Chapter Two: Vector-Valued Functions and Motion in Space

When a body (or object) travels through space, the equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ that give the body's coordinates as functions of time serve as parametric equations for body's motion and path. With vector notation,

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

gives the body's position as a vector function of time. For example, an object moving in *xy*-plane, the component function $h(t) = 0$ for all time.

2.1 Vector Functions

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

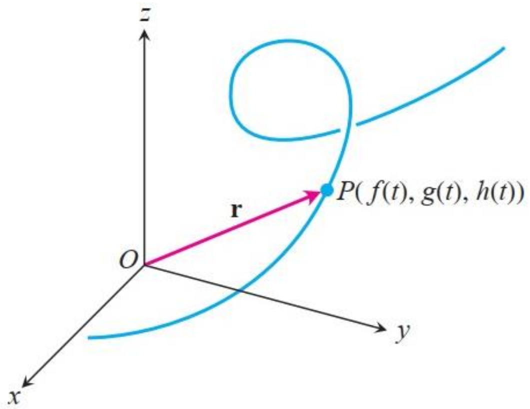

Figure 2.1 The position vector.

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$$

The point $(x, y, z) = \big(f(t), g(t), h(t)\big)$, $t \in I$, make up the curve in space that we call the particle's path. Thus, a curve in space can be represented by in vector form

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's position $P\big(f(t), g(t), h(t)\big)$ at time t is the particle's position vector (see Figure 2.1). The functions f, g, and h are the component functions of the position vector. $\mathbf{r}$ is a vector function (vector-valued function) of the real variable t.

Examples: Figure 2.2 shows the graph of the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

The curve rises as the $\mathbf{k}$-component $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder.

More helices can be seen in Figure 2.3.

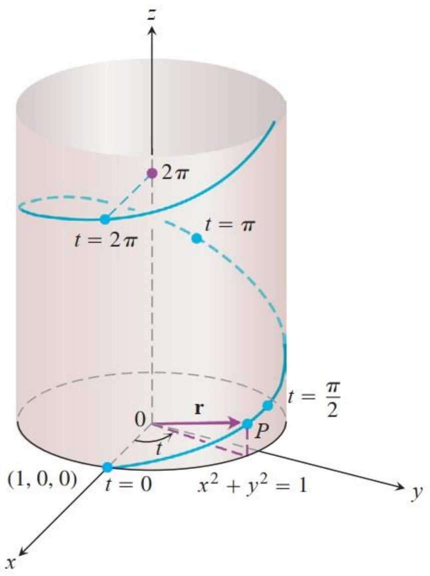

Figure 2.2 The upper half of the helix.

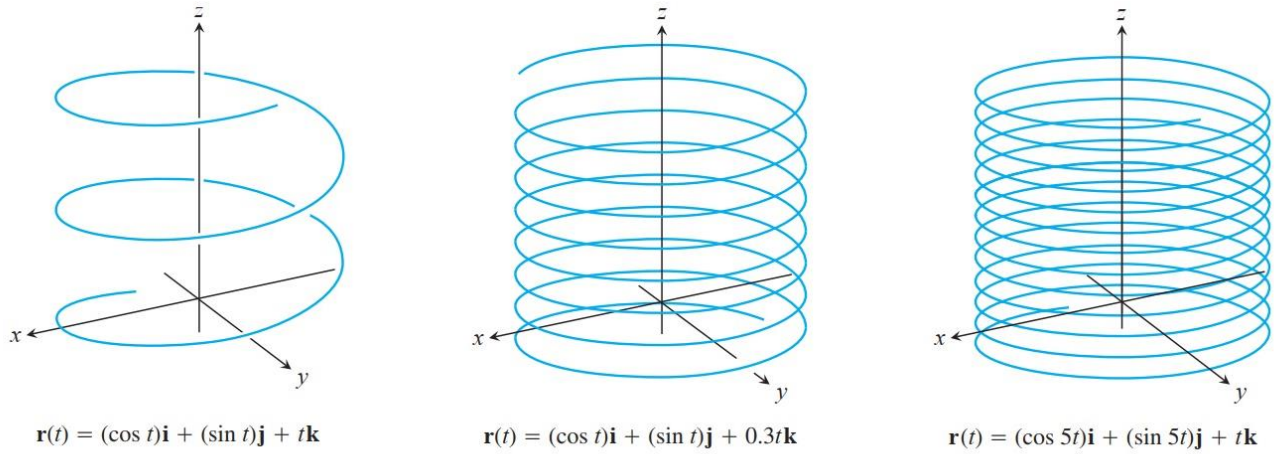

Figure 2.3 Helices.

Limits and Continuity: Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has limit $\mathbf{L}$ at t approaches t_o and write

$$\lim_{t\to t_o} \mathbf{r}(t) = \mathbf{L}\,.$$

If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then $\lim_{t\to t_o} \mathbf{r}(t) = \mathbf{L}$ precisely when

$$\lim_{t\to t_o} f(t) = L_1, \quad \lim_{t\to t_o} g(t) = L_2, \quad \text{and} \quad \lim_{t\to t_o} h(t) = L_3\,.$$

Limit $\lim_{t\to t_o} \mathbf{r}(t)$ exists if limits of all components exist. If one of limits does not exist, then $\lim_{t\to t_o} \mathbf{r}(t)$ does not exist.

$$\lim_{t\to t_o} \mathbf{r}(t) = \left(\lim_{t\to t_o} f(t)\right)\mathbf{i} + \left(\lim_{t\to t_o} g(t)\right)\mathbf{j} + \left(\lim_{t\to t_o} h(t)\right)\mathbf{k}\,.$$

Example1: Find the limit of vector function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ as t approaches $\frac{\pi}{4}$.

Solution: $\lim_{t\to\pi/4} \mathbf{r}(t) = \left(\lim_{t\to\pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}$.

Example2: Find

$$\lim_{t\to 0}\mathbf{r}(t) = \left(\frac{1}{t^2+1}\right)\mathbf{i} + (\ln(t+1))\mathbf{j} + \left(\frac{1}{t}\right)\mathbf{k}.$$

Solution: $\lim_{t\to 0}\mathbf{r}(t) = \mathbf{i} + 0\mathbf{j} + \infty\mathbf{k}$. The limit does not exist (DNE).

For continuity, a vector function $\mathbf{r}(t)$ is continuous at a point $t = t_o$ in its domain if $\lim_{t\to t_o} \mathbf{r}(t) = \mathbf{r}(t_o)$. The function is continuous if it is continuous at every point in its domain. So, $\mathbf{r}(t)$ is continuous at $t = t_o$ if and only if

$$\lim_{t\to t_o} f(t) = f(t_o), \quad \lim_{t\to t_o} g(t) = g(t_o), \text{ and } \quad \lim_{t\to t_o} h(t) = h(t_o),$$

that is, all $f, g, and\ h$ are continuous at $t = t_o$. If one of them is not continuous at $t = t_o$, then $\mathbf{r}(t)$ is not continuous at $t = t_o$.

Example3: What the value of the t in which the vector function

$\mathbf{r}(t) = (\tan t)\mathbf{i} + (\ln t)\mathbf{j} + \left(\sqrt{1-t}\right)\mathbf{k}$ is continuous.

Solution: Each component function of $\mathbf{r}(t)$ is continuous on its domain. So, let us find firstly the domain of $f(t), g(t), \text{and } h(t)$:

$$\left.\begin{aligned} D_f &= \left\{t \mid t \neq \cdots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\} \\ D_g &= \{t > 0\} \\ D_h &= \{t \leq 1\} \end{aligned}\right. , D_{\mathbf{r}} = \{0 < t \leq 1\}.$$

Derivatives and Motion: The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a derivative (is differentiable) at t if $f, g,$ and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

1. Velocity is the derivative of position: $\mathbf{v}(t) = \mathbf{v} = \frac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of the velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Example4: A particle is moving on a path having position vector $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$, find: (a) the velocity and acceleration vectors; (b) the particle's speed at any time t; (c) the time, when the particle's acceleration is orthogonal to its velocity.

Solution: (a)

$$\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (2t)\mathbf{k}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$$

(b) Speed is the magnitude of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2} = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9 + 4t^2}.$$

(c) $\mathbf{v} \cdot \mathbf{a} = 9\sin t\cos t - 9\sin t\cos t + 4t = 0 \Rightarrow t = 0.$

Differentiation Rules for Vector Functions: Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector Functions of Constant Length: If $\mathbf{r}$ is a differentiable vector function of t of constant length (see Figure 2.4), then

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0\,.$$

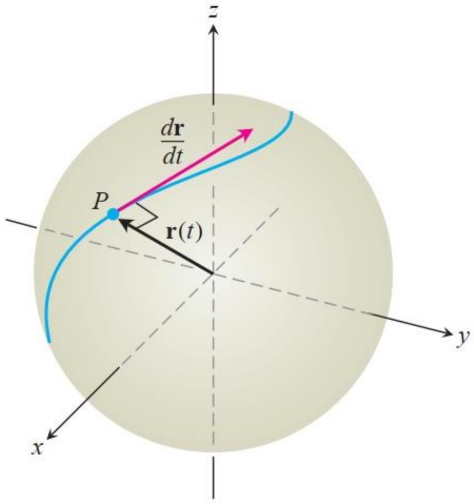

Figure 2.4.

Example5: Show that $\mathbf{r}(t)=(\sin t)\mathbf{i}+(\cos t)\mathbf{j}+\sqrt{3}\mathbf{k}$ has constant length and is orthogonal to its derivative.

Solution:

$$\mathbf{r}(t)=(\sin t)\mathbf{i}+(\cos t)\mathbf{j}+\sqrt{3}\mathbf{k}$$

$$|\mathbf{r}(t)|=\sqrt{(\sin t)^2+(\cos t)^2+\left(\sqrt{3}\right)^2}=\sqrt{1+3}=2$$

$$\frac{d\mathbf{r}}{dt}=(\cos t)\mathbf{i}-(\sin t)\mathbf{j}$$

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=\sin t\cos t-\sin t\cos t=0$$

Integrals of Vector Functions (definite and indefinite integrals)

Indefinite Integral: The indefinite integral of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int\mathbf{r}(t)dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int\mathbf{r}(t)dt=\mathbf{R}(t)+\mathbf{C}\,.$$

Example6: Find indefinite integral $\int\big((\cos t)\mathbf{i}+\mathbf{j}-(2t)\mathbf{k}\big)dt$.

Solution: $\displaystyle\int\big((\cos t)\mathbf{i}+\mathbf{j}-2t\mathbf{k}\big)dt=\left(\int\cos t\,dt\right)\mathbf{i}+\left(\int dt\right)\mathbf{j}-\left(\int 2t\,dt\right)\mathbf{k}$

$$=(\sin t+C_1)\mathbf{i}+(t+C_2)\mathbf{j}-(t^2+C_3)\mathbf{k}$$

$$=\sin t\,\mathbf{i}+t\mathbf{j}-t^2\mathbf{k}+\mathbf{C}\,,\qquad \text{where }\ \mathbf{C}=C_1\mathbf{i}+C_2\mathbf{j}+C_3\mathbf{k}\,.$$

Definite Integral: If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the definite integral of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}\,.$$

Example7: Evaluate definite integral $\int_0^\pi \big((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}\big)dt$.

Solution: $\displaystyle\int_0^\pi \big((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}\big)dt = \left(\int_0^\pi \cos t\, dt\right)\mathbf{i} + \left(\int_0^\pi dt\right)\mathbf{j} - \left(\int_0^\pi 2t\, dt\right)\mathbf{k}$

$$= [\sin t]_0^\pi\,\mathbf{i} + [t]_0^\pi\,\mathbf{j} - [t^2]_0^\pi\,\mathbf{k}$$
$$= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k}$$
$$= \pi\mathbf{j} - \pi^2\mathbf{k}\,.$$

Example8: Find the position vector function of the particle departed initially (at time $t = 0$) from point (3, 0, 0) with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Also, the acceleration vector of particle is $\mathbf{a} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$.

Solution: The goal is to find $\mathbf{r}(t)$, and we have $\mathbf{v}(0) = 3\mathbf{j}$, $\mathbf{r}(0) = 3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, and

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$$

Integrating both sides of acceleration equation with respect to t gives

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (2t)\mathbf{k} + \mathbf{C}_1\,.$$

We use $\mathbf{v}(0) = 3\mathbf{j}$ to find $\mathbf{C}_1$:

$$3\mathbf{j} = -(3\sin 0)\mathbf{i} + (3\cos 0)\mathbf{j} + (0)\mathbf{k} + \mathbf{C}_1 \;\Rightarrow\; 3\mathbf{j} = 3\mathbf{j} + \mathbf{C}_1 \;\Rightarrow\; \mathbf{C}_1 = 0\,.$$

So, the velocity vector as function of time is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (2t)\mathbf{k}$$

Again, integrating both sides of velocity equation with respect to t gives

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k} + \mathbf{C}_2.$$

We use $\mathbf{r}(0) = 3\mathbf{i}$ to find $\mathbf{C}_2$:

$$3\mathbf{i} = (3\cos 0)\mathbf{i} + (3\sin 0)\mathbf{j} + 0\mathbf{k} + \mathbf{C}_2 \;\Rightarrow\; 3\mathbf{i} = 3\mathbf{i} + +\mathbf{C}_2 \;\Rightarrow\; \mathbf{C}_2 = 0\,.$$

So, the position vector as function of time is $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$.

2.2 Arc Length and the Unit Tangent Vector $\mathbf{T}$

Arc Length Along a Space Curve: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the length of a smooth curve $\mathbf{r}(t)$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt, \qquad \text{or} \qquad L = \int_a^b |\mathbf{v}| dt.$$

where $|\mathbf{v}|$ is the length of a velocity vector $\frac{d\mathbf{r}}{dt}$.

Example1: Find the length of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = 2\pi$.

Solution:

$$L = \int_a^b |\mathbf{v}| dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2} \text{ units of length.}$$

If we choose a base point $P_o(t)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = \big(x(t), y(t), z(t)\big)$ on C and a “directed distance” $s(t)$, measured along C from the base point (Figure 2.5). We call s an arc length parameter for the curve. The Arc Length Parameter with Base Point $P(t_o)$ is

$$s(t) = \int_{t_o}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_o}^{t} |\mathbf{v}(\tau)| d\tau.$$

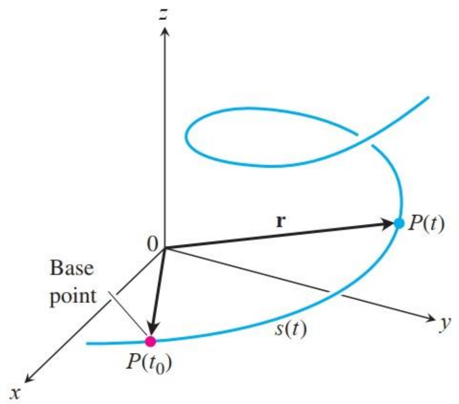

Figure 2.5.

If a curve $\mathbf{r}(t)$ is given in terms of some parameter t and $s(t)$ is the arc length function given by equation above, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}\big(t(s)\big)$.

Example2: Find arc length parameterization of helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ if $t_o = 0$.

Solution: $s(t) = \int_{t_o}^{t} |\mathbf{v}(\tau)| d\tau = \int_0^t \sqrt{(-\sin\tau)^2 + (\cos\tau)^2 + (1)^2} d\tau = \int_0^t \sqrt{2} d\tau = \sqrt{2}\, t.$

$$t = \frac{s}{\sqrt{2}}, \mathbf{r}\big(t(s)\big) = \left(\cos\frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

Speed on a Smooth Curve:

$$\frac{ds}{dt} = |\mathbf{v}(t)|\,.$$

Unit Tangent Vector T: The unit tangent vector of a smooth curve $\mathbf{r}(t)$ is

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}\,.$$

The unit tangent vector $\mathbf{T}$ is a differentiable function of t whenever $\mathbf{v}$ is a differentiable function of t (see Figure 2.6).

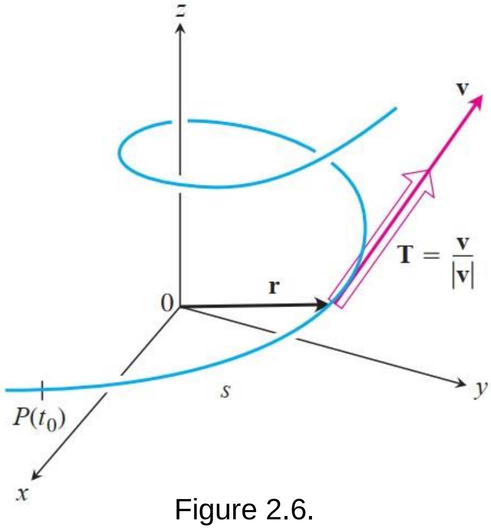

Figure 2.6.

Example3: Find the unit tangent vector of the curve $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$.

Solution: $\dfrac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$, and $|\mathbf{v}| = \sqrt{9 + 4t^2}$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9+4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9+4t^2}}\mathbf{k}\,.$$

2.3 Curvature and Unit Normal Vector N

Curvature of a Plane Curve: Since $\mathbf{T}$ is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the curvature (Figure 2.7). The symbol for the curvature function is the Greek letter κ ("kappa"). If $\mathbf{T}$ is the unit vector of a smooth curve, the curvature function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

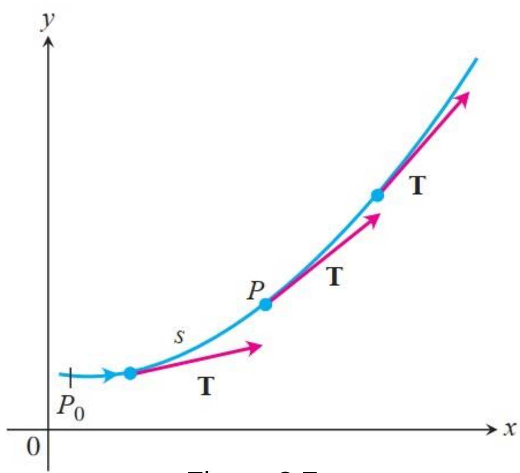

Figure 2.7.

If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can calculate the curvature as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\,\frac{dt}{ds}\right| = \frac{1}{ds/dt}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

So, the formula for calculating curvature: If $\mathbf{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is the unit tangent vector.

In Figure 2.8, on a straight line, the unit tangent vector $\mathbf{T}$ always points in the same direction, so its components are constants. Therefore,

$$\left|\frac{d\mathbf{T}}{ds}\right| = |0| = 0\,.$$

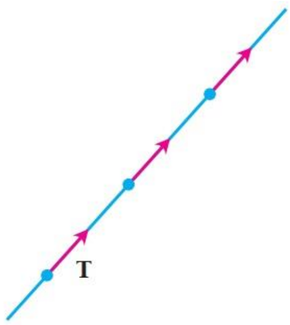

Figure 2.8.

Example1: Show that the curvature of a circle of radius a is $1/a$.

Solution: the vector function of the circle of radius a is $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}\,, |\mathbf{v}| = \sqrt{= (-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2} = a\,.$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}}{a} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}\,, \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1\,.$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}\,(1) = \frac{1}{a}.$$

Among the vectors orthogonal to the unit tangent vector $\mathbf{T}$ is one of particular significance because it points in the direction in which the curve is turning. Since $\mathbf{T}$ has constant length, the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$. Therefore, if we divide $d\mathbf{T}/ds$ by its length κ we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$ (Figure 2.9).

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

The principal normal vector $\mathbf{N}$ will point toward the concave side of the curve.

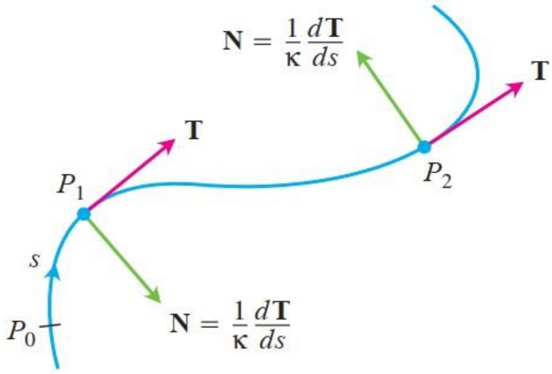

Figure 2.9.

If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s, we can use the Chain Rule to calculate $\mathbf{N}$ directly:

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)\left(\frac{dt}{ds}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|\left|\frac{dt}{ds}\right|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

This formula enables us to find $\mathbf{N}$ without having to find and κ and s first.

So, if $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

where $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is the unit tangent vector.

Example2: Find $\mathbf{T}$ and $\mathbf{N}$ for the circular motion $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}$.

Solution: We first find $\mathbf{T}$:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}\,, |\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}\,.$$

$$\frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}\,, \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}\,.$$

Notice that $\mathbf{T}\cdot\mathbf{N} = 0$, verifying that $\mathbf{N}$ is orthogonal to $\mathbf{T}$. Notice too, that for the circular motion here, $\mathbf{N}$ points from $\mathbf{r}(t)$ towards the circle's center at the origin.

Circle of Curvature for Plane Curves: The circle of curvature or osculating circle at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has);
2. has the same curvature the curve has at P;
3. lies toward the concave or inner side of the curve (as in Figure 2.10).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which, according to Example1, is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

To find ρ, we find κ and take the reciprocal. The **centre of curvature** of the curve at P is the centre of the circle of curvature.

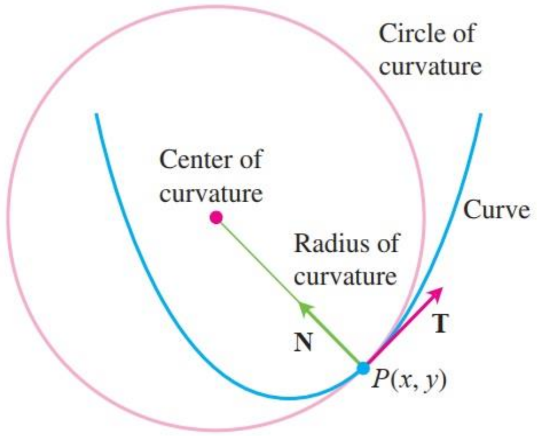

Figure 2.10.

Example3: Find and graph the osculating circle of the parabola $y = x^2$ at the origin, where the vector function of the parabola is $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$.

Solution: First we find the curvature of the parabola at the origin,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}\,, |\mathbf{v}| = \sqrt{1 + 4t^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}} = (1 + 4t^2)^{-\frac{1}{2}}\mathbf{i} + 2t(1 + 4t^2)^{-\frac{1}{2}}\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = 4t(1 + 4t^2)^{-\frac{3}{2}}\mathbf{i} + \left[2(1 + 4t^2)^{-\frac{1}{2}} - 8t(1 + 4t^2)^{-\frac{3}{2}}\right]\mathbf{j}\,.$$

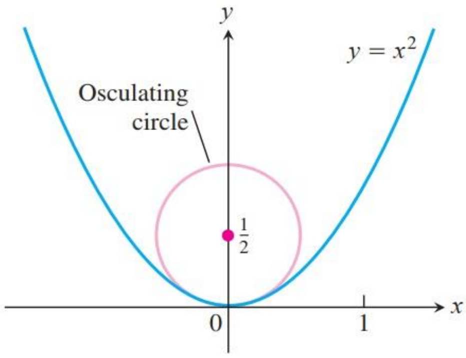

Figure 2.11.

At the origin, $t = 0$, so the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

$$\kappa(0) = \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}\,| = (1)\sqrt{0^2 + 2^2} = 2\,.$$

Therefore, the radius of curvature is $\frac{1}{\kappa} = \frac{1}{2}$ and the centre of the circle is $\left(0, \frac{1}{2}\right)$ (see Figure 2.11). The equation of the osculating circle is

$$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \quad \text{or} \quad x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

Curvature and Normal Vectors for Space Curves: If a smooth curve in space is specified by the position vector $\mathbf{r}(t)$ as a function of some parameter t, and if s is the arc length parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$. The curvature in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

just as for plane curves. The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$, and we define the **principal unit normal** to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Example4: Find the curvature and $\mathbf{N}$ for the helix:

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \qquad a, b \geq 0, \qquad a^2 + b^2 \neq 0.$$

Solution: We calculate $\mathbf{T}$ from the velocity vector $\mathbf{v}$:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}, |\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}]$$

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}],$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}]\right| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}.$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = \frac{\sqrt{a^2+b^2}}{a}\cdot\frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}] = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

2.4 Torsion and the Unit Binormal Vector $\mathbf{B}$

If you are traveling along a space curve, the Cartesian $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ coordinate system for representing the vectors describing your motion are not truly relevant to you. What is meaningful instead are the vectors representative of your forward direction (the unit tangent vector $\mathbf{T}$), the direction in which your path is turning (the unit normal vector $\mathbf{N}$), and the tendency of your motion to "twist" out of the plane created by these vectors in the direction perpendicular to this plane (defined by the **unit binormal vector** $\mathbf{B} = \mathbf{T} \times \mathbf{N}$) (see Figure 2.12).

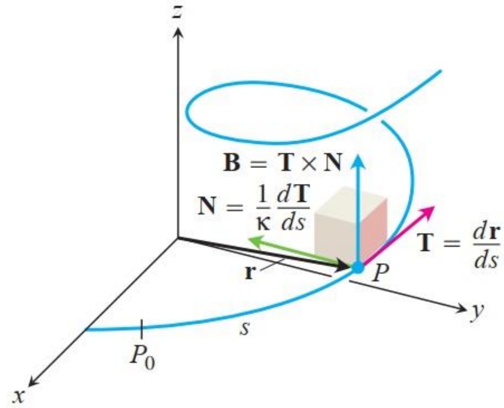

Figure 2.12.

Torsion: The binormal vector of a curve in space is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, a unit vector orthogonal to both $\mathbf{T}$ and $\mathbf{N}$ (Figure 2.13). Together $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space. It is called the Frenet ("fre-nay") frame, or the $\mathbf{TNB}$ frame.

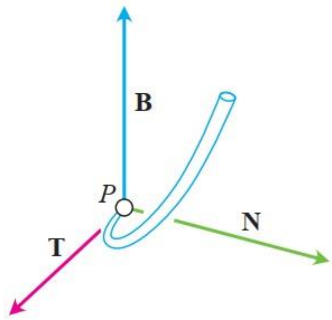

Figure 2.13.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \xRightarrow{\text{differentiate } \mathbf{B} \text{ with respect to } s} \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

Since $\mathbf{N}$ is the direction of $d\mathbf{T}/ds$, $d\mathbf{T}/ds \times \mathbf{N} = 0$ and

$$\frac{d\mathbf{B}}{ds} = 0 + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

From this we see that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ since a cross product is orthogonal to its factors. Since $d\mathbf{B}/ds$ is also orthogonal to $\mathbf{B}$ (the latter has constant length), it follows that $d\mathbf{B}/ds$ is orthogonal to the plane of $\mathbf{B}$ and $\mathbf{T}$. In other words, $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$, so $d\mathbf{B}/ds$ is a scalar multiple of $\mathbf{N}$. In symbols,

$$\frac{d\mathbf{B}}{ds} = -\tau\,\mathbf{N}\,.$$

The negative sign in this equation is traditional. The scalar is called the **torsion** along the curve. Notice that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\,\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau \xRightarrow{\text{so that}} \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Unlike the curvature κ, which is never negative, the torsion may be positive, negative, or zero.

The three planes determined by $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are named and shown in Figure 2.14. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve. Torsion measures how the curve twists.

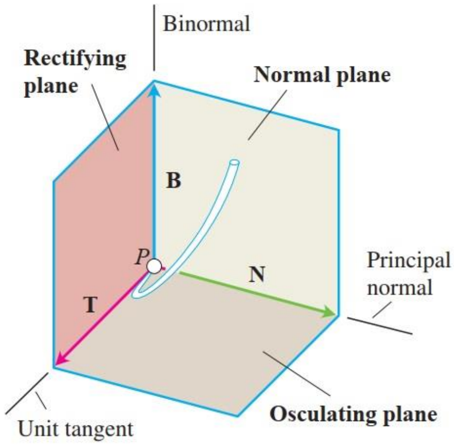

Figure 2.14.

If we think of the curve as the path of a moving body, then $|d\mathbf{T}/ds|$ tells how much the path turns to the left or right as the object moves along; it is called the **curvature** of the object's path. The number $-(d\mathbf{B}/ds) \cdot \mathbf{N}$ tells how much a body's path rotates or twists out of its plane of motion as the object moves along; it is called the **torsion** of the body's path. Look at Figure 2.15. If P is a train climbing up a curved track, the rate at which the headlight turns from side to side per unit distance is the curvature of the track. The rate at which the engine tends to twist out of the plane formed by $\mathbf{T}$ and $\mathbf{N}$ is the torsion.

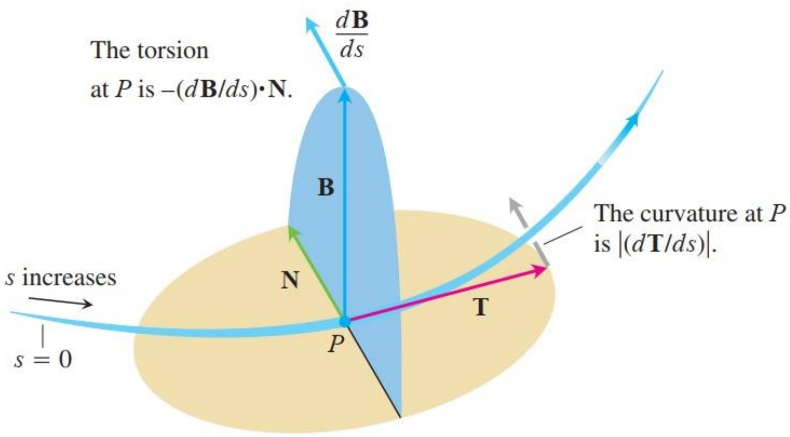

Figure 2.15.

Tangential and Normal Components of Acceleration: When a body is accelerated, we usually want to know how much of the acceleration acts in the direction of motion, in the tangential direction $\mathbf{T}$. We can calculate this using the Chain Rule to rewrite $\mathbf{v}$ as

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=\mathbf{T}\frac{ds}{dt}$$

and differentiating both sides $\qquad \boxed{\frac{d\mathbf{T}}{ds}=\kappa\mathbf{N}}$

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}\,\mathbf{T}+\frac{ds}{dt}\frac{d\mathbf{T}}{dt}=\frac{d^2s}{dt^2}\,\mathbf{T}+\frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}\,\mathbf{T}+\frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right)$$

$$=\frac{d^2s}{dt^2}\,\mathbf{T}+\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\,.$$

So, the **Tangential** and **Normal** Components of Acceleration

$$\mathbf{a}=a_{\mathrm{T}}\mathbf{T}+a_{\mathrm{N}}\mathbf{N}\,,$$

where

$$a_{\mathrm{T}}=\frac{d^2s}{dt^2}=\frac{d}{dt}|\mathbf{v}| \qquad \text{and} \qquad a_{\mathrm{N}}=\kappa\left(\frac{ds}{dt}\right)^2=\kappa|\mathbf{v}|^2$$

are the tangential and normal scalar components of acceleration.

Notice that the binormal vector $\mathbf{B}$ does not appear in the equation above. No matter how the path of the moving body we are watching may appear to twist and turn in space, the acceleration $\mathbf{a}$ always lies in the plane of $\mathbf{T}$ and $\mathbf{N}$ orthogonal to $\mathbf{B}$. The equation also tells us exactly how much of the acceleration takes place tangent to the motion (d^2s/dt^2) and how much takes place normal to the motion $[\kappa(ds/dt)^2]$ (Figure 2.16). $\mathbf{a}$ is the rate of change of velocity $\mathbf{v}$, and in general, both the length and direction of $\mathbf{v}$ change as a body moves along its path. The tangential component of acceleration a_{T} measures the rate of change of the length of $\mathbf{v}$ (that is, the change in the speed). The normal component of acceleration a_{N} measures the rate of change of the direction of $\mathbf{v}$.

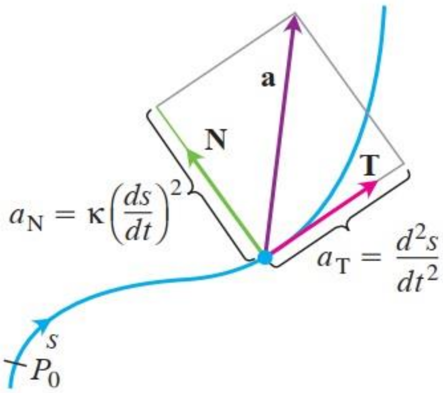

Figure 2.16.

If a body moves in a circle at a constant speed, d^2s/dt^2 is zero and all the acceleration points along $\mathbf{N}$ toward the circle's centre. If the body is speeding up or slowing down, $\mathbf{a}$ has a nonzero tangential component (Figure 2.17). To calculate a_{N} we usually use the formula $a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2}$, which comes from solving the equation $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_{\mathrm{T}}{}^2 + a_{\mathrm{N}}{}^2$ for a_{N}. With this formula, we can find a_{N} without having to calculate κ first.

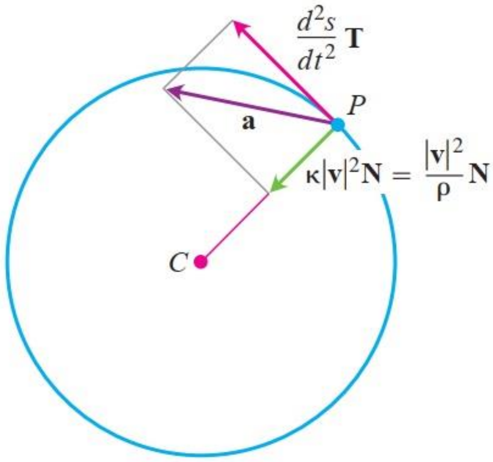

Figure 2.17.

So, the Formula for Calculating the Normal Component of Acceleration is

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2}\,.$$

Example1: Without finding $\mathbf{T}$ and $\mathbf{N}$, write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}\,, \qquad t > 0$$

in the form of tangential and normal components.

Solution: First, we find a_{T}:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t\,, \qquad t > 0$$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1$$

To find a_{N}:

$$\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$$

$$|\mathbf{a}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{t^2 + 1}$$

$$|\mathbf{a}|^2 = t^2 + 1$$

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2} = \sqrt{(t^2+1) - 1} = \sqrt{t^2} = t$$

Then, $\quad \mathbf{a} = a_{\mathrm{T}} + a_{\mathrm{N}} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}\,.$

Formulas for Computing Curvature and Torsion: We now give some easy-to-use formulas for computing the curvature and torsion of a smooth curve.

$$\mathbf{v}\times\mathbf{a}=\left(\frac{ds}{dt}\mathbf{T}\right)\times\left[\frac{d^2s}{dt^2}\,\mathbf{T}+\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right]=\left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T}\times\mathbf{T})+\kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T}\times\mathbf{N})$$

$$\boxed{\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{ds}{dt}\mathbf{T}}$$

$$=\kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}\,.$$

$$\boxed{\mathbf{T}\times\mathbf{T}=0 \quad \text{and} \quad \mathbf{T}\times\mathbf{N}=\mathbf{B}}$$

It follows that

$$|\mathbf{v}\times\mathbf{a}|=\kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}|=\kappa|\mathbf{v}|^3\,.$$

$$\boxed{\frac{ds}{dt}=|\mathbf{v}| \quad \text{and} \quad |\mathbf{B}|=1}$$

Solving for κ gives the following formula: Vector Formula for Curvature is

$$\kappa=\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}\,.$$

This equation calculates the curvature, a geometric property of the curve, from the velocity and acceleration of any vector representation of the curve in which $|\mathbf{v}|$ is different from zero. From any formula for motion along a curve, no matter how variable the motion may be (as long as $\mathbf{v}$ is never zero), we can calculate a physical property of the curve that seems to have nothing to do with the way the curve is traversed.

The most widely used formula for torsion, derived in more advanced texts, is

$$\tau=\frac{\begin{vmatrix}\dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\\ \dddot{x} & \dddot{y} & \dddot{z}\end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} \qquad (\text{if } \mathbf{v}\times\mathbf{a}\neq 0)$$

This formula calculates the torsion directly from the derivatives of the component functions $x=f(t), y=g(t), z=h(t)$ that make up $\mathbf{r}$. The determinant's first row comes from $\mathbf{v}$, the second row comes from $\mathbf{a}$, and the third row comes from $\dot{\mathbf{a}}=d\mathbf{a}/dt$.

Example2: Use vector formula for curvature and torsion formula to find κ and τ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k} \ .$$

Solution: First, we calculate the curvature:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + (b)\mathbf{k}$$

$$\mathbf{a} = -(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ab\sin t)\mathbf{i} - (ab\cos t)\mathbf{j} + a^2\mathbf{k}$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2+b^2)^{3/2}} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2} .$$

Notice that the result agrees with in **Example4** (section 2.3, page 12), where we calculated the curvature directly from its definition.

To evaluate torsion, we find the entries in the determinant by differentiating $\mathbf{r}$ with respect to t. We already have $\mathbf{v}$ and $\mathbf{a}$, and

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a\sin t)\mathbf{i} - (a\cos t)\mathbf{j} \ .$$

Hence,

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2} = \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2} .$$

From this last equation we see that the torsion of a helix about a circular cylinder is constant. In fact, constant curvature and constant torsion characterise the helix among all curves in space.

Formulas for Curves in Space:

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration:

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2}$$

Solved Problems:

2.1 Vector Functions

Prob1: Find the angle between the velocity and acceleration vectors at time $t = 0$ for position vector:

$\mathbf{r}(t) = (\ln(t^2+1))\mathbf{i} + (\tan^{-1} t)\mathbf{j} + \sqrt{t^2+1}\mathbf{k}.$

Solution:

$$\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-\frac{1}{2}}\mathbf{k} \;\Rightarrow\; \mathbf{v}(0) = \mathbf{j} \;\Rightarrow\; |\mathbf{v}(0)| = 1$$

$$\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{\frac{3}{2}}}\right]\mathbf{k} \;\Rightarrow\; \mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \quad |\mathbf{a}(0)| = \sqrt{(2)^2+1} = \sqrt{5}$$

$$\theta = \cos^{-1}\left[\frac{\mathbf{v}(0)\cdot\mathbf{a}(0)}{|\mathbf{v}(0)||\mathbf{a}(0)|}\right] = \cos^{-1} 0 = \frac{\pi}{2}.$$

Prob2: Find parametric equations for the line that is tangent to the given curve at $t = 0$ $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}.$

Solution:

$$\mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k} \quad \overset{t=0}{\Longrightarrow} \quad \mathbf{v}(0) = \mathbf{i} + \mathbf{k}$$

$$\mathbf{r}(0) = -\mathbf{j} + \mathbf{k} \Rightarrow P_o(0, -1, 1)$$

$$\left.\begin{array}{l} x = t \\ y = -1 \\ z = 1 + t \end{array}\right\} \text{ are the parametric equations of the tangent line.}$$

Prob3: A particle is moving along the path of the unit circle with position vector $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $t \geq 0$, answer the following:

(a) Does the particle have constant speed? If so, what is its constant speed?

(b) Is the particle's acceleration vector always orthogonal to its velocity vector?

(c) Does the particle move clockwise or counterclockwise around the circle?

(d) Does the particle begin at the point (1, 0)?

Solution: $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \;\Rightarrow\; \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$

(a) $|\mathbf{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1 \;\Rightarrow$ constant speed.

(b) $\mathbf{v} \cdot \mathbf{a} = \sin t \cos t - \sin t \cos t = 0 \;\Rightarrow$ yes, orthogonal.

(c) Clockwise movement.

(d) Yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$.

2.2 Arc Length and the Unit Tangent Vector T

Prob1: Find the length of the curve $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, -\ln 4 \le t \le 0$.

Solution: $\mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$$|\mathbf{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} = \sqrt{3e^{2t}} = \sqrt{3}e^t$$

$$L = \int_a^b |\mathbf{v}| dt = \int_{-\ln 4}^{0} \sqrt{3}e^t dt = \sqrt{3}e^t\Big|_{-\ln 4}^{0} = \sqrt{3} - \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} \text{ unit length.}$$

2.3 Curvature and Unit Normal Vector N

Prob1: Find $\mathbf{T}$, $\mathbf{N}$, and κ for the space curve $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

Solution: $\mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \;\Rightarrow\; |\mathbf{v}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2}\cosh t$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k}}{\sqrt{2}\cosh t} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\right)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$$

$$\frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\mathbf{k} \;\Rightarrow\; \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)^2 + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)^2}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t \tanh^2 t} = \frac{1}{\sqrt{2}}\operatorname{sech} t$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\mathbf{k}}{\frac{1}{\sqrt{2}}\operatorname{sech} t} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2}\cosh t}\,\frac{1}{\sqrt{2}}\operatorname{sech} t = \frac{1}{2}\operatorname{sech}^2 t\,.$$

Prob2: Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = t\mathbf{i} + (\sin t)\mathbf{j}$ at the point $\left(\frac{\pi}{2}, 1\right)$. (The curve parametrises the graph of $y = \sin x$ in the xy-plane.)

Solution: $\mathbf{v} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+\cos^2 t} \Rightarrow \left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = 1$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + (\cos t)\mathbf{j}}{\sqrt{1+\cos^2 t}} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{\sin t \cos t}{(1+\cos^2 t)^{\frac{3}{2}}}\mathbf{i} + \frac{-\sin t}{(1+\cos^2 t)^{\frac{3}{2}}}\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{\sin^2 t\cos^2 t}{(1+\cos^2 t)^3} + \frac{\sin^2 t}{(1+\cos^2 t)^3}} = \frac{|\sin t|}{1+\cos^2 t} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|_{t=\frac{\pi}{2}} = 1$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| \Rightarrow \kappa\left(\frac{\pi}{2}\right) = 1 \Rightarrow \rho = \frac{1}{\kappa} = 1 \text{ and the centre is } \left(\frac{\pi}{2}, 0\right) \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1\,.$$

2.4 Torsion and the Unit Binormal Vector B

Prob1: Find $\mathbf{T}$, $\mathbf{N}$, κ, $\mathbf{B}$ and τ for the space curve $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$.

Solution: $\mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2}\cosh t$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k}}{\sqrt{2}\cosh t} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\right)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$$

$$\frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)^2 + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)^2}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t\tanh^2 t} = \frac{1}{\sqrt{2}}\operatorname{sech} t$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)\mathbf{k}}{\frac{1}{\sqrt{2}}\operatorname{sech} t} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2}\cosh t}\,\frac{1}{\sqrt{2}}\operatorname{sech} t = \frac{1}{2}\operatorname{sech}^2 t\,.$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\tanh t & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \left(-\frac{1}{\sqrt{2}}\tanh^2 t - \frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$$

$$= \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$$

$$\mathbf{a} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \quad \Rightarrow \quad \dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j}$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix} = (\sinh t)\mathbf{i} - (-\cosh t)\mathbf{j} + (-\sinh^2 t + \cosh^2 t)\mathbf{k}$$

$$= (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + \mathbf{k} \quad \Rightarrow \quad |\mathbf{v}\times\mathbf{a}|^2 = \sinh^2 t + \cosh^2 t + 1$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{2\cosh^2 t}.$$