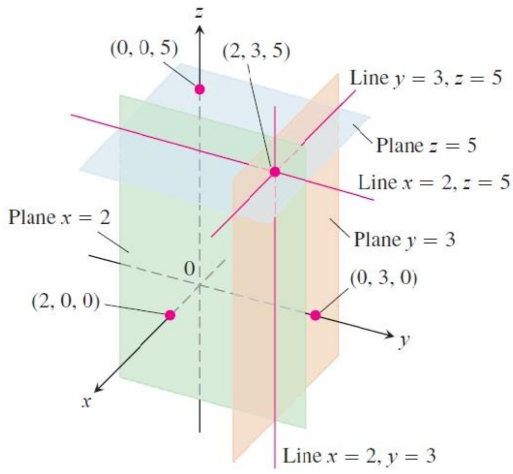


Figure 1.3 The planes x = 2, y = 3, and z = 5 determine three lines through the point (2, 3, 5).

The planes x = 2 and y = 3 in Figure 1.3 intersect in a line parallel to the z-axis. This line is described by the *pair* of equations x = 2, y = 3. A point (x, y, z) lies on the line if and only if x = 2 and y = 3. Similarly, the line of intersection of the planes y = 3 and z = 5 is described by the equation pair y = 3, z = 5. This line runs parallel to the x-axis. The line of intersection of the planes x = 2 and z = 5, parallel to the y-axis, is described by the equation pair x = 2, z = 5. In the following examples, we match coordinate equations and inequalities with the sets of points they define in space.

Example 1: Interpreting Equations and Inequalities Geometrically

(a)	$z \ge 0$	The half-space consisting of the points on and above the xy-plane.
(b)	$x = -3$	The plane perpendicular to the x-axis at x = -3. This plane lies parallel to the yz-plane and 3 units behind it.
(c)	$z = 0,\ x \le 0,\ y \ge 0$	The second quadrant of the xy-plane.
(d)	$x \ge 0,\ y \ge 0,\ z \ge 0$	The first octant.
(e)	$-1 \le y \le 1$	The slab between the planes y = -1 and y = 1 (planes included).
(f)	$y = -2,\ z = 2$	The line in which the planes y = -2 and z = 2 intersect. Alternatively, the line through the point (0, -2, 2) parallel to the x-axis.

Example 2: Graphing Equations

What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \text{ and } z = 3?$$

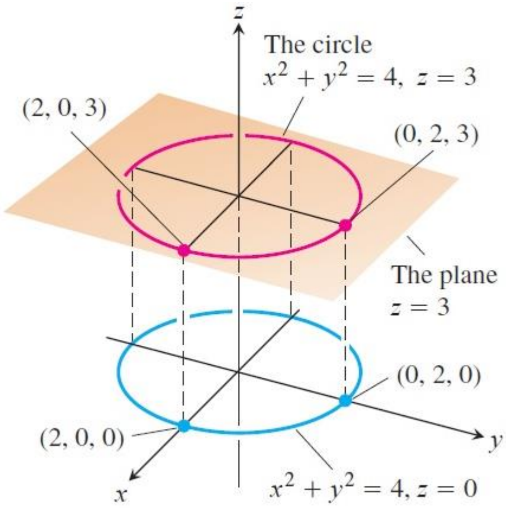


Figure 1.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

Solution: The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points "the circle $x^2 + y^2 = 4$ in the plane $z = 3$" or, more simply, "the circle $x^2 + y^2 = 4$, $z = 3$" (Figure 1.4).

Distance and Spheres in Space

The formula for the distance between two points in the *xy*-plane extends to points in space.

The Distance Between $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof: We construct a rectangular box with faces parallel to the coordinate planes and the points P_1 and P_2 at opposite corners of the box (Figure 1.5). If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then the three box edges P_1A, AB, and BP_2 have lengths

$$|P_1A| = |x_2 - x_1|, |AB| = |y_2 - y_1|, |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 \text{ and } |P_1B|^2 = |P_1A|^2 + |AB|^2$$

(see Figure 1.5)

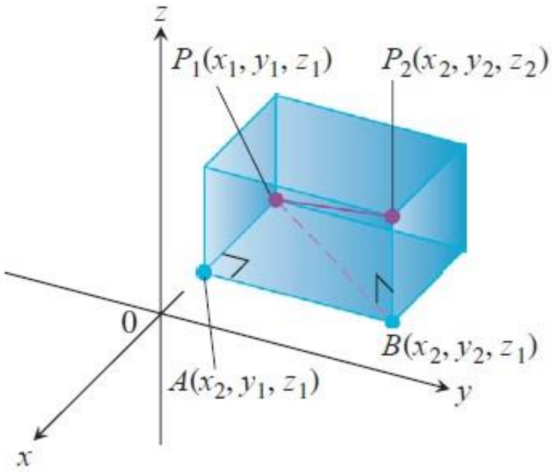


Figure 1.5 We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2

So,

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$
$$= |P_1A|^2 + |AB|^2 + |BP_2|^2 \text{ substitute } |P_1B|^2 = |P_1A|^2 + |AB|^2$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Therefore, $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 3: Finding the Distance Between Two Points

The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$|P_1P_2| = \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2}$$
$$= \sqrt{16 + 4 + 25}$$
$$= \sqrt{45} \approx 6.708$$

We can use the distance formula to write equations for spheres in space (Figure 1.6). A point $P(x, y, z)$ lies on the sphere of radius a centered at $P_o(x_o, y_o, z_o)$ precisely when $|P_oP| = a$ or

$$(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 = a^2$$

The Standard Equation for the Sphere of Radius a and Centre (x_o, y_o, z_o)

$$(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 = a^2$$

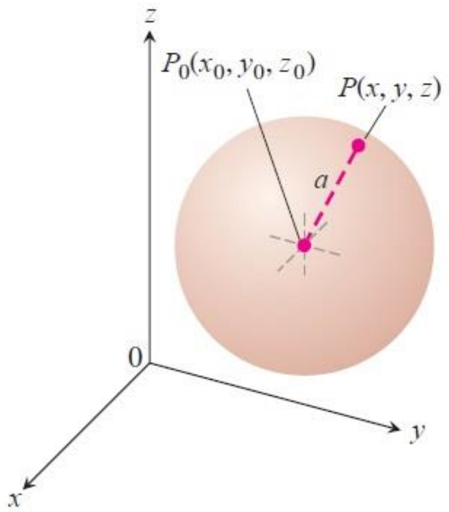


Figure 1.6 The standard equation of the sphere of radius a centred at the point (x_o, y_o, z_o) is

$$(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 = a^2.$$

Example 4: Finding the Centre and Radius of a Sphere

Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution: We find the centre and radius of a sphere the way we find the centre and radius of a circle: Complete the squares on the *x*-, *y*-, and *z*-terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the centre and radius. For the sphere here, we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(-\frac{4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(-\frac{4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

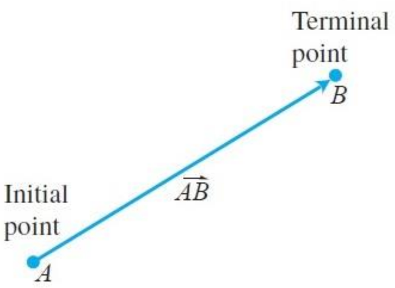
From this standard form, we read that x_o=-3/2, y_o=0, z_o=2, and $a = \frac{\sqrt{21}}{2}$. The centre is (-3/2, 0, 2). The radius is $\frac{\sqrt{21}}{2}$.

Example 5: Interpreting Equations and Inequalities

(a)	$x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
(b)	$x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.

(c)	$x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
(d)	$x^2 + y^2 + z^2 = 4$, $z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the *xy*-plane (the plane *z* = 0).

1.2 Vectors:



A vector in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point *A* and terminal point *B*; its length is denoted by $|\overrightarrow{AB}|$ (see Figure 1.7). Two vectors are equal if they have the same length and direction.

Figure 1.7 the directed line segment $|\overrightarrow{AB}|$.

Examples of two- and three-dimensions vectors (see Figure 1.8):

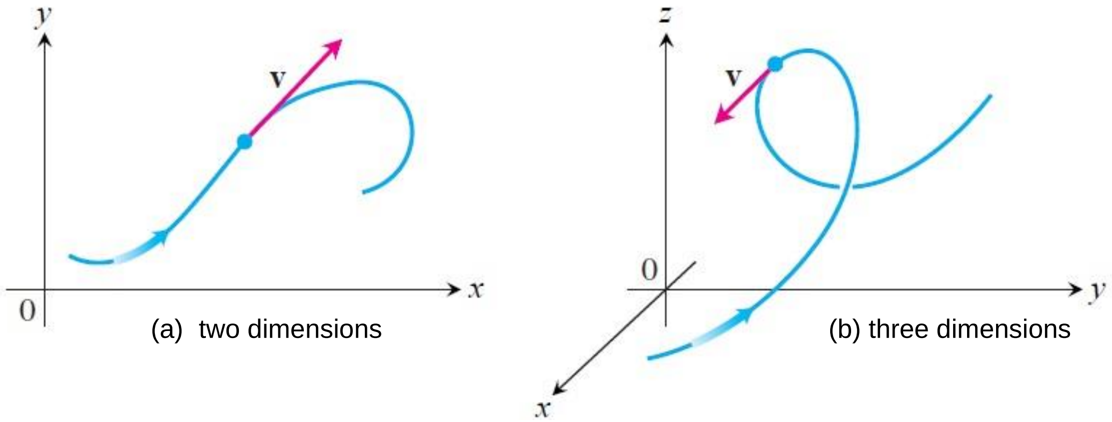


Figure 1.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

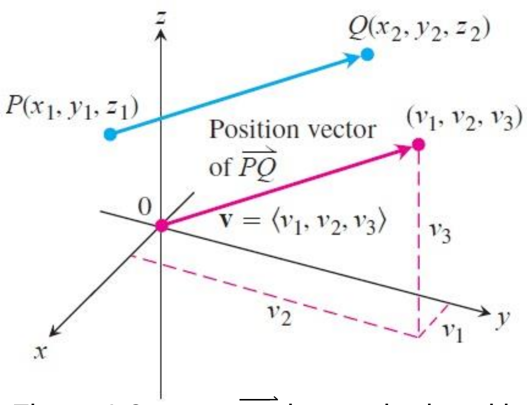


Figure 1.9 vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

We need a way to represent vectors algebraically so that we can be more precise about the direction of a vector. Let $\mathbf{v} = \overrightarrow{PQ}$. There is one directed line segment equal to $\overrightarrow{PQ}$ whose initial point is the origin (Figure 1.9). It is the representative of $\mathbf{v}$ in standard position and is the vector we normally use to represent $\mathbf{v}$.

We can specify **v** by writing the coordinates of its terminal point (v_1, v_2, v_3) when **v** is in standard position. If **v** is a vector in the plane its terminal point (v_1, v_2) has two coordinates.

So, we can define the Component Form of the vector:
If **v** is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the component form of **v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

If **v** is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of **v** is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

Given the points (Figure 1.9) $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

If **v** is two-dimensional with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as points in the plane, then $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. There is no third component for planar vectors.

Two vectors are equal if and only if their standard position vectors are identical. Thus $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$. The **magnitude** or **length** of the vector $\overrightarrow{PQ}$ is the length of any of its equivalent directed line segment representations. In particular, if $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the standard position vector for $\overrightarrow{PQ}$ then the distance formula gives the magnitude or length of **v**, denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. Therefore, the magnitude or length of the vector is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1: Component Form and Length of a Vector
Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$and terminal point $Q(-5, 2, 2)$.

Solution:

(a) The standard position vector v representing $\overrightarrow{PQ}$ has components

$v_1 = x_2 - x_1 = -5 - (-3) = -2$, $v_2 = y_2 - y_1 = 2 - 4 = -2$, and $v_3 = z_2 - z_1 = 2 - 1 = 1$.

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

Example 2: Force Moving a Cart

A small cart is being pulled along a smooth horizontal floor with a 20-lb force $\mathbf{F}$ making a 45° angle to the floor (Figure 1.10). What is the effective force moving the cart forward?

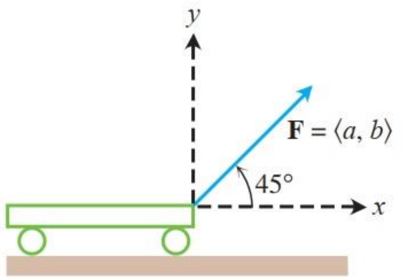


Figure 1.10 The force pulling the cart

Solution: The effective force is the horizontal component of $\mathbf{F} = \langle a, b\rangle$, given by

$$a = |\mathbf{F}| \cos 45^o = 20\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ Ib}$$

Notice that $\mathbf{F}$ is a two-dimensional vector.

Vector Algebra Operation

Two principal operations involving vectors are vector addition and scalar multiplication.

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ vectors with k a scalar. A scalar is a real number and can be positive, negative, or zero.

A. Vector addition: $\mathbf{u} \pm \mathbf{v} = \langle u_1 \pm v_1, u_2 \pm v_2, u_3 \pm u_3\rangle$;

B. Scalar multiplication: $k\,\mathbf{u} = \langle ku_1, ku_2, ku_3\rangle$

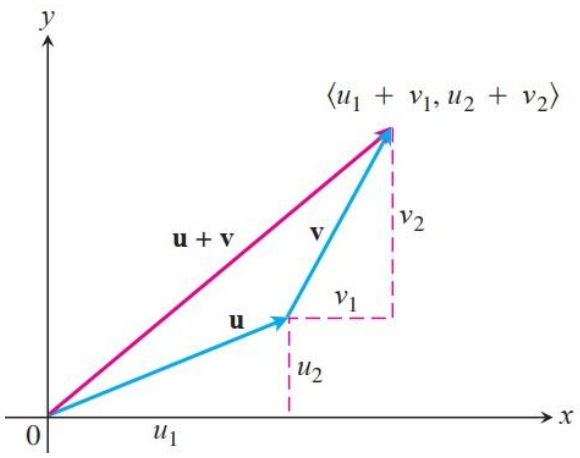
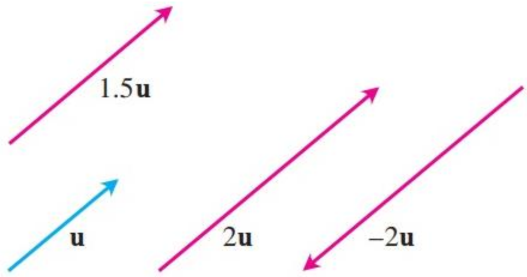


Figure 1.11 Geometric interpretation of the vector sum.

The definition of vector addition is illustrated geometrically for planar vectors in Figure 1.11, where the initial point of one vector is placed at the terminal point of the other.

Figure 1.12 displays a geometric interpretation of the product $k\mathbf{u}$ of the scalar k and vector $\mathbf{u}$. If $k > 0$, then $k\mathbf{u}$ has the same direction as $\mathbf{u}$; if $k < 0$, then the direction of $k\mathbf{u}$ is opposite to that of $\mathbf{u}$. Comparing the lengths of $\mathbf{u}$ and $k\mathbf{u}$, we see that

$$|k\mathbf{u}| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} = |k|\sqrt{u_1^2 + u_2^2 + u_3^2} = |k||\mathbf{u}|$$



The length of $k\mathbf{u}$ is the absolute value of the scalar k times the length of $\mathbf{u}$. The vector (-1) $\mathbf{u}$ = - $\mathbf{u}$ has the same length as $\mathbf{u}$ but points in the opposite direction.

Figure 1.12 Scalar multiples of **u.**

By the difference $\mathbf{u} - \mathbf{v}$ of two vectors, we mean $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Note that $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}$, so adding the vector $(\mathbf{u} - \mathbf{v})$ to $\mathbf{v}$ gives $\mathbf{u}$ (Figure 1.13). Figure 1.14 shows the difference $\mathbf{u} - \mathbf{v}$ as the sum $\mathbf{u} + (-\mathbf{v})$.

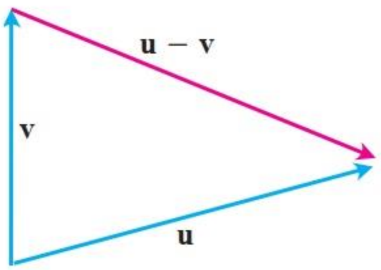


Figure 1.13 The vector $\mathbf{u} - \mathbf{v}$, when added when added to $\mathbf{v}$, gives $\mathbf{u}$.

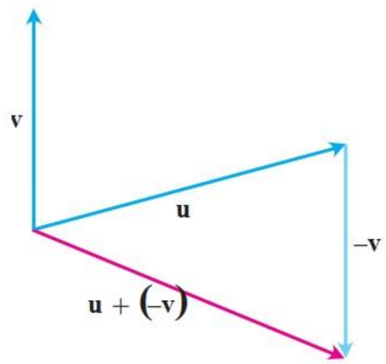


Figure 1.14 $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

Example 3: Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1\rangle$ and $\mathbf{v} = \langle 4, 7, 0\rangle$. Find (a) 2$\mathbf{u}$ + 3$\mathbf{v}$; (b) $\mathbf{u} - \mathbf{v}$; (c) $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution: (a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1\rangle + 3\langle 4, 7, 0\rangle = \langle -2, 6, 2\rangle + \langle 12, 21, 0\rangle = \langle 10, 27, 2\rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1\rangle - \langle 4, 7, 0\rangle = \langle -1 - 4, 3 - 7, 1 - 0\rangle = \langle -5, -4, 1\rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a unit vector. The standard unit vectors are

$\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$

Any vector can be written as follows: $\mathbf{v} = \langle v_1, v_2, v_3\rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

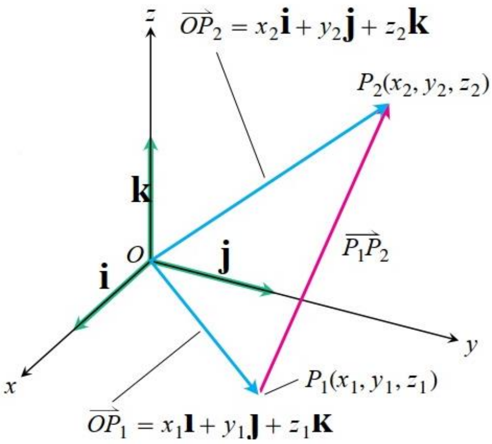


Figure 1.15 the vector from P_1 to P_2.

We call the scalar (or number) v_1 component of the vector $\mathbf{v}$, v_2 the $\mathbf{j}$-component, and v_3 the $\mathbf{k}$-component. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is (see Figure 1.15)

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Whenever $\mathbf{v} \neq 0$, its length $|\mathbf{v}|$ is not zero and

$$\left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$.

Example 4: Finding a Vector's Direction

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution: we divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3-1)\mathbf{i} + (2-0)\mathbf{j} + (0-1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$.

If $\mathbf{v} \neq 0$, then $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$; the equation $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ in terms of its length and direction.

Midpoint of a Line Segment

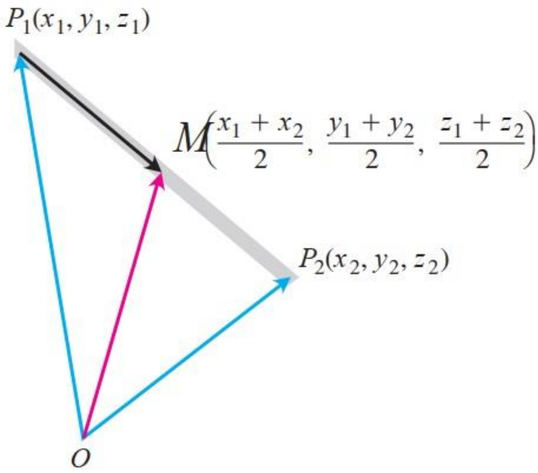


Figure 1.16 Midpoint of line segment.

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The midpoint *M* of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point (see Figure 1.16)

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 5: Finding Midpoints

The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2)$$

1.3 The Dot Product

Dot products are also called inner or scalar products because the product results in a scalar, not a vector. It is used to calculate the angle between two vectors directly from their components; show whether two vectors are orthogonal or not; find projection vector.

The dot product $\mathbf{u} \bullet \mathbf{v}$ ($\mathbf{u}$ dot $\mathbf{v}$) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Example 1: Finding Dot Product

(a) $\langle 1, -2, -1 \rangle \bullet \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \bullet (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1.$

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

Angle Between Vectors

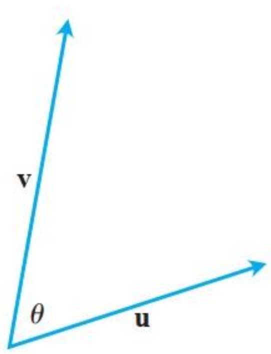


Figure 1.17 The angle between $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathrm{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$,

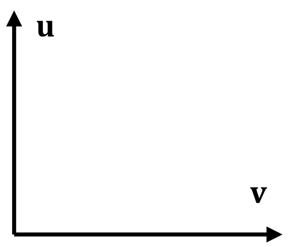
$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

$$0 \leq \theta \leq \pi$$

Perpendicular (Orthogonal) Vectors



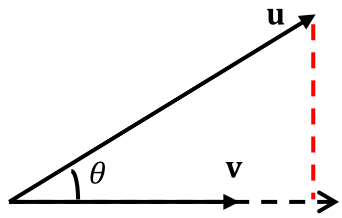
Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (or perpendicular) if and only if $\mathbf{u} \bullet \mathbf{v} = 0$.

If we have two vectors $\mathbf{u}$ and $\mathbf{v}$**,** from dot product, we can know:

1. $\mathbf{u} \bullet \mathbf{v} = (+)$, acute angle;
2. $\mathbf{u} \bullet \mathbf{v} = (-)$, obtuse angle;
3. $\mathbf{u} \bullet \mathbf{v} = (0)$, right angle.

Vector Projections

The vector projection of $\mathbf{u}$ onto a nonzero vector $\mathbf{v}$ is



$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|}$$
$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$
$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

Scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$$

Example 2: Finding the Vector Projection

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution:

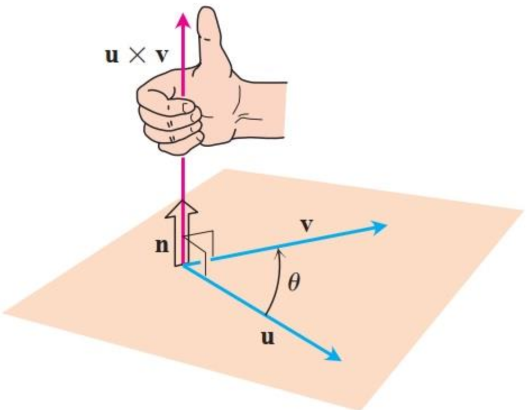
$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = \frac{-4}{9}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = \left(-\frac{4}{9}\mathbf{i}+\frac{8}{9}\mathbf{j}+\frac{8}{9}\mathbf{k}\right)$$

We find the scalar component

$$|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}\right) = 2-2-\frac{4}{3} = -\frac{4}{3}$$

1.4 The Cross Product

The Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics.



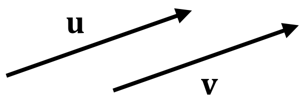
We start with two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in space. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they determine a plane. We select a unit vector $\mathbf{n}$ perpendicular to the plane by the right-hand rule. This means that we choose $\mathbf{n}$ to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle from $\mathbf{u}$ to $\mathbf{v}$ (Figure 1.18). Then the cross product $\mathbf{u}\times\mathbf{v}$ ("$\mathbf{u}$ cross $\mathbf{v}$") is the vector defined as follows:

Figure 1.18 The construction $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}$$

Unlike the dot product, the cross product is a vector. For this reason, it is also called the vector product of $\mathbf{u}$ and $\mathbf{v}$, and applies only to vectors in space. The vector is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because it is a scalar multiple of $\mathbf{n}$.

Parallel Vectors: Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = 0$.



Properties of the Cross Product: If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

Figure 1.19 visualises property 4.

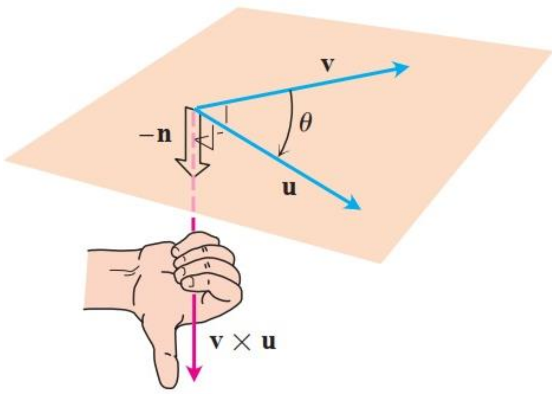


Figure 1.19 The construction $\mathbf{v} \times \mathbf{u}$.

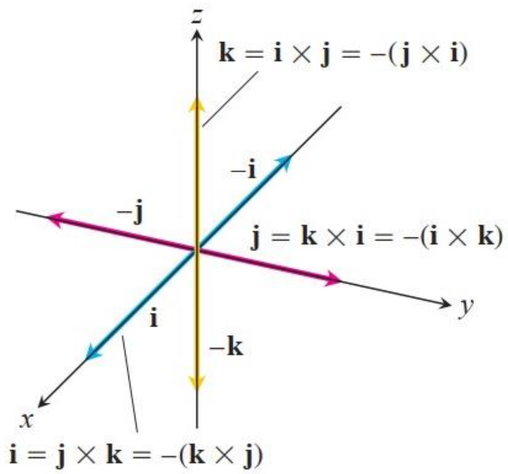
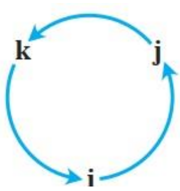


Figure 1.20 The pairwise cross product of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

When we apply the definition to calculate the pairwise cross products of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we find (Figure 1.20)

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

and $$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

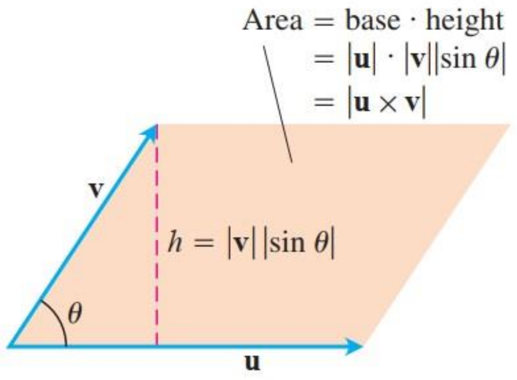


Calculating Cross Product Using Determinants:

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$



This is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ (Figure 1.21), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}||\sin\theta|$ the height. Because $\mathbf{n}$ is a unit vector, the area of a parallelogram is

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

Figure 1.21 The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Example 1: Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Solution:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

Example 2: Find unit vector orthogonal to the vectors $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{j} + \mathbf{k}$

Solution:

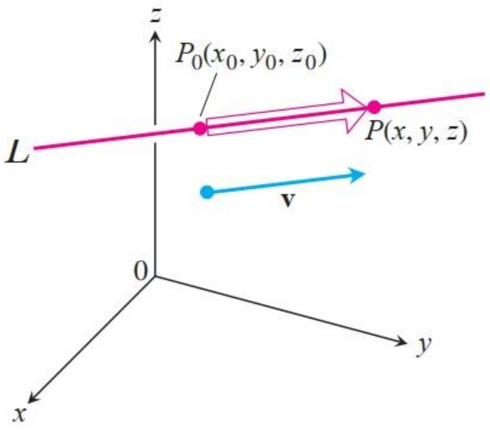
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$$

$$|\mathbf{u}\times\mathbf{v}| = \sqrt{(-5)^2 + (-3)^2 + (12)^2} = \sqrt{178}$$

$$\mathbf{z} = \frac{1}{\sqrt{178}}(-5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k})$$

1.5 Lines and Planes in Space

Lines: In the plane, a line is determined by a point and a number giving the slope of the line. In space, a line is determined by a point and a vector giving the direction of the line.



Figured 1.22 A point *P* lies on L.

Suppose that L is a line in space passing through a point $P_o(x_o, y_o, z_o)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_oP}$ parallel to $\mathbf{v}$ (Figure 1.22). Thus, $\overrightarrow{P_oP} = t\mathbf{v}$, t is scalar parameter $(-\infty, \infty)$.

$$(x - x_o)\mathbf{i} + (y - y_o)\mathbf{j} + (z - z_o)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

or

$$\frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3} = t$$

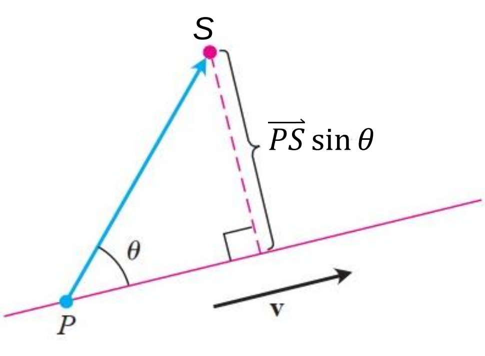
Parametric Equations of a Line

The standard parametrization of the line through $P_o(x_o, y_o, z_o)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is $x = x_o + v_1 t,\ y = y_o + v_2 t,\ z = z_o + v_3 t$.

Example 1: Find parametric equations for the line through (-2, 0, 4) parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Solution: $x = -2 + 2t,\ y = 4t,\ z = 4 - 2t$.

Distance from a Point S to a Line Through *P* Parallel to $\mathbf{v}$ (see Figure 1.23)



$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

Figure 1.23 The distance from point to a line.

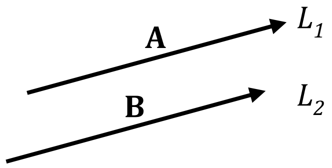
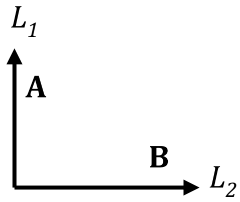
Example 2: Find the distance from the point *S*(1, 1, 5) to the line

$L: x = 1 + t,\ y = 3 - t,\ z = 2t$

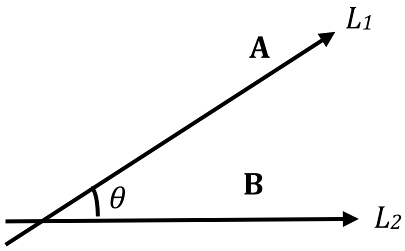
Solution: from the equation for the *L*, *L* passes through *P*(1, 3, 0) and parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. $\overrightarrow{PS} = (1-1)\mathbf{i} + (1-3)\mathbf{j} + (5-0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$ and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}\,.$$

Parallel Lines: $L_1 \| L_2 \Rightarrow \mathbf{A} \times \mathbf{B} = 0.$	**Orthogonal Lines:** $L_1 \perp L_2 \Rightarrow \mathbf{A} \cdot \mathbf{B} = 0.$
	

Angle Between Two Lines:

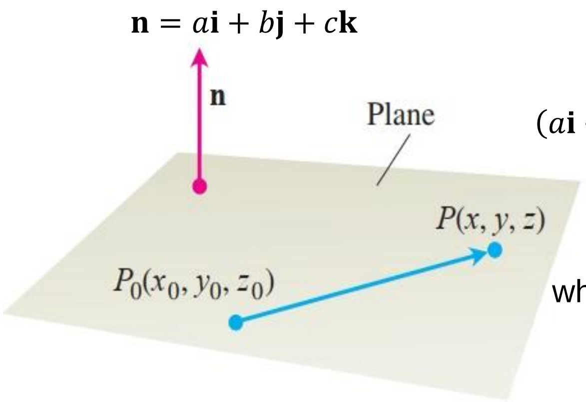


$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)$$

Skew Lines: are two lines that do not intersect and not parallel.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and vector normal (perpendicular) to the plane (see Figure 1.24).



$$\mathbf{n}\cdot\overrightarrow{P_oP}=0$$

$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot[(x-x_o)\mathbf{i}+(y-y_o)\mathbf{j}+(z-z_o)\mathbf{k}]=0$$

$$a(x-x_o)+b(y-y_o)+c(z-z_o)=0$$

$$ax+by+cz=d,$$

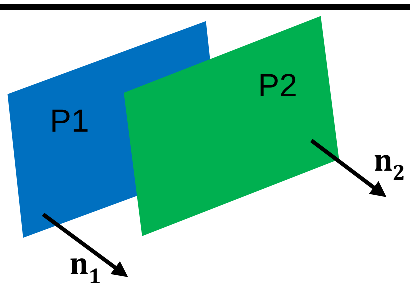
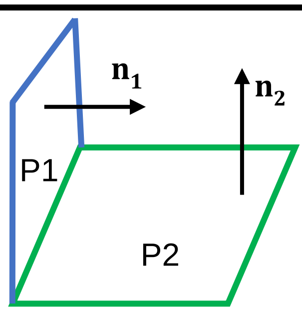
where $$d=ax_o+by_o+cz_o$$

Figure 1.24 A plane in space.

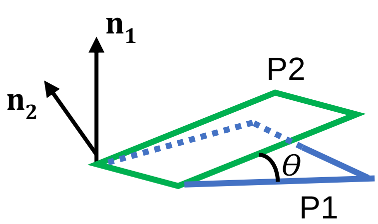
Example 3: Find an equation for the plane through $P_o(-3, 0, 7)$ perpendicular to $\mathbf{n}=5\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Solution: the component equation is $5\big(x-(-3)\big)+2(y-0)+(-1)(z-7)=0$

$$5x+2y-z=-22\,.$$

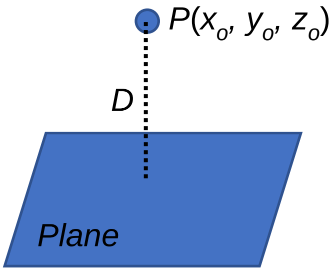
Parallel Planes: $P_1 \parallel P_2 \Rightarrow \mathbf{n_1}\times\mathbf{n_2}=0$.	**Orthogonal Planes**: $P1 \perp P2 \Rightarrow \mathbf{n_1}\cdot\mathbf{n_2}=0$.
	

Angle Between Two Planes:



$$\theta=\cos^{-1}\left(\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}\right)$$

Distance Between Point and Plane:



$$D = \frac{ax_o + by_o + cz_o - d}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \begin{cases} + & \text{point } P \text{ lies above} \\ - & \text{point } P \text{ lies below} \\ 0 & \text{point } P \text{ lies on the plane} \end{cases}$$

Or as you can see in the next example:

$$D = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Example 4: Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

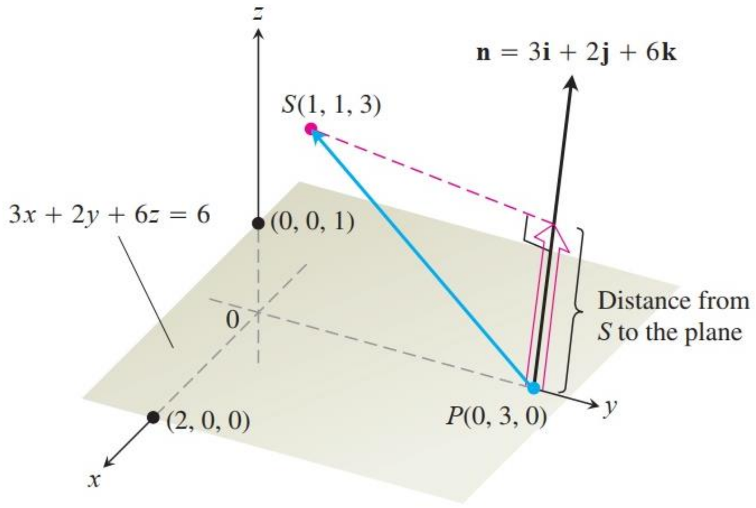


Figure 1.25 The distance from point S to the plane

Solution: First, we find a point in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Figure 1.25). The coefficients in the equation give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

We can find interception points from the plane's equation. If we take *P* to be the *y*-intercept (0, 3, 0), then

$$\overrightarrow{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{z}$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7$$

The distance from S to the plane is

$$D = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{z}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{z}\right)\right| = \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.$$

Solved Problems:

Prob1. Find a vector has length 15 in the direction of $\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution: $|\mathbf{B}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$, $\quad \dfrac{\mathbf{B}}{|\mathbf{B}|} = \dfrac{1}{\sqrt{6}}\mathbf{i} + \dfrac{2}{\sqrt{6}}\mathbf{j} - \dfrac{1}{\sqrt{6}}\mathbf{k}$,

$$\mathbf{v} = 15\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) = \frac{15}{\sqrt{6}}\mathbf{i} + \frac{30}{\sqrt{6}}\mathbf{j} - \frac{15}{\sqrt{6}}\mathbf{k}.$$

Prob2. Find a vector has length 22 in the opposite direction of $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j}$.

Solution: $|\mathbf{A}| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$, $\quad \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{2}{\sqrt{13}}\mathbf{i} - \dfrac{3}{\sqrt{13}}\mathbf{j}$,

$$\mathbf{v} = -22\left(\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}\right) = -\frac{44}{\sqrt{13}}\mathbf{i} + \frac{66}{\sqrt{13}}\mathbf{j}.$$

Prob3. Using vectors, show that the sum of triangle angles is 180°, the points of triangle are (1, 1), (4, 3), and (2, 5). Then, find the area of the triangle.

Solution: $\mathbf{A} = (2-1)\mathbf{i} + (5-1)\mathbf{j} = \mathbf{i} + 4\mathbf{j}$, $\quad |\mathbf{A}| = \sqrt{1+16} = \sqrt{17}$

$\mathbf{B} = (4-1)\mathbf{i} + (3-1)\mathbf{j} = 3\mathbf{i} + 2\mathbf{j}$, $\quad |\mathbf{B}| = \sqrt{9+4} = \sqrt{13}$

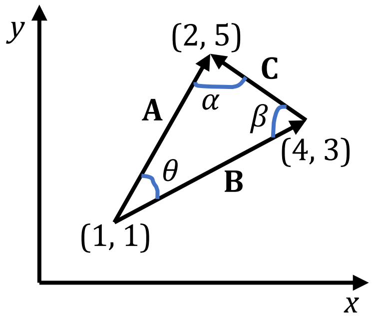
$\mathbf{C} = (2-4)\mathbf{i} + (5-3)\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$, $|\mathbf{C}| = \sqrt{4+4} = \sqrt{8}$

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{3+8}{\sqrt{17}\sqrt{13}}\right) = \cos^{-1}\left(\frac{11}{\sqrt{221}}\right) = 42.27^\circ$$

$$\beta = \cos^{-1}\left(\frac{-\mathbf{B}\cdot\mathbf{C}}{|\mathbf{B}||\mathbf{C}|}\right) = \cos^{-1}\left(\frac{6-4}{\sqrt{13}\sqrt{8}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}}\right) = 78.69^\circ$$

$$\alpha = \cos^{-1}\left(\frac{-\mathbf{A}\cdot-\mathbf{C}}{|\mathbf{A}||\mathbf{C}|}\right) = \cos^{-1}\left(\frac{-2+8}{\sqrt{17}\sqrt{8}}\right) = \cos^{-1}\left(\frac{6}{\sqrt{136}}\right) = 59.03^\circ$$

$$\theta + \beta + \alpha = 179.99 \approx 180^\circ$$

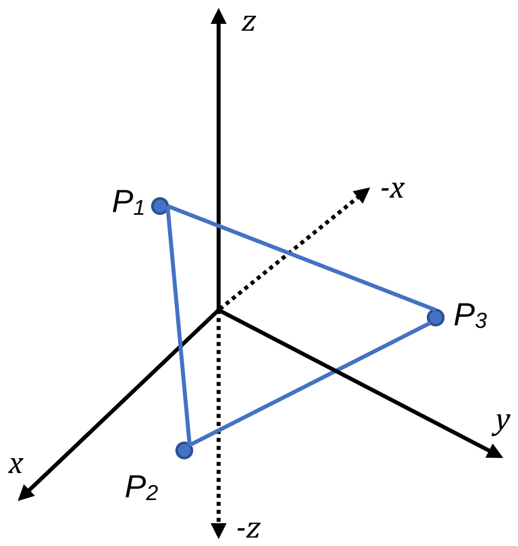


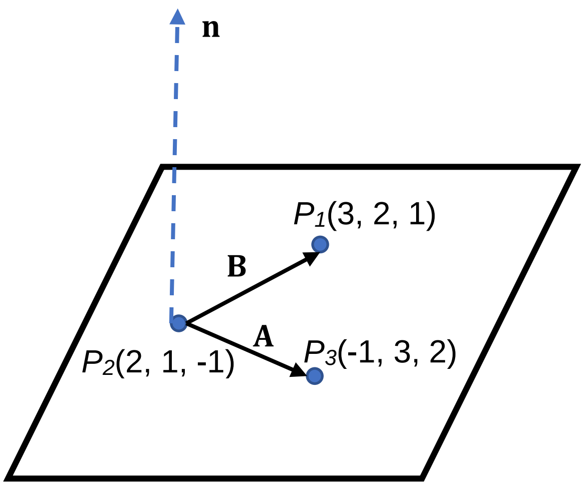
$$\text{Area of traingle} = \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= \frac{1}{2}|-10\mathbf{k}| = 5\ \text{unit}^2.$$

Prob4. Find equation of the plane has $P_1(3, 2, 1)$, $P_2(2, 1, -1)$, and $P_3(-1, 3, 2)$.

Solution:





$$\left.\begin{array}{l}\mathbf{A} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ \mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}\end{array}\right\} \quad \mathbf{n} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)\mathbf{i} - (-6-3)\mathbf{j} + (-3-2)\mathbf{k}$$

$\mathbf{n} = \mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, point $P_2(2, 1, -1)$, or any point, we find

$$x + 9y - 5z = 1(2) + 9(1) - 5(-1) \xrightarrow{\text{Equ. of plane is}} x + 9y - 5z = 16\,.$$

Prob5: Given equations of two planes, plane1 $(x + y + z = 1)$ and plane2 $(2x - 3y + z = 4)$, find: **(a)** point ∈ plane1; **(b)** whether the two planes are parallel or not; **(c)** the intersection point, if they are intersecting; **(d)** equation of the line of intersection for the two planes.

Solution: **(a)** $y = z = 0, \Rightarrow x = 1, \Rightarrow$ point is $(1, 0, 0)$.

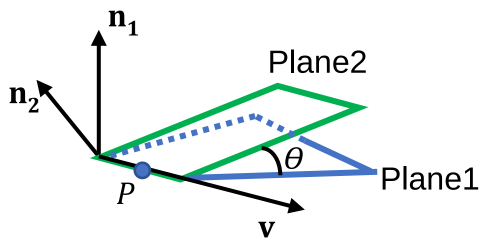
$$\text{(b)} \quad \left.\begin{array}{l}\mathbf{n_1} = \mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{n_2} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\end{array}\right\} \quad \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = (1+3)\mathbf{i} \ldots\ldots \neq 0, \text{so they are not parallel.}$$

$$\text{(c)} \quad z = 0, \Longrightarrow \left.\begin{array}{r} x + y = 1 \\ 2x - 3y = 4\end{array}\right\} \xrightarrow{\text{Multiply }1^{\text{st}}\text{ equ. by 3}} \begin{array}{r} 3x + 3y = 3 \\ \underline{2x - 3y = 4} \\ 5x = 7 \end{array}$$

$$x = \frac{7}{5}, y = -\frac{2}{5}, \text{point is } P\left(\frac{7}{5}, -\frac{2}{5}, 0\right).$$

$$\text{(d)} \quad \mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = (1+3)\mathbf{i} - (1-2)\mathbf{j} + (-3-2)\mathbf{k} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}\,,$$

we have $P\left(\frac{7}{5}, -\frac{2}{5}, 0\right)$ and the vector $\mathbf{v}$ $\xRightarrow{\text{Equ. of line is}}$ $\dfrac{x-\frac{7}{5}}{4} = \dfrac{y+\frac{2}{5}}{1} = \dfrac{z}{-5}$.



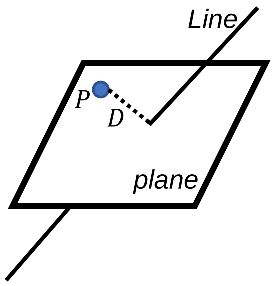
Prob6: Determine a point $P \in$ plane $(x - 2y + 3z = 0)$, then find the distance between that point P and an intersection point of line (L) with the plane.

$$L: \ \frac{4-z}{-3} = \frac{2x}{3} = \frac{1-\frac{1}{2}y}{4}$$

Solution: $y = z = 0, \Longrightarrow x = 0, \Longrightarrow P(0,0,0)$,

$$\frac{4-z}{-3} = \frac{2x}{3} = \frac{1-\frac{1}{2}y}{4} \xRightarrow{\text{rearrange}} \frac{x}{3/2} = \frac{y-2}{-8} = \frac{z-4}{3} = t\,,$$

$$\text{parametric equ.} \quad \left.\begin{array}{l} x = \frac{3}{2}t \\ y = 2 - 8t \\ z = 4 + 3t \end{array}\right\} \xRightarrow{\text{substitute in plane}} \left\{\begin{array}{l} \frac{3}{2}t - 2(2-8t) + 3(4+3t) = 0 \\ \frac{3}{2}t - 4 + 16t + 12 + 9t = 0 \\ \frac{53}{2}t = -8 \Rightarrow t = -\frac{16}{53} \end{array}\right.$$



$$\text{substitute } t \text{ in parametric equ.} \quad \left.\begin{array}{l} x = \frac{3}{2}\left(-\frac{16}{53}\right) = -\frac{24}{53} \\ y = 2 - 8\left(-\frac{16}{53}\right) = \frac{234}{53} \\ z = 4 + 3\left(-\frac{16}{53}\right) = \frac{164}{53} \end{array}\right\}, \quad \text{point of intersection } \left(-\frac{24}{53}, \frac{234}{53}, \frac{164}{53}\right)$$

$$D = \sqrt{\left(-\frac{24}{53} - 0\right)^2 + \left(\frac{234}{53} - 0\right)^2 + \left(\frac{164}{53} - 0\right)^2}$$

$$D = \frac{1}{53}\sqrt{(24)^2 + (234)^2 + (164)^2} = 5.41 \text{ unit}$$

Prob7: Find the angle between two planes, plane1 $(2x-3y+3z=1)$ and plane2 $(x-y+\frac{1}{3}z=0)$. Then, find the distance between *P*(1, 1, -2) and plane1.

Solution: $\left.\begin{matrix}\mathbf{n_1}=2\mathbf{i}-3\mathbf{j}+3\mathbf{k}\\ \mathbf{n_2}=\mathbf{i}-\mathbf{j}+\frac{1}{3}\mathbf{k}\end{matrix}\right\}, \theta=\cos^{-1}\left(\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}\right)=\cos^{-1}\left(\frac{2+3+1}{\sqrt{4+9+9}\sqrt{1+1+\frac{1}{9}}}\right)$

$$=\cos^{-1}\left(\frac{6}{\sqrt{22}\sqrt{\frac{19}{9}}}\right)=28.3°,$$

$$D=\frac{2(1)-3(1)+3(-2)-1}{\sqrt{4+9+9}}=-\frac{8}{\sqrt{22}}=\frac{8}{\sqrt{22}}\text{ , the point lies below the plane.}$$

Prob8: Check whether these two planes are parallel or not and find the distance between them: plane1 $(x-2y+4z=1)$ and plane2 $(3x-6y+12z=5)$.

Solution: $\left.\begin{matrix}\mathbf{n_1}=\mathbf{i}-2\mathbf{j}+4\mathbf{k}\\ \mathbf{n_2}=3\mathbf{i}-6\mathbf{j}+12\mathbf{k}\end{matrix}\right\}, \mathbf{n_1}\times\mathbf{n_2}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 4\\ 3 & -6 & 12\end{vmatrix}$

$$=(-24+24)\mathbf{i}-(12-12)\mathbf{j}+(-6+6)\mathbf{k}=0\text{ ,}$$

so, plane1 ∥ plane2,

Then, find point ∈ plane1, $y=z=0,\implies x=1, P(1,0,0)$,

$$D=\frac{3(1)-6(0)+12(0)-5}{\sqrt{9+36+144}}=-\frac{2}{\sqrt{189}}=-0.145=0.145\text{ unit, plane1 lies below plane2.}$$

Prob9: Find the intersection point of the line passes through (2, 4, -1), (5, 0, 7) with *xz*-plane.

Solution: $\mathbf{v}=3\mathbf{i}-4\mathbf{j}+8\mathbf{k}$, with initial point $(2,4,-1),\Rightarrow \frac{x-2}{3}=\frac{y-4}{-4}=\frac{z+1}{8}=t$,

$xz-plane \Rightarrow y=0,\Rightarrow y=4-4t \overset{y=0}{\Longrightarrow} 4-4t=0 \Rightarrow t=1$,

$$\left.\begin{matrix}x=2+3(1)=5\\ y=4-4(1)=0\\ z=-1+8(1)=7\end{matrix}\right\}\xrightarrow{\text{the point is}}(5,0,7).$$

Prob10: Find the equation of plane through (1, 2, -1) and perpendicular to line of intersection of these two planes $(2x+y+z=2),(x+2y+z=3)$.

Solution: $\left.\begin{matrix}\mathbf{n_1}=2\mathbf{i}+\mathbf{j}+\mathbf{k}\\ \mathbf{n_2}=\mathbf{i}+2\mathbf{j}+\mathbf{k}\end{matrix}\right\}, \mathbf{v}=\mathbf{n_1}\times\mathbf{n_2}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 1\\ 1 & 2 & 1\end{vmatrix}=(1-2)\mathbf{i}-(2-1)\mathbf{j}+(4-1)\mathbf{k}$

$\mathbf{v} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, which is the normal vector to the plane3,

$$-x - y + 3z = -1(1) - 1(2) + 3(-1) \Rightarrow -x - y + 3z = -6 \xrightarrow{\text{Equ. of plane}} x + y - 3z = 6.$$

1.5 Cylinders and Quadric Surfaces:

Cylinders:

A cylinder is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a generating curve for the cylinder. The cylinder can be generated by different curves.

Cylinder is generated by parabola: $y = x^2$, suppose that the point $P_o(x_o, x_o^2, 0)$ lies on the parabola $y = x^2$ in the *xy*-plane. Then, for any value of *z*, the point $Q(x_o, x_o^2, z)$ will lie on the cylinder because it lies on the line $x = x_o, y = x_o^2$ through P_o parallel to the *z*-axis. Conversely, any point $Q(x_o, x_o^2, z)$ whose *y*-coordinate is the square of its *x*-coordinate lies on the cylinder because it lies on the line $x = x_o, y = x_o^2$ through P_o parallel to the *z*-axis (see Figures 1.25 and 1.26).

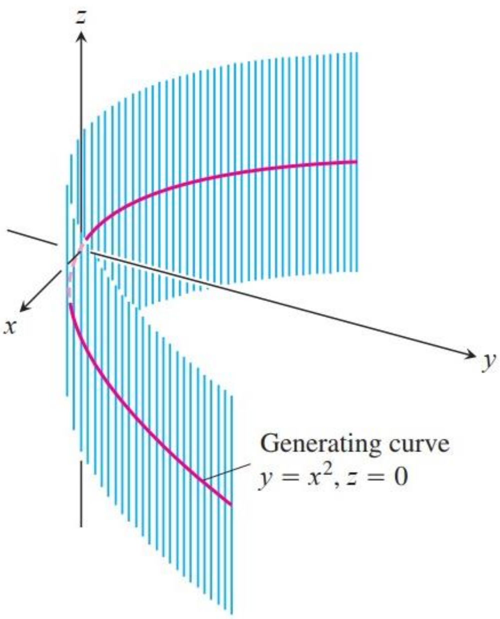


Figure 1.25 The cylinder of lines passing through the parabola $y = x^2$ in the *xy*-plane parallel to the *z*-axis.

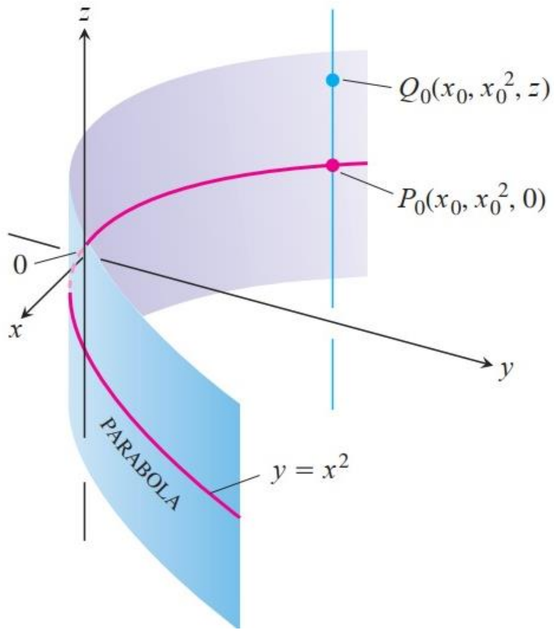


Figure 1.26 Every point of the cylinder in Figure 1.25 has coordinates of (x_o, x_o^2, z). We call it the cylinder $y = x^2$.

Cylinder is generated by circle (circular cylinder): The equation $x^2 + y^2 = r^2$ defines the circular cylinder made by the lines parallel to the *z*-axis that pass through the circle $x^2 + y^2 = r^2$ in the *xy*-plane (see Figure 1.27).

Cylinder is generated by ellipse (elliptical cylinder): For example, the equation $x^2 + 4z^2 = 4$ defines the elliptical cylinder made by the lines parallel to the *y*-axis that pass through the ellipse $x^2 + 4z^2 = 4$ in the *xz*-plane (see Figure 1.28).

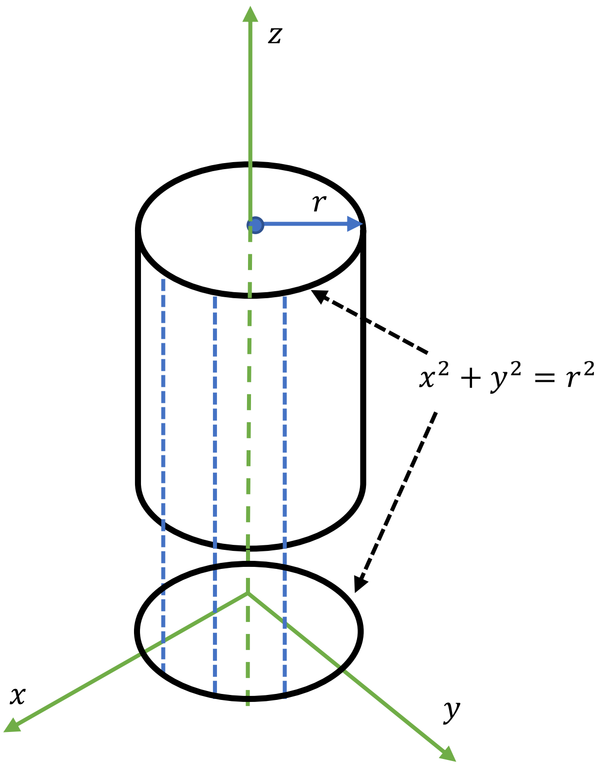


Figure 1.27 Cylinder is generated from circle.

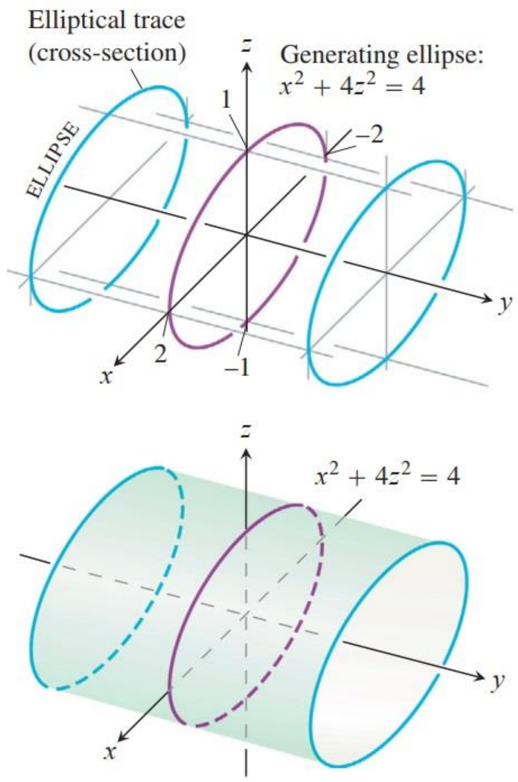


Figure 1.28 Elliptical cylinder.

Cylinder is generated from hyperbola: For example, the hyperbolic cylinder $y^2 - z^2 = 1$ is made of lines parallel to the *x*-axis and passing through the hyperbola $y^2 - z^2 = 1$ in the *yz*-plane. The cross-sections of the cylinder in planes perpendicular to the *x*-axis are hyperbolas congruent to the generating hyperbola (see Figure 1.29).

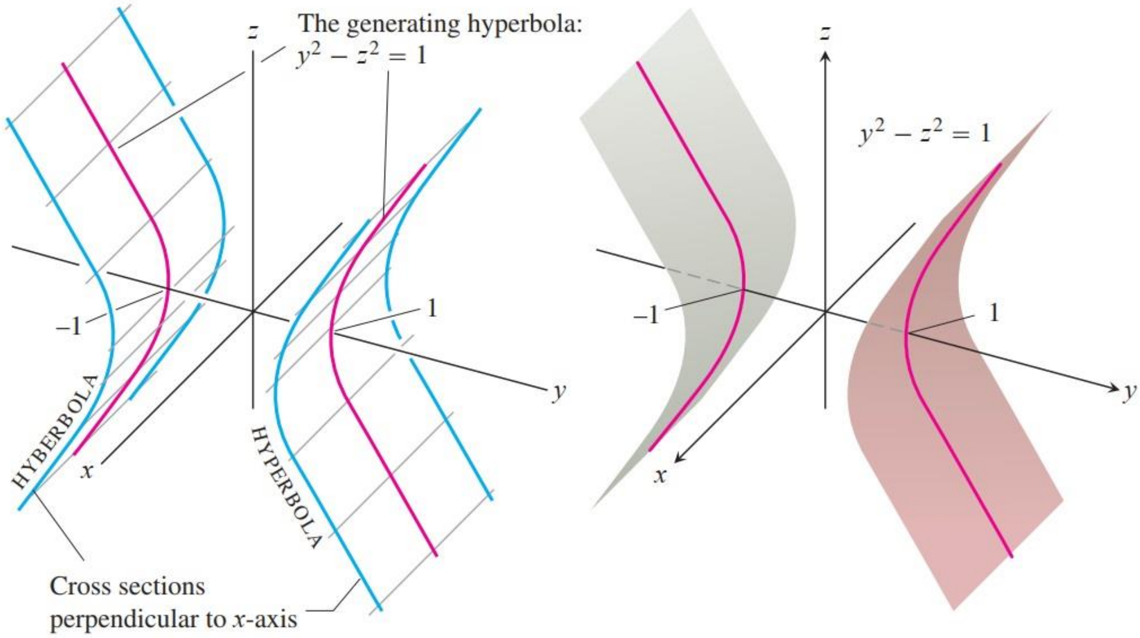


Figure 1.29 The hyperbolic cylinder.

Quadric Surfaces:

Quadric surfaces are surfaces defined by second-degree equations in *x*, *y*, and *z*. These surfaces are the three-dimensional analogues of ellipses, parabolas, and hyperbolas. The most general form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0\,,$$

where A, B, C, and so on are constants. The basic quadric surfaces are ellipsoids, paraboloids, elliptical cones, and hyperboloids. Each type is presented by an example.

Ellipsoids: The ellipsoid below cuts the coordinate axes at $(\pm a, 0, 0), (0, \pm b, 0), (0, 0, \pm c)$ (see Figure 1.30).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

It lies within the rectangular box defined by the inequalities $|x| \le a, |y| \le b,$ and $|z| \le c$. The surface is symmetric with respect to each of the coordinate planes because each variable in the defining equation is squared. It can be seen from Figure 1.30 that the ellipsoid has elliptical cross-sections in each of the three coordinate planes. The curves in which the three coordinate planes cut the surface are ellipses. For example,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ when } z = 0$$

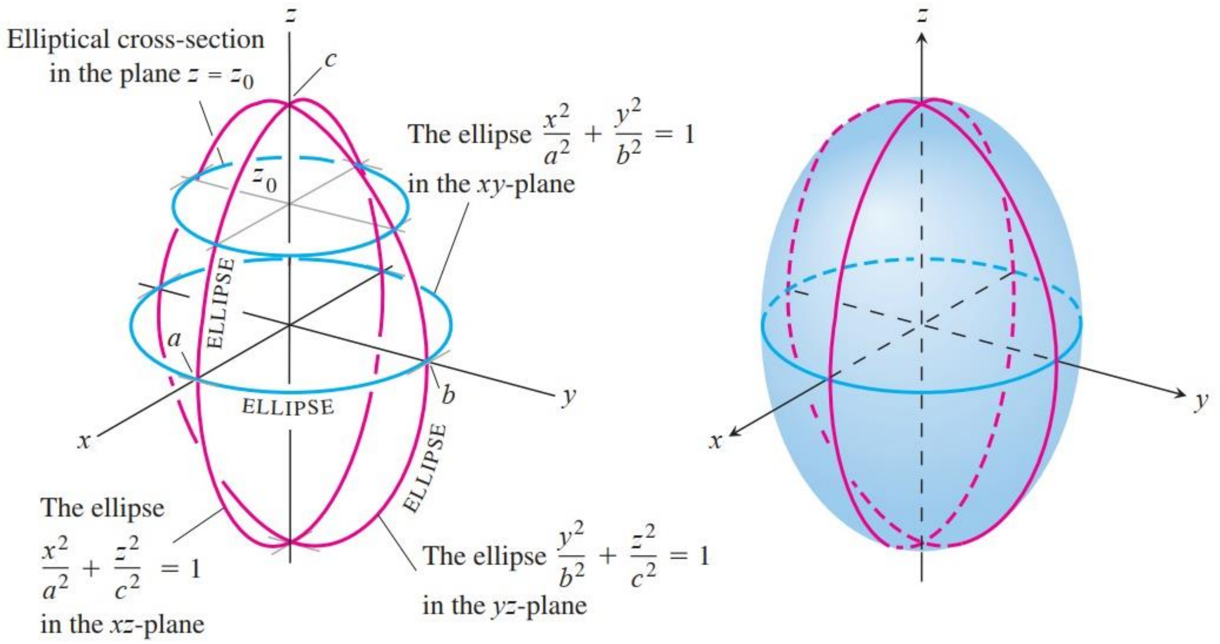


Figure 1.30 The ellipsoid.

The section cut from the surface by the plane $z = z_o, |z_o| < c,$ is the ellipse

$$\frac{x^2}{a^2\left(1-\left(\frac{z_o}{c}\right)^2\right)} + \frac{y^2}{b^2\left(1-\left(\frac{z_o}{c}\right)^2\right)} = 1$$

If any two of the semi-axes a, b, and c are equal, the surface is an ellipsoid of revolution. If all three are equal, the surface is a sphere.

Paraboloids: The elliptical paraboloid below is symmetric with respect to the planes $x = 0$ and $y = 0$ (see Figure 1.31).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

The only intercept on the axes is the origin. Except for this point, the surface lies above (if $c > 0$) or entirely below (if $c < 0$) the *xy*-plane, depending on the sign of c. The sections cut by the coordinate planes are

$x = 0$: the parabola $z = \frac{c}{b^2}y^2$

$y = 0$: the parabola $z = \frac{c}{a^2}x^2$

$z = 0$: the point $(0, 0, 0)$.

Figure 1.31 is shown for $c > 0$. The cross-sections perpendicular to the *z*-axis above the *xy*-plane are ellipses. The cross-sections in the planes that contain the *z*-axis are parabolas.

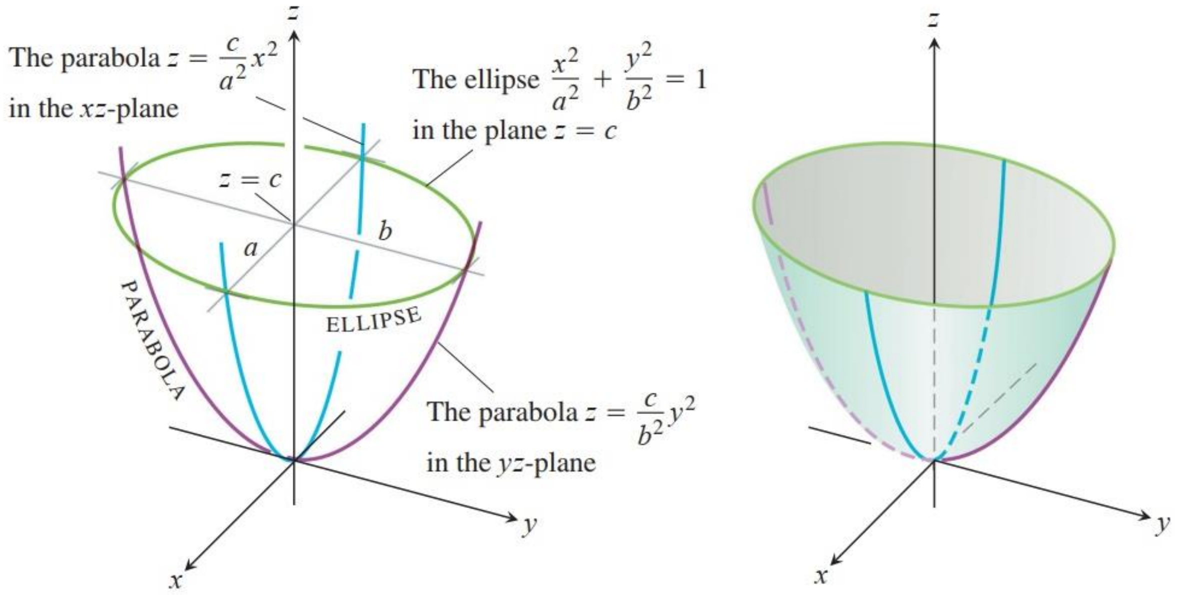


Figure 1.31 The elliptical paraboloid.

Each plane $z = z_o$ above the *xy*-plane cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_o}{c}.$$

Cones: The elliptical cone below is symmetric with respect to the three coordinate planes (see Figure 1.32).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

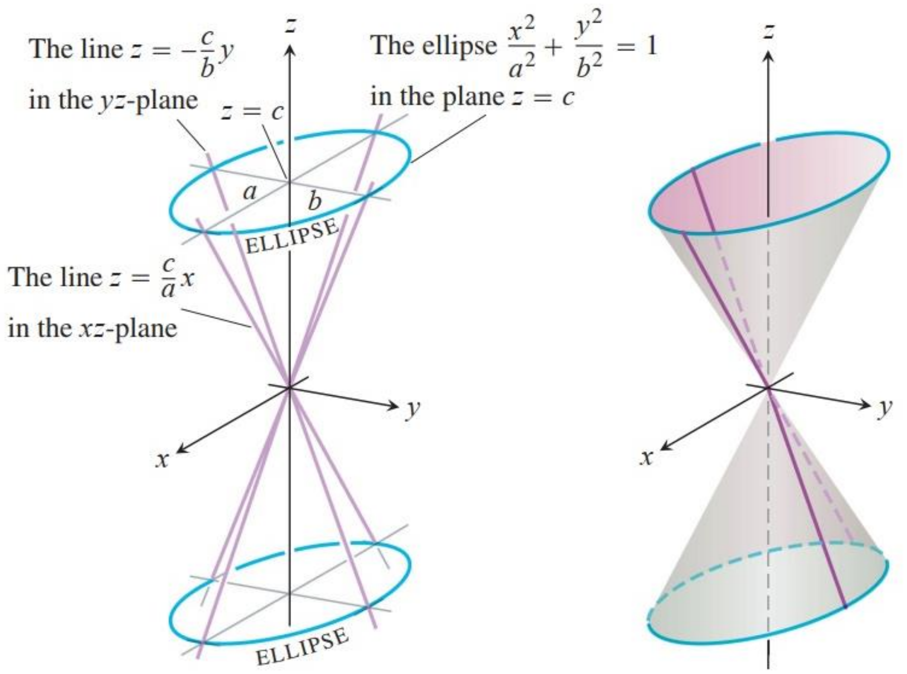


Figure 1.32 The elliptical cone.

Planes perpendicular to the *z*-axis cut the cone in ellipses above and below the *xy*-plane. Vertical planes that contain the *z*-axis cut it in pairs of intersecting lines.

The sections cut by the coordinate planes are

$x=0$: the lines $z=\pm\frac{c}{b}y$

$y=0$: the lines $z=\pm\frac{c}{a}x$

$z=0$: the point $(0,0,0)$.

The sections cut by planes $z=z_0$ above and below the *xy*-plane are ellipses whose centres lie on the *z*-axis and whose vertices lie on the lines given above. If $a=b$, the cone is a right circular cone.

Hyperboloids: The hyperboloid of one sheet is symmetric with respect to each of the three coordinate planes (see Figure 1.32). Planes perpendicular to the *z*-axis cut it in ellipses. Vertical planes containing the *z*-axis cut it in hyperbolas.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

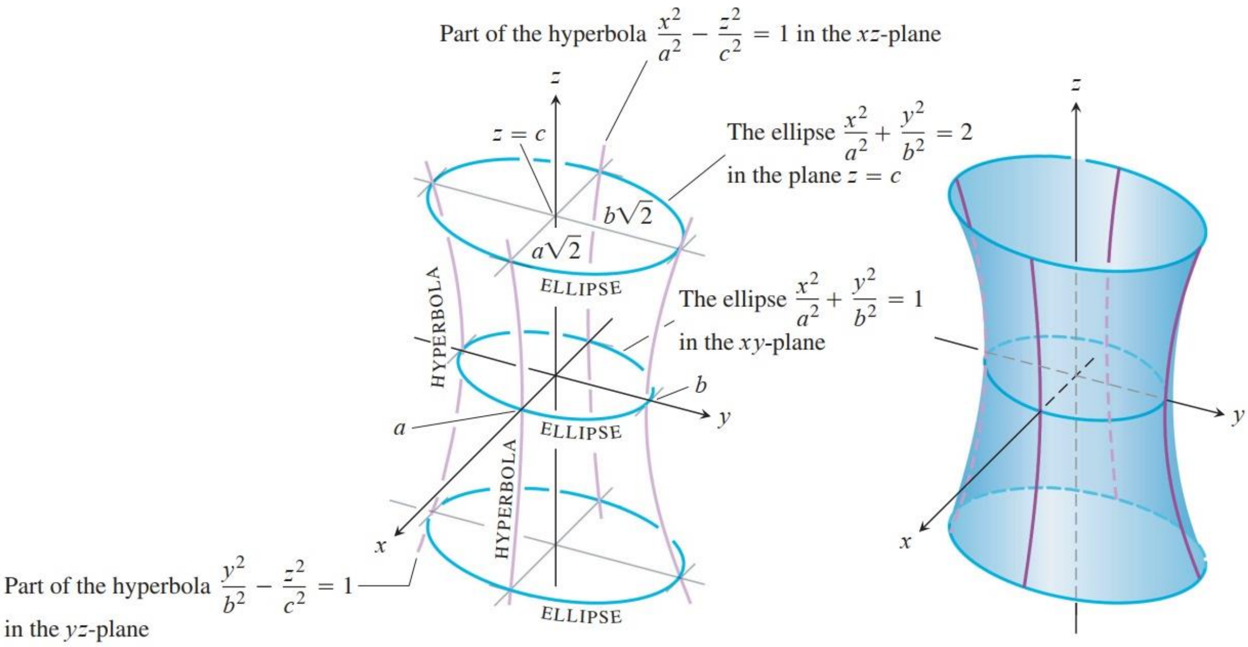


Figure 1.33 The hyperboloid.

The sections cut out by the coordinate planes are

$x=0$: the hyperbola $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

$y=0$: the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$

$z=0$: the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The plane $z=z_o$ cuts the surface in an ellipse with centre on the *z*-axis and vertices on one of the hyperbolic sections above. The surface is connected, meaning that it is possible to travel from one point on it to any other without leaving the surface. For this reason, it is said to have one sheet, in contrast to the hyperboloid in the next example, which has two sheets. If the $a=b$, hyperboloid is a surface of revolution.

The hyperboloid of two sheets is symmetric with respect to the three coordinate planes (see Figure 1.33).

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The plane $z=0$ does not intersect the surface; in fact, for a horizontal plane to intersect the surface, we must have $z \geq c$. The hyperbolic sections

$x=0$: $\frac{z^2}{c^2}-\frac{y^2}{b^2}=1$

$y=0$: $\frac{z^2}{c^2}-\frac{x^2}{a^2}=1$

have their vertices and foci on the *z*-axis. The surface is separated into two portions, one above the plane $z=c$ and the other below the plane $z=-c$.

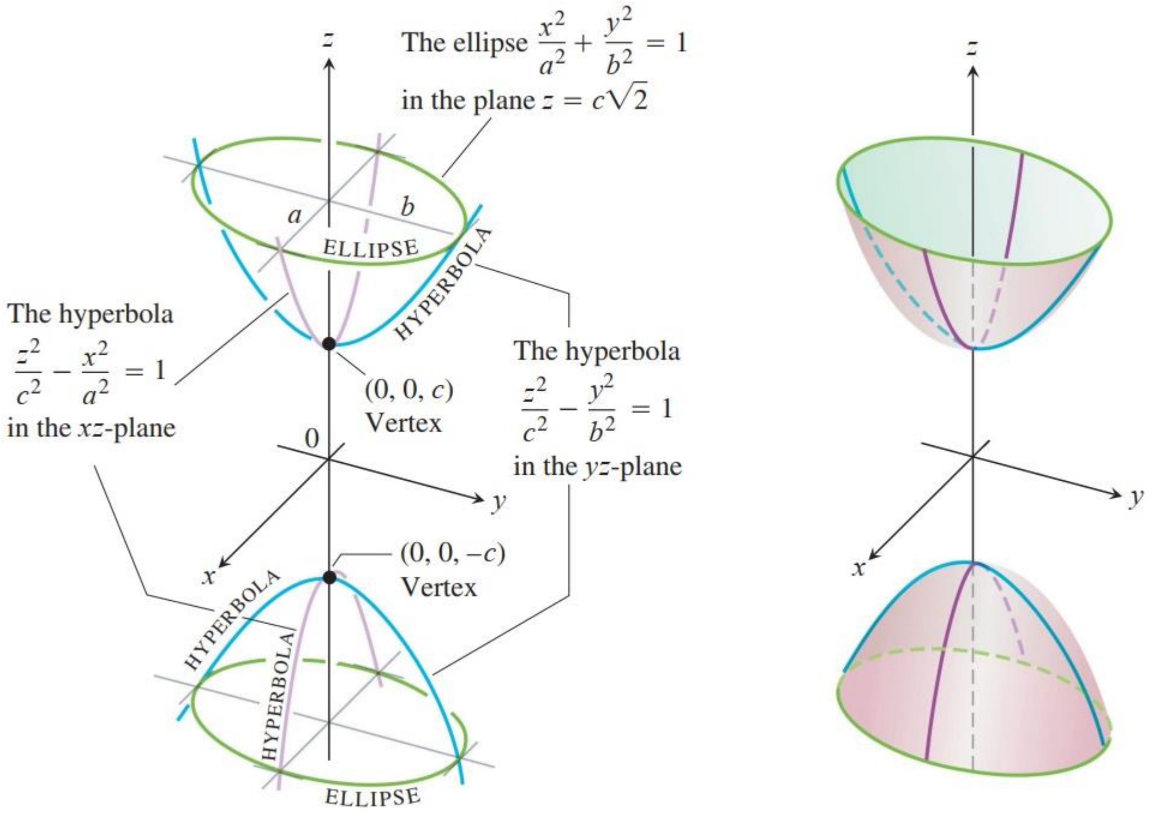


Figure 1.33 The hyperboloid of two sheets.

Planes perpendicular to the *z*-axis above and below the vertices cut it in ellipses. Vertical planes containing the *z*-axis cut it in hyperbolas.

The hyperbolic paraboloid (A Saddle Point): The hyperbolic paraboloid has symmetry with respect to the planes $x=0$ and $y=0$ (see Figure 1.34).

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}\text{ , }c>0$$

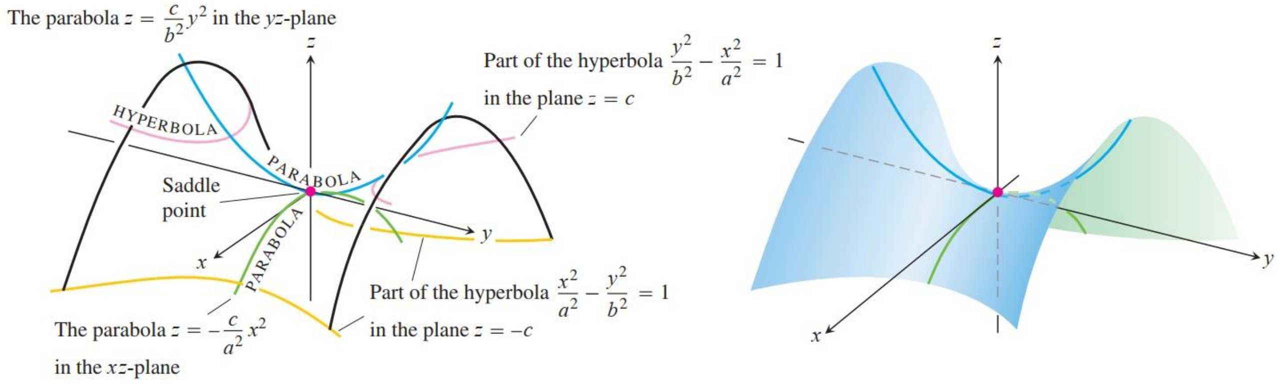


Figure 1.34 The hyperbolic paraboloid.

The sections in these planes are

$x = 0$: the parabola $z = \frac{c}{b^2}y^2$

$y = 0$: the parabola $z = -\frac{c}{a^2}x^2$

The cross-sections in planes perpendicular to the *z*-axis above and below the *xy*-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

In the plane $x = 0$, the parabola opens upward from the origin. The parabola in the plane $y = 0$ opens downward. If we cut the surface by a plane $z = z_o > 0$, the section is a hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_o}{c}$$

Near the origin, the surface is shaped like a saddle or mountain pass. To a person traveling along the surface in the *yz*-plane the origin looks like a minimum. To a person traveling in the *xz*-plane the origin looks like a maximum. Such a point is called a saddle point of a surface.