Chapter Three: Partial Derivatives

In this chapter, we will discuss the functions of several variables and the derivates of functions of several variables.

3.1 Functions of Several Variables

Function of n Independent Variables: Suppose D is a set of ordered pairs of real numbers $(x_1, x_2, \dots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** $x_1$to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

If f is a function of two independent variables, we usually call the independent variables x and y and picture the domain of f as a region in the xy-plane. If f is a function of three independent variables, we call the variables x, y, and z and picture the domain as a region in space.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable.

Example1: Evaluate the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$.

Solution: $f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5$.

Domains and Ranges: In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}, y$ cannot be less than x^2. If $f(x, y) = 1/xy$, xy cannot be zero. The domain of a function is assumed to be the largest set for which the defining rule generates real numbers. The range consists of the set of output values for the dependent variable.

Example2: (a) Functions of two variables

Function	Domain	Range
$w=\sqrt{y-x^2}$	$y\geq x^2$	$[0,\infty)$
$w=\dfrac{1}{xy}$	$xy\neq 0$	$(-\infty,0)\cup(0,\infty)$
$w=\sin xy$	Entire plane	$[-1,1]$

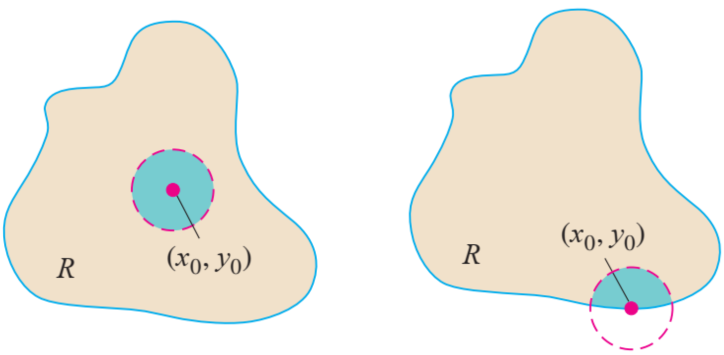
(b) Functions of three variables

Function	Domain	Range
$w=\sqrt{x^2+y^2+z^2}$	Entire space	$[0,\infty)$
$w=\dfrac{1}{x^2+y^2+z^2}$	$(x,y,z)\neq 0$	$(0,\infty)$
$w=xy\ln z$	Half-space $z>0$	$(-\infty,\infty)$

Functions of two variables: Regions in the plane can have interior points and boundary points just like intervals on the real line. Closed intervals $[a,b]$ include their boundary points, open intervals (a,b) do not include their boundary points, and intervals such as $[a,b)$ are neither open nor closed.

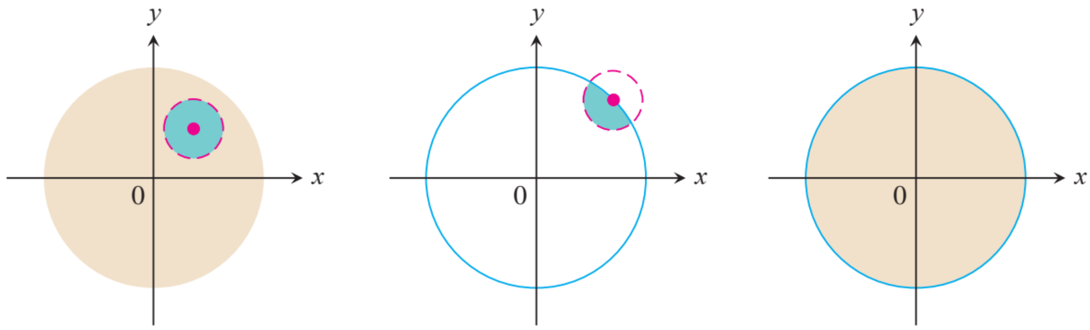
A point (x_o,y_o) in a region (set) R in the xy-plane is an **interior point** of R if it is the centre of a disk of positive radius that lies entirely in R (Figure 3.1). A point (x_o,y_o) is a **boundary point** of R if every disk centred at (x_o,y_o) contains points that lie outside of R as well as points that lie in R.

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 3.2).



(a) Interior point. (b) Boundary point.

Figure 3.1.



$\{(x,y) \mid x^2 + y^2 < 1\}$
Open unit disk. Every point an interior point.

$\{(x,y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x,y) \mid x^2 + y^2 \le 1\}$
Closed unit disk. Contains all boundary points.

Figure 3.2.

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Examples of bounded sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of unbounded sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

Example3: Describe the domain of the function $f(x,y) = \sqrt{y - x^2}$.

Solution: Since f is defined only where $y - x^2 \ge 0$, the domain is the closed, unbounded region shown in Figure 3.3. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior.

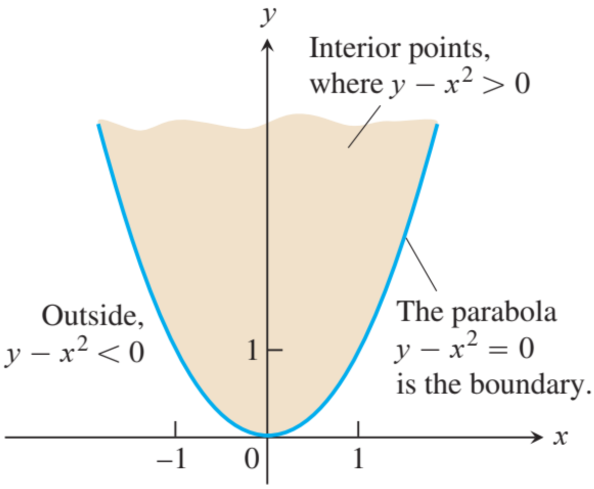


Figure 3.3.

Graphs, Level Curves, and Contours of Functions of Two Variables: There are two ways to picture the values of a function $f(x,y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x,y)$ in space.

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called a **level curve** of f. The set of all points $(x,y,f(x,y))$ in space, for (x,y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z=f(x,y)$.

Example4: Graph $f(x,y)=100-x^2-y^2$ and plot the level curves $f(x,y)=0, f(x,y)=51$ and $f(x,y)=75$ in the domain of f in the plane.

Solution: The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z=100-x^2-y^2$, a portion of which is shown in Figure 3.4.

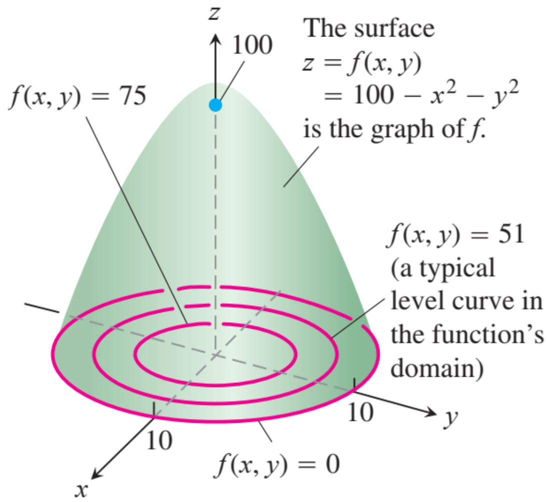


Figure 3.4.

The level curve $f(x,y)=0$ is the set of points in the xy-plane at which

$$f(x,y)=100-x^2-y^2=0, \text{ or } x^2+y^2=100,$$

which is the circle of radius 10 centred at the origin. Similarly, the level curves $f(x,y)=51$ and $f(x,y)=75$ (Figure 3.4) are the circles

$$f(x,y)=100-x^2-y^2=51, \text{or } x^2+y^2=49,$$
$$f(x,y)=100-x^2-y^2=75, \text{or } x^2+y^2=25,$$

The curve in space in which the plane $z=c$ cuts a surface $z=f(x,y)$ is made up of the points that represent the function value $f(x,y)=c$. It is called the **contour curve** in the domain of f. Figure 3.5 shows the contour curve $f(x,y)=75$ on the surface $z=100-x^2-y^2$ defined by the function $f(x,y)=100-x^2-y^2$. The contour curve lies directly above the circle $x^2+y^2=25$ which is the level curve $f(x,y)=75$ in the function's domain.

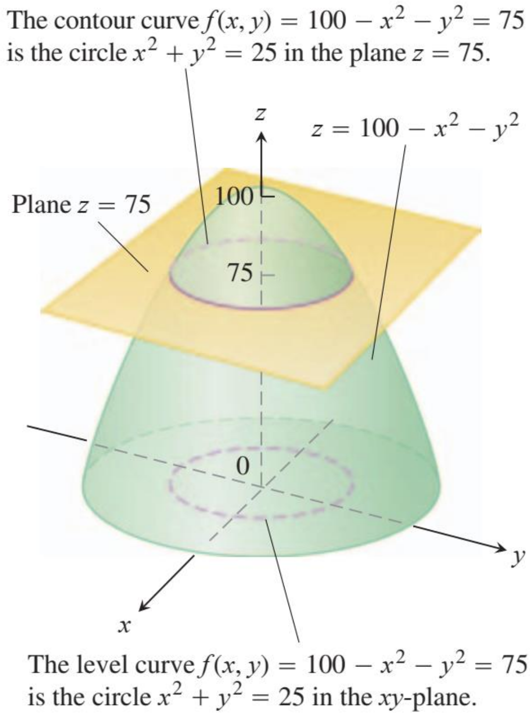


Figure 3.5.

Functions of Three Variables: In the plane, the points where a function of two independent variables has a constant value $f(x, y) = c$ make a curve in the function's domain. In space, the points where a function of three independent variables has a constant value $f(x, y, z) = c$ make a surface in the function's domain.

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in a four-dimensional space, we cannot sketch them effectively in our three-dimensional frame of reference. We can see how the function behaves, however, by looking at its three-dimensional level surfaces.

Example5: Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Solution: The value of f is the distance from the origin to the point (x, y, z). Each level surface $\sqrt{x^2 + y^2 + z^2} = c,\ c > 0$ is a sphere of radius c centred at the origin. Figure 3.6 shows a cutaway view of three of these spheres. The level surface $\sqrt{x^2 + y^2 + z^2} = 0$ consists of the origin alone.

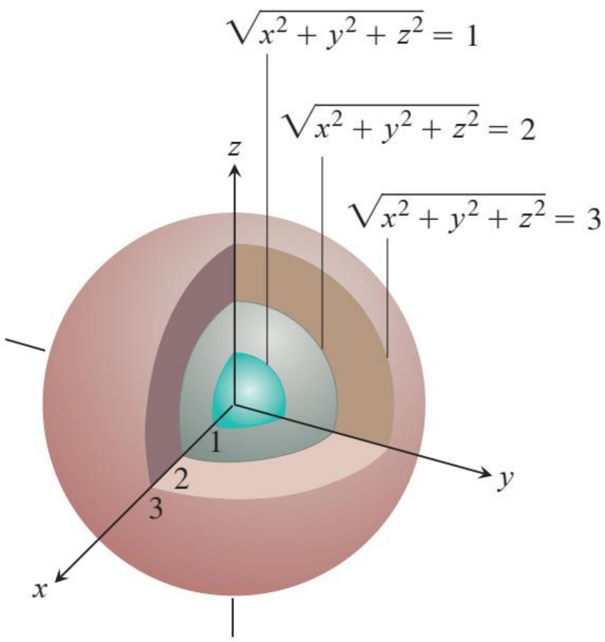


Figure 3.6.

We are not graphing the function here; we are looking at level surfaces in the function's domain. The level surfaces show how the function's values change as we move through its domain. If we remain on a sphere of radius c centred at the origin, the function maintains a constant value c. If we move from one sphere to another, the function's value changes. It increases if we move away from the origin and decreases if we move toward the origin.

The definitions of interior, boundary, open, closed, bounded, and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid balls of positive radius instead of disks.

A point in a region R in space is an **interior point** of R if it is the centre of a solid ball that lies entirely in R (Figure 3.7(a)). A point is a **boundary point** of R if every sphere centred at encloses points that lie outside of R as well as points that lie inside R (Figure 3.7(b)). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

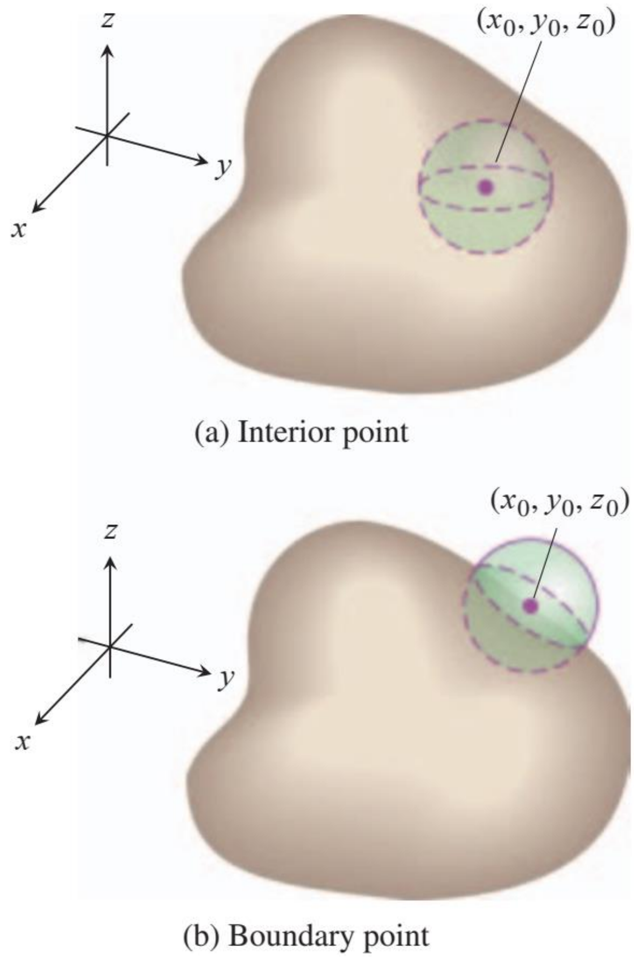


Figure 3.7.

Examples of open sets in space include the interior of a sphere, the open half-space $z > 0$, the first octant (where x, y, and z are all positive), and space itself.

Examples of closed sets in space include lines, planes, the closed half-space $z \geq 0$ the first octant together with its bounding planes, and space itself (since it has no boundary points).

A solid sphere with part of its boundary removed or a solid cube with a missing face, edge, or corner point would be neither open nor closed.

Functions of more than three independent variables are also important. For example, the temperature on a surface in space may depend not only on the location of the point $P(x, y, z)$ on the surface, but also on time t when it is visited, so we would write

$$T = f(x, y, z, t) .$$

3.2 Limits and Continuity in Higher Dimensions

Limits: We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_o, y_o) and write

$$\lim_{(x,y)\to(x_o,y_o)} f(x, y) = L$$

if, for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta.$$

Properties of Limits of Functions of Two Variables:

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} (kf(x, y)) = kL$ (any number k)
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$ $M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Example1: Calculate the limits:

(a) $\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$; (b) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$

Solution: (a) $\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$

(b) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5.$

Example2: Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y}$$

$$\lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0(\sqrt{0} + \sqrt{0}) = 0.$$

Continuity: A function $f(x,y)$ is **continuous at the point** (x_o, y_o) if

1. f defined at (x_o, y_o),
2. $\lim_{(x,y)\to(x_o,y_0)} f(x,y)$ exists,
3. $\lim_{(x,y)\to(x_o,y_0)} f(x,y) = f(x_0, y_o)$.

A function is **continuous** if it is continuous at every point of its domain.

Example3: Find the points at which the function is continuous

$$f(x,y) = \frac{2x^2 + y}{\sin(xy)}.$$

Solution: $f(x,y)$ is continuous when $\sin(xy) \neq 0$, that is, $xy \neq n\pi,\ n = 0, \pm1, \pm2, \dots$.

Functions of More Than Two Variables: The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \quad \text{and} \quad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P\to(1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

3.3 Partial Derivatives

Partial Derivatives of a Function of Two Variables: The partial derivative of $f(x, y)$ with respect to x at the point (x_o, y_o) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} = \left.\frac{d}{dx} f(x, y_o)\right|_{x=x_o} = \lim_{h\to 0} \frac{f(x_o + h, y_o) - f(x_o, y_o)}{h},$$

provided the limit exists.

The notation for a partial derivative of the surface $z = f(x, y)$:

$$f_x, \frac{\partial f}{\partial x}, z_x, \text{or } \frac{\partial z}{\partial x}$$

"Partial derivative of f (or z) with respect to x."

The partial derivative of $f(x, y)$ with respect to y at the point (x_o, y_o) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} = \left.\frac{d}{dy} f(x_o, y)\right|_{y=y_o} = \lim_{h\to 0} \frac{f(x_o, y_o + h) - f(x_o, y_o)}{h},$$

provided the limit exists.

The notation for a partial derivative of the surface $z = f(x, y)$:

$$f_y, \frac{\partial f}{\partial y}, z_y, \text{or } \frac{\partial z}{\partial y}$$

"Partial derivative of f (or z) with respect to y."

Calculation: The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant and with respect to y in the usual way while treating x as constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_o, y_o).

Example1: Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1\,.$$

Solution: To find $\partial f/\partial x$ we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0 = 2x + 3y\,.$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4) + 3(-5) = -7\,.$$

To find $\partial f/\partial y$ we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+3xy+y-1)=0+3x+1-0=3x+1\,.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)}=3(4)+1=13\,.$$

Example2: Find $\partial f/\partial y$ if $f(x,y)=y\sin xy$.

Solution: We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(y\sin xy)=y\frac{\partial}{\partial y}\sin xy+(\sin xy)\frac{\partial}{\partial y}(y)=(y\cos xy)\frac{\partial}{\partial y}(xy)+\sin xy$$

$$=xy\cos xy+\sin xy\,.$$

Example3: Find f_x and f_y if

$$f(x,y)=\frac{2y}{y+\cos x}.$$

Solution: We treat f as a quotient. With y held constant, we get

$$f_x=\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}$$

$$=\frac{2y\sin x}{(y+\cos x)^2}.$$

With x held constant, we get

$$f_y=\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial y}(2y)-2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}$$

$$=\frac{2\cos x}{(y+\cos x)^2}.$$

Implicit Partial Differentiation: Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

Example4: Find $\partial z/\partial x$ if the equation $yz-\ln z=x+y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution: We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

Example5: The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 3.8).

Solution: The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = 2y|_{(1,2)}$$

$$= 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = 2y|_{y=2} = 4.$$

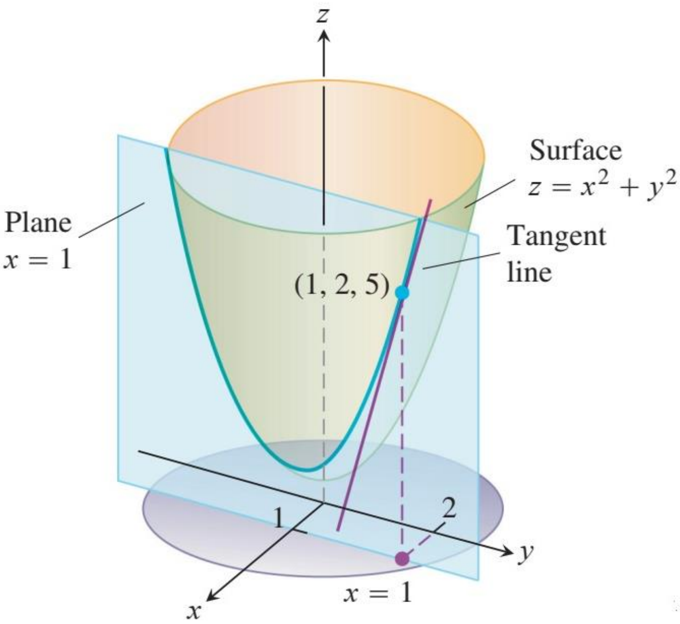


Figure 3.8.

Functions of More Than Two Variables: The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example6: If $x, y,$ and z are independent variables and $f(x, y, z) = x\sin(y + 3z)$, find $\partial f/\partial z$.

Solution:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x\sin(y + 3z)] = x\frac{\partial}{\partial z}\sin(y + 3z) = x\cos(y + 3z)\frac{\partial}{\partial z}(y + 3z) = 3x\cos(y + 3z)\,.$$

Second-Order Partial Derivatives: When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

$\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

$\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}

$\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on.

Notice the order in which the derivatives are taken:

$\frac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

Example7: If $f(x, y) = x\cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y \partial x},\ \frac{\partial^2 f}{\partial y^2}, \text{and } \frac{\partial^2 f}{\partial x \partial y}.$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x$$

So

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x\,.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

So

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y\,.$$

The Mixed Derivative Theorem: If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)\,.$$

Example8: Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution: The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. If we postpone the differentiation with respect to y and differentiate first with respect to x, however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad and \quad \frac{\partial^2 w}{\partial y \partial x} = 1\,.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x \partial y = 1$ as well.

Partial Derivatives of Still Higher Order:

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \quad , \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}\,.$$

Example9: Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2 z + x^2 y$.

Solution: We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4\,.$

3.4 The Chain Rule

The Chain Rule for functions of a single variable said that when $w = f(x)$ was a differentiable function of x and $x = f(t)$ was a differentiable function of t, w became a differentiable function of t and dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For functions of two or more variables the Chain Rule has several forms. The form depends on how many variables are involved but works like the Chain Rule in single variable functions once we account for the presence of additional variables.

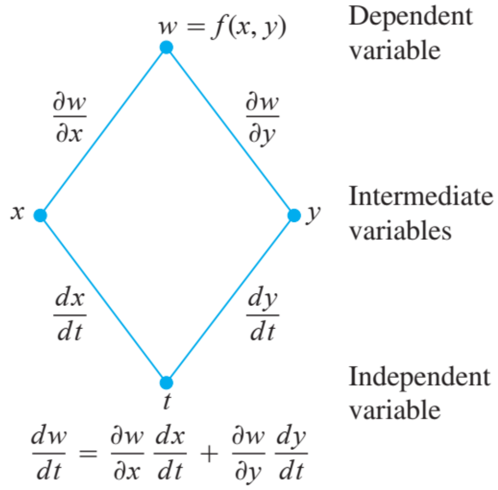
Functions of Two Variables: The Chain Rule formula for a function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

Chain Rule Theorem for Functions of Two Independent Variables: If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite $w = f\big(x(t), y(t)\big)$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

The **tree diagram** provides a convenient way to remember the Chain Rule. To remember the Chain Rule picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



Example1: Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution: We apply the Chain Rule to find dw/dt as follows:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) = (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t .$$

In this example, we can check the result with a more direct calculation. As a function of t,

$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t$,

so

$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2} \cdot 2\cos 2t = \cos 2t.$

In either case, at the given value of t,

$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1$.

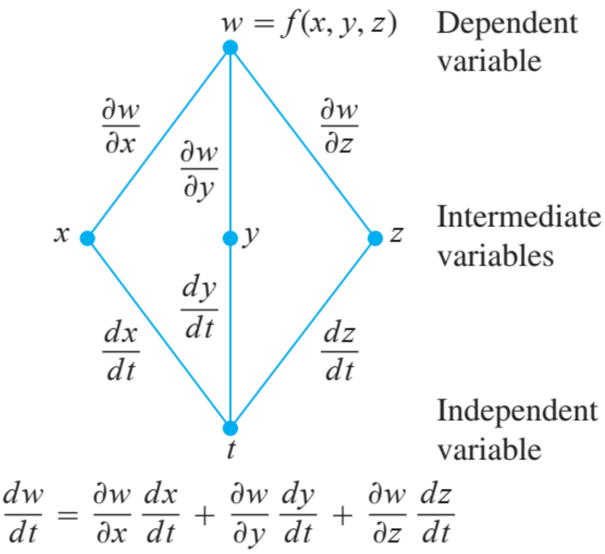
Functions of Three Variables: it only involves adding the expected third term to the two-variable formula.

Chain Rule Theorem for Functions of Three Independent Variables: If $w = f(x, y, z)$ is differentiable and $x, y,$ and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

The diagram we use for remembering the new equation is similar to the previous one, with three routes from w to t. Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



Example2: Find dw/dt if

$w = xy + z, \;\; x = \cos t \,, \;\; y = \sin t \;, \;\; z = t$.

What is the derivative's value at $t = 0$?

Solution:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1) = (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t\,.$$

$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2$.

Functions Defined on Surfaces: See the following theorem.

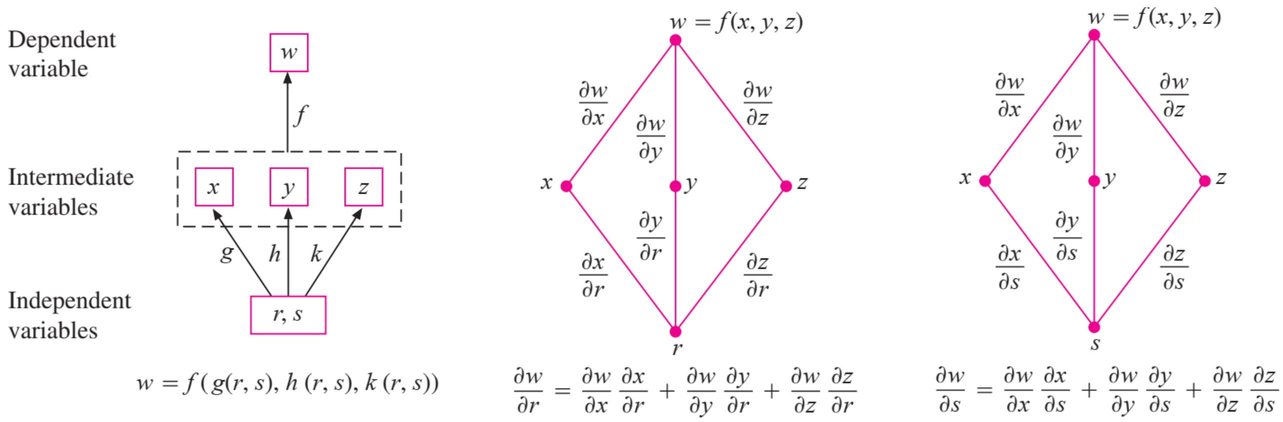
Chain Rule Theorem for Two Independent Variables and Three Intermediate Variables:

Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s),$ and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The first of these equations can be derived from the Chain Rule in previous Theorem by holding s fixed and treating r as t. The second can be derived in the same way, holding r fixed and treating s as t. The tree diagrams for both equations are shown in Figure below.



Example3: Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.$$

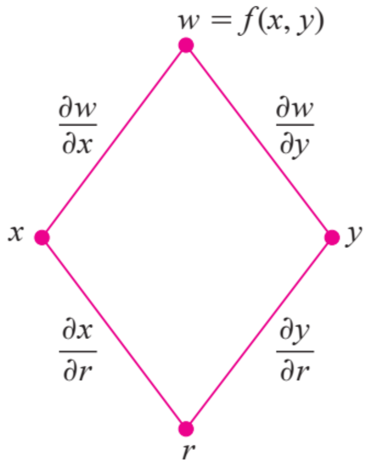
If f is a function of two variables instead of three, each equation in previous Theorem becomes correspondingly one term shorter.

If $w = f(x,y),\ x = g(r,s)$, and $y = h(r,s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

The tree diagram for the first of these equations. The diagram for the second equation is similar; just replace r with s.

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

Example4: Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s\,.$$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1)$$

$$= 2(r - s) + 2(r + s) = 4r\,.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x)(-1) + (2y)(1)$$

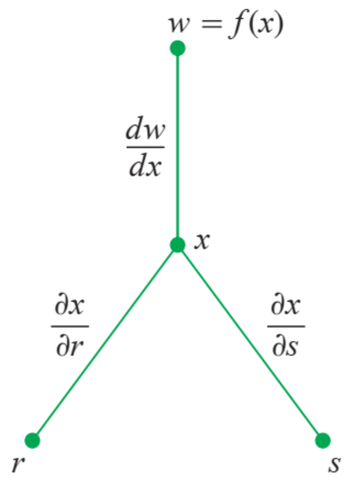
$$= -2(r - s) + 2(r + s) = 4s\,.$$

If f is a function of x alone, our equations become even simpler.

If $w = f(x)$ and $x = g(r,s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

In this case, we can use the ordinary (single-variable) derivative, dw/dx as you can see in the tree diagram.

Implicit Differentiation Revisited: The two-variable Chain Rule leads to a formula that takes most of the work out of implicit differentiation. Suppose that

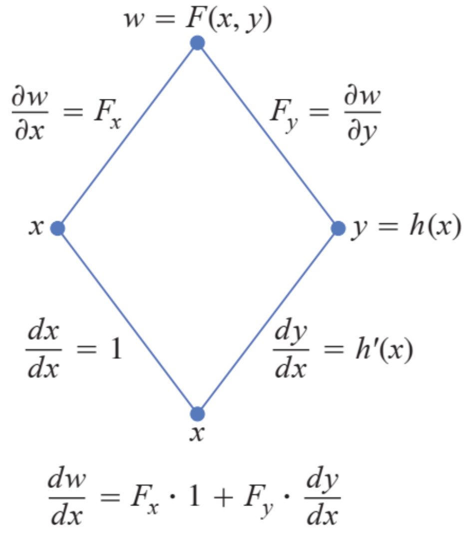
1. The function $F(x,y)$ is differentiable and
2. The equation $F(x,y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Since $w = F(x, y) = 0$ the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (as shown in tree diagram), we find (as before, but with $t = x$ and $f = F$)

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

If $F_y = \partial w/\partial y \neq 0$ we can solve this equation for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$



This relationship gives a surprisingly simple shortcut to finding derivatives of implicitly defined functions, which we state here.

A Formula for Implicit Differentiation: Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example5: Find dy/dx if $y^2 - x^2 - \sin xy = 0$

Solution: Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

Functions of Many Variables: In general, suppose that $w = f(x, y, \dots, v)$ is a differentiable function of the variables $x, y, \dots, v$ (a finite set) and $x, y, \dots, v$ the are differentiable functions of $p, q, \dots, t$ (another finite set). Then w is a differentiable function of the variables p through t and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

The other equations are obtained by replacing p by $q, \dots, t$ one at a time.

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial v}\right)}_{\text{Derivatives of } w \text{ with respect to the intermediate variables}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial q}, \dots, \frac{\partial v}{\partial t}\right)}_{\text{Derivatives of intermediate variables with respect to the selected independent variables}}$$

3.5 Directional Derivatives and Gradient Vectors

Directional Derivatives in the Plane: Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_o(x_o, y_o)$ is a point in R, and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_o + su_1, \qquad y = y_o + su_2$$

parametrise the line through parallel to $\mathbf{u}$. If the parameter s measures arc length from P_o in the direction of $\mathbf{u}$, we find the rate of change of f at in the direction of $\mathbf{u}$ by calculating df/ds at P_o (Figure 3.9).

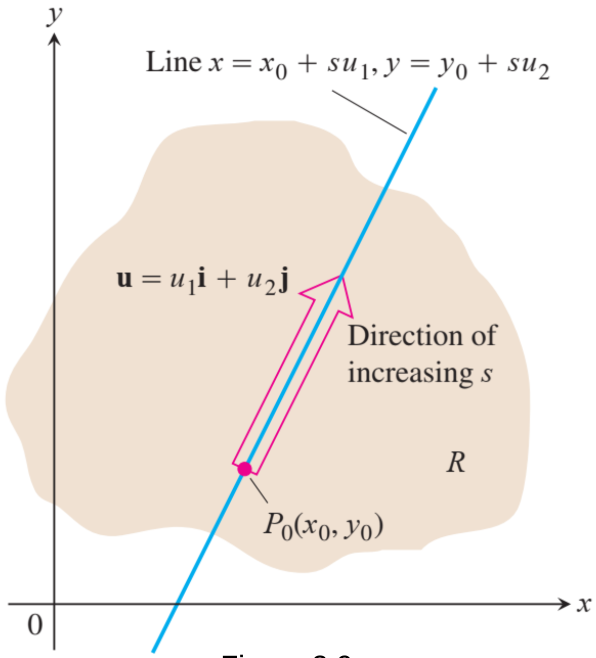


Figure 3.9.

So, the directional derivative can be defined:

The derivative of f at P_o in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_o} = \lim_{s\to 0} \frac{f(x_o + su_1, y_o + su_2) - f(x_o, y_o)}{s},$$

provide the limit exists.

The directional derivative is also denoted by $(D_{\mathbf{u}}f)_{P_o}$ “the derivative of f at P_o in the direction of $\mathbf{u}$”.

Example1: Find the derivative of $f(x, y) = x^2 + xy$ at $P_o(1,2)$ in the direction of the unit vector $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{j}$.

Solution:

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_o} = \lim_{s\to 0} \frac{f(x_o + su_1, y_o + su_2) - f(x_o, y_o)}{s} = \lim_{s\to 0} \frac{f\left(1 + s\frac{1}{\sqrt{2}}, 2 + s\frac{1}{\sqrt{2}}\right) - f(1,2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1(2))}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right)^2 + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} = \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0}\left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

Calculation and Gradients: We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_o + su_1, \qquad y = y_o + su_2$$

through parametrised with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then

$$\begin{aligned}(D_{\mathbf{u}}f)_{P_o} = \left(\frac{df}{ds}\right)_{\mathbf{u},P_o} &= \left(\frac{\partial f}{\partial x}\right)_{P_o}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_o}\frac{dy}{ds} \\ &= \left(\frac{\partial f}{\partial x}\right)_{P_o}u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_o}u_2 \\ &= \left[\left(\frac{\partial f}{\partial x}\right)_{P_o}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_o}\mathbf{j}\right]\cdot[u_1\mathbf{i} + u_2\mathbf{j}] = \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_o}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_o}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_o}\cdot\underbrace{[u_1\mathbf{i} + u_2\mathbf{j}]}_{\text{Direction } \mathbf{u}}.\end{aligned}$$

Gradient Vector: The gradient vector (gradient) of $f(x, y)$ at a point $P_o(x_o, y_o)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_o.

The notation is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f, read the way it is written.

The Directional Derivative Is a Dot Product: If $f(x, y)$ is differentiable in an open region containing P_o, then

$$(D_{\mathbf{u}}f)_{P_o} = \left(\frac{df}{ds}\right)_{\mathbf{u},P_o} = (\nabla f)_{P_o}\cdot\mathbf{u},$$

the dot product of the gradient f at P_o and $\mathbf{u}$.

Example2: Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution: The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$

$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2(0) = 2.$

The gradient of f at $(2,0)$ is

$\nabla f|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$.

The derivative of f at $(2,0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{P_o}\Big|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1\,.$$

Evaluating the dot product in the formula

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta\,,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f reveals the following properties.

Properties of the Directional Derivative:

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f|\cos(0) = |\nabla f|\,.$$

2. Similarly, f decreases most rapidly in the direction of $-|\nabla f|$. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$$

3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f|(0) = 0\,.$$

As we discuss later, these properties hold in three dimensions as well as two.

Example3: Find the directions in which

$$f(x,y) = \left(\frac{x^2}{2}\right) + \left(\frac{y^2}{2}\right)$$

(a) Increases most rapidly at the point $(1,1)$

(b) Decreases most rapidly at $(1,1)$.

(c) What are the directions of zero change in f at $(1,1)$?

Solution: (a) The function increases most rapidly in the direction of at $(1, 1)$.

The gradient there is $(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}$.

Its direction is $\mathbf{u} = \dfrac{\mathbf{i}+\mathbf{j}}{|\mathbf{i}+\mathbf{j}|} = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{(1)^2+(1)^2}} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$.

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$-\mathbf{u} = -\dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$.

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f:

$\mathbf{n} = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{n} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$.

Gradients and Tangents to Level Curves: If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve a level curve of f), then $f\big(g(t), h(t)\big) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f\big(g(t), h(t)\big) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0$$

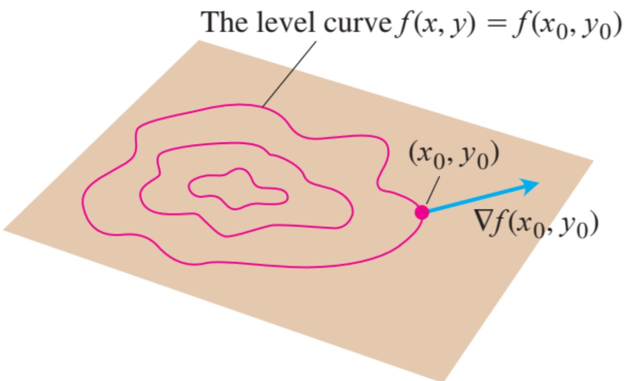


Figure 3.10.

This equation says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$ so it is normal to the curve.

At every point (x_o, y_o) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_o, y_o) (Figure 3.10).

This enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_o(x_o, y_o)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_o) + B(y - y_o) = 0$$

If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_o,y_o)} = f_x(x_o, y_o)\mathbf{i} + f_y(x_o, y_o)\mathbf{j}$, the equation is the tangent line given by

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) = 0.$$

Example4: Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Figure 3.11) at the point $(-2, 1)$.

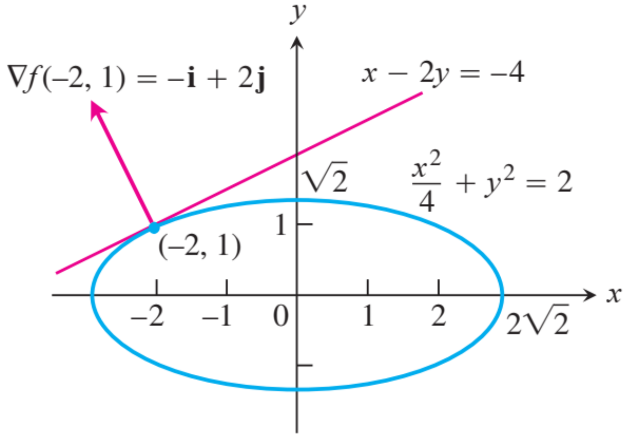


Figure 3.11.

Solution: The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2$$

The gradient of f at $(-2, 1)$ is

$$(\nabla f)|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}$$

The tangent is the line

$$-1(x + 2) + 2(y - 1) = 0 \;\Rightarrow\; x - 2y = -4 .$$

Algebra Rules for Gradients:

If we know the gradients of two functions f and g, we automatically know the gradients of their constant multiples, sum, difference, product, and quotient. Notice that these rules have the same form as the corresponding rules for derivatives of single-variable functions.

1. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
2. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
3. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Example5: We illustrate the rules with

$$f(x, y) = x - y \quad \Rightarrow \quad \nabla f = \mathbf{i} - \mathbf{j}, \qquad\qquad g(x, y) = 3y \quad \Rightarrow \quad \nabla f = 3\mathbf{j}$$

We have

1. $\nabla(2f) = \nabla(2x - 2y) = 2\mathbf{i} - 2\mathbf{j} = 2\nabla f$

2. $\nabla(f + g) = \nabla(x + 2y) = \mathbf{i} + 2\mathbf{j} = \nabla f + \nabla g$

3. $\nabla(f-g) = \nabla(x-4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$

4. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x-6y)\mathbf{j}$

$$= 3y(\mathbf{i}-\mathbf{j}) + 3y\mathbf{j} + (3x-6y)\mathbf{j}$$

$$= 3y(\mathbf{i}-\mathbf{j}) + (3x-3y)\mathbf{j}$$

$$= 3y(\mathbf{i}-\mathbf{j}) + (x-y)3\mathbf{j} = g\nabla f + f\nabla g$$

5. $\nabla\left(\frac{f}{g}\right) = \nabla\left(\frac{x-y}{3y}\right) = \nabla\left(\frac{x}{3y} - \frac{1}{3}\right)$

$$= \frac{1}{3y}\mathbf{i} - \frac{x}{3y^2}\mathbf{j}$$

$$= \frac{3y\mathbf{i} - 3x\mathbf{j}}{9y^2} = \frac{3y(\mathbf{i}-\mathbf{j}) - (3x-3y)\mathbf{j}}{9y^2}$$

$$= \frac{3y(\mathbf{i}-\mathbf{j}) - (x-y)3\mathbf{j}}{9y^2} = \frac{g\nabla f - f\nabla g}{g^2}.$$

Functions of Three Variables:

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cdot |\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

so the properties listed earlier for functions of two variables continue to hold. At any given point, f increases most rapidly in the direction of∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f the derivative is zero.

Example6: (a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_o(1,1,0)$ in the direction od $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_o and what are the rates of change in these directions?

Solution: (a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

The partial derivatives of f at P_o are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy|_{(1,1,0)} = -2, \qquad f_z = -1|_{(1,1,0)} = -1\,.$$

The gradient of f at P_o is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$ The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

3.6 Tangent Planes

In this section we define the tangent plane at a point on a smooth surface in space. We calculate an equation of the tangent plane from the partial derivatives of the function defining the surface. This idea is similar to the definition of the tangent line at a point on a curve in the coordinate plane for single-variable functions. We then study the total differential of functions of several variables.

Tangent Planes and Normal Lines: If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function f, then $f\big(g(t), h(t), k(t)\big) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt}f\big(g(t),h(t),k(t)\big)=\frac{d}{dt}c$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt}+\frac{\partial f}{\partial y}\frac{dh}{dt}+\frac{\partial f}{\partial z}\frac{dk}{dt}=0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)}_{\nabla f}\cdot\underbrace{\left(\frac{dg}{dt}\mathbf{i}+\frac{dh}{dt}\mathbf{j}+\frac{dk}{dt}\mathbf{k}\right)}_{d\mathbf{r}/dt}=0$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

Now let us restrict our attention to the curves that pass through P_o (Figure 3.12). All the velocity vectors at P_o are orthogonal to ∇f at P_o, so the curves' tangent lines all lie in the plane through P_o normal to ∇f. We call this plane the tangent plane of the surface at P_o. The line through P_o perpendicular to the plane is the surface's normal line at P_o.

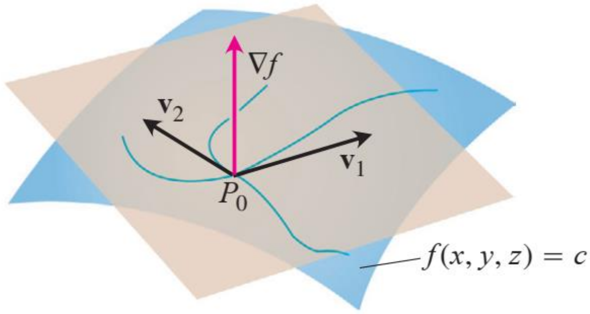


Figure 3.12.

Tangent Plane, Normal Line (Definition)

The **tangent plane** at the point $P_o(x_0, y_o, z_o)$ on the level surface $f(x,y,z)=c$ of a differentiable function f is the plane through P_o normal to $\nabla f|_{P_o}$.

The **normal line** of the surface at P_o is the line through P_o parallel to $\nabla f|_{P_o}$.

Thus, the tangent plane and normal line have the following equations:

Tangent Plane to $f(x,y,z)=c$ at $P_o(x_o, y_o, z_o)$

$$f_x(P_o)(x-x_o)+f_y(P_o)(y-y_o)+f_z(P_o)(z-z_o)=0$$

Normal Line to $f(x,y,z)=c$ at $P_o(x_o, y_o, z_o)$

$$x=x_o+f_x(P_o)t, \qquad y=y_o+f_y(P_o)t, \qquad z=z_o+f_z(P_o)t$$

Example1: Find the tangent plane and normal line of the surface

$$f(x,y,z)=x^2+y^2+z-9=0$$

at the point $P_o(1,2,4)$.

Solution: The surface is shown in Figure 3.13.

The tangent plane is the plane through P_o perpendicular to the gradient of f at P_o. The gradient is

$$\nabla f|_{P_o} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}\,.$$

The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0$$

$$2x + 4y + z = 14$$

The line normal to the surface at P_o is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t$$

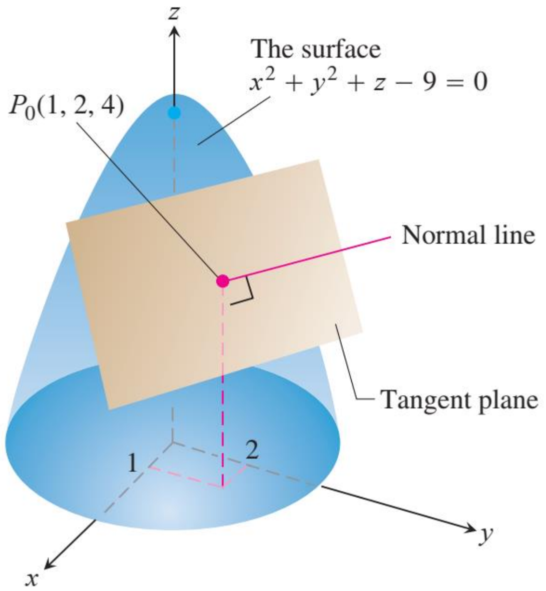


Figure 3.13.

Plane Tangent to a Surface: To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_o(x_o, y_o, z_o)$ where $z_o = f(x_o, y_o)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$ The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x,y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x,y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x,y) - z) = 0 - 1 = -1$$

The formula

$$F_x(P_o)(x - x_o) + F_y(P_o)(y - y_o) + F_z(P_o)(z - z_o) = 0$$

for the plane tangent to the level surface at P_o therefore reduces to

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) - (z - z_o) = 0$$

Plane Tangent to a Surface $z = f(x, y)$ at $\left(x_o, y_o, f(x_o, y_o)\right)$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_o(x_o, y_o, z_o) = \left(x_o, y_o, f(x_o, y_o)\right)$ is

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) - (z - z_o) = 0\,.$$

Example2: Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$.

Solution: We calculate the partial derivatives of $f(x,y) = x\cos y - ye^x$ first.

$$f_x(0,0) = (\cos y - ye^x)_{(0,0)} = 1 - 0(1) = 1$$

$$f_y(0,0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1$$

The tangent plane is therefore

$$(1)(x-0) + (-1)(y-0) - (z-0) = 0 \quad \Rightarrow \quad x - y - z = 0\,.$$

Example3: The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse E (Figure 3.14). Find parametric equations for the line tangent to E at the point $P_o(1,1,3)$.

Solution: The tangent line is orthogonal to both ∇f and ∇g at P_o, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and coordinates of P_o give us equations for the line. We have

$$\nabla f|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

The tangent line is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t\,.$$

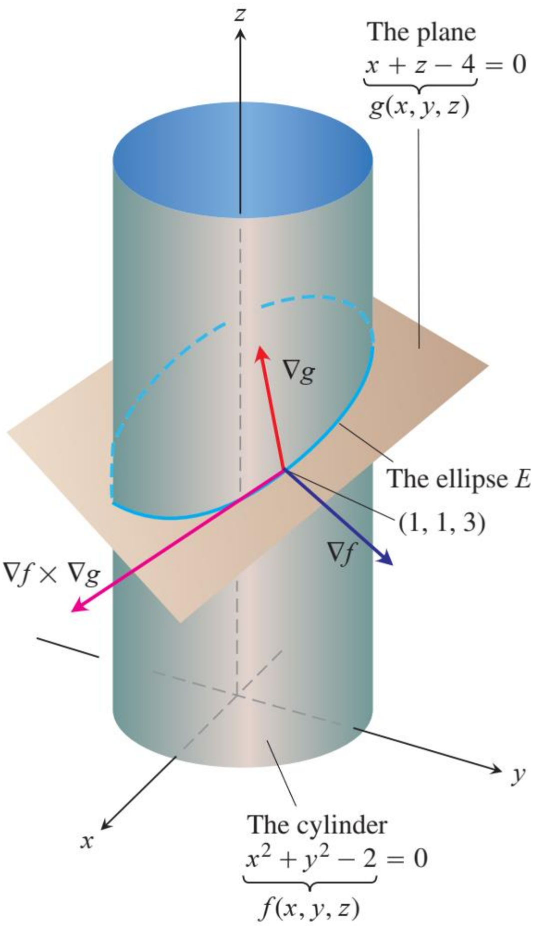


Figure 3.14.

3.7 Extreme Values and Saddle Points

Derivative Tests for Local Extreme Values: To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points, we then look for local maxima, local minima, and points of inflection. For a function $f(x,y)$ of two variables, we look for points where the surface $z = f(x,y)$ has a horizontal tangent plane. At such points, we then look for local maxima, local minima, and saddle points.

Local Maxima and Local Minima: Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centred at (a, b).

2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centred at (a, b).

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms (Figure 3.15). At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema**.

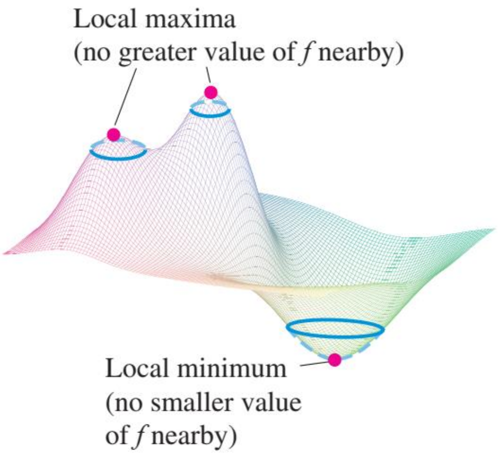


Figure 3.15

First Derivative Test for Local Extreme Values: If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Critical Point: An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Saddle Point: A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centred at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 3.16).

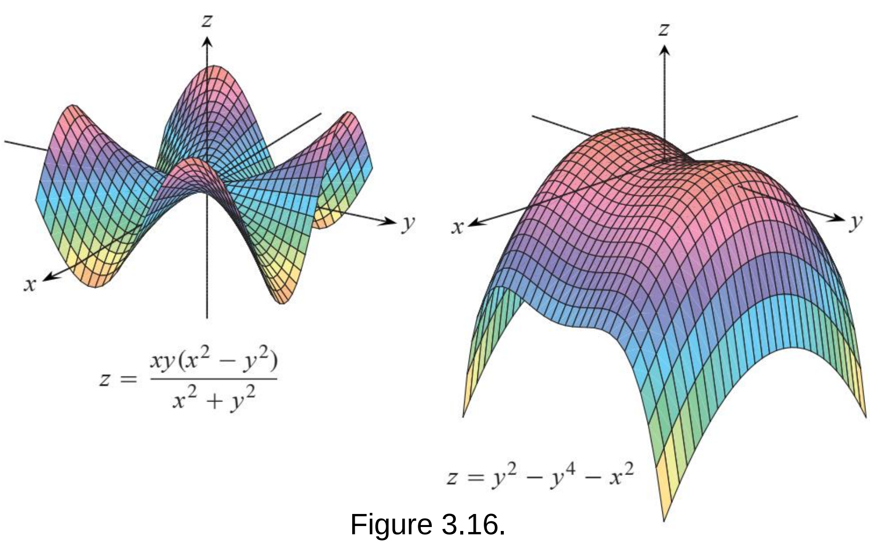


Figure 3.16.

Example1: Find the local extreme values of $f(x,y) = x^2 + y^2$.

Solution: The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0, \quad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum (Figure 3.17).

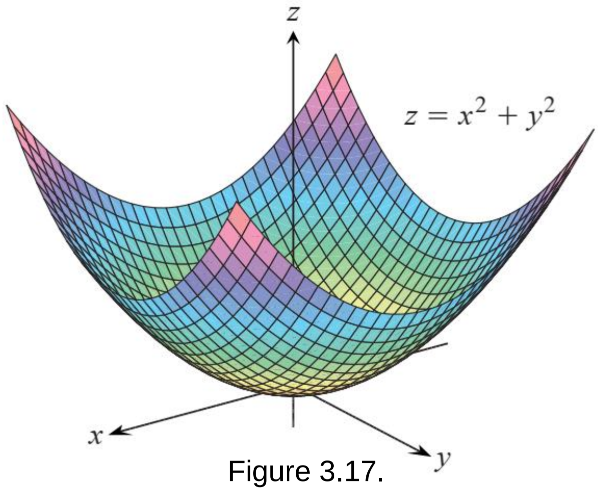


Figure 3.17.

Example2: Find the local extreme values of $f(x,y) = y^2 - x^2$.

Solution: The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0,0)$. Along the positive x-axis, however, f has the value $f(x,0) = -x^2 < 0$; along the positive y-axis, f has the value $f(0,y) = y^2 > 0$. Therefore, every open disk in the xy-plane centred at $(0,0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin (Figure 3.18) instead of a local extreme value. We conclude that the function has no local extreme values.

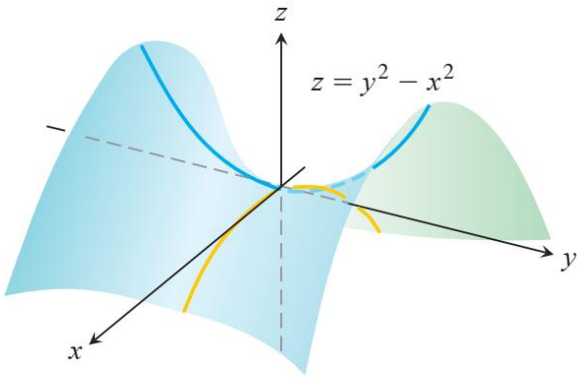


Figure 3.18.

Second Derivative Test for Local Extreme Values: Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centred at (a,b) and that $f_x(x,y) = f_y(x,y) = 0$. Then

1. f has a **local maximum** at (a,b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).

2. f has a **local minimum** at (a,b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).

3. f has a **saddle point** at (a,b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b).

4. The **test is inconclusive** at (a,b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Second Derivative Test says that if the discriminant is positive at the point (a, b), then the surface curves the same way in all directions: downward if $f_{xx} < 0$ giving rise to a local maximum, and upward if $f_{xx} > 0$ giving a local minimum. On the other hand, if the discriminant is negative at (a, b), then the surface curves up in some directions and down in others, so we have a saddle point.

Example3: Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4\,.$$

Solution: The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0$$

solving both equations to find x and y

$$x = y = -2\,.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1\,.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3\,.$$

The combination

$$f_{xx} < 0 \quad \text{and} \quad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.

Example4: Find the local extreme values of $f(x, y) = xy$.

Solution: Since f is differentiable everywhere (Figure 3.19), it can assume extreme values only where

$$f_x = y = 0 \quad \text{and} \quad f_y = x = 0\,.$$

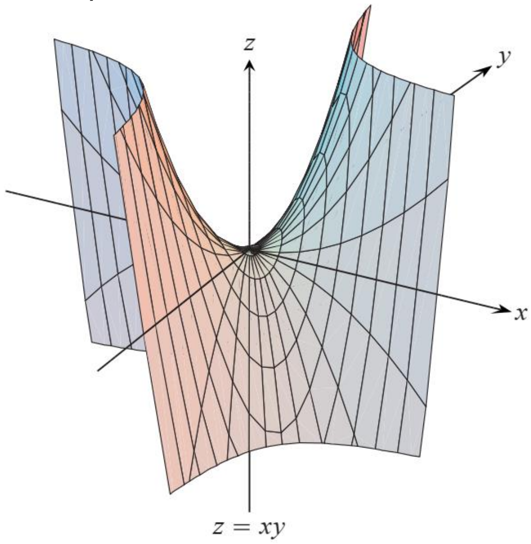


Figure 3.19.

Thus, the origin is the only point where f might have an extreme value. To see what happens there, we calculate

$$f_{xx} = 0, \qquad f_{yy} = 0, \qquad f_{xy} = 1\,.$$

The discriminant

$$f_{xx}f_{yy} - f_{xy}^2 = -1\,,$$

is negative. Therefore, the function has a saddle point at $(0, 0)$. We conclude that $f(x, y) = xy$ has no local extreme values.

Summary of Max-Min Tests:

The extreme values of $f(x, y)$ can occur only at

1. boundary points of the domain of f

2. critical points (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centred at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

1. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

2. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

3. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

4. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive**.

Solved Problems:

3.1 Functions of Several Variables

Prob1: Find the domain and the range of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution: Domain is all (x, y) satisfying $x^2 + y^2 \leq 9$, Range is $0 \leq z \leq 3$.

Prob2: Find an equation for the level curve of the function $f(x, y) = \sqrt{x^2 - 1}$ that passes through the given point $(1, 0)$.

Solution: At $(1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \text{ or } x = -1$.

Prob3: Find an equation for the level surface of the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ through the given point $(-1, 2, 1)$.

Solution:

At $(-1, 2, 1) \Rightarrow w = \ln(1 + 2 + 1) = \ln 4 \Rightarrow \ln 4 = \ln(x^2 + y^2 + z^2) \Rightarrow x^2 + y^2 + z^2 = 4$.

3.2 Limits and Continuity in Higher Dimensions

Prob1: Find the limit

$$\lim_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan y\,.$$

Solution: $\displaystyle \lim_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan y = (\sec 0)\left(\tan\frac{\pi}{4}\right) = (1)(1) = 1\,.$

Prob2: Find the limit by rewriting the fraction first

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - y^2}{x - y}\,.$$

Solution: $\displaystyle \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x - y} = \lim_{(x,y)\to(1,1)} (x + y) = (1 + 1) = 2\,.$

Prob3: Find the limit

$$\lim_{P\to(\pi,0,3)} z\, e^{-2y} \cos 2x\,.$$

Solution: $\displaystyle \lim_{P\to(\pi,0,3)} z\, e^{-2y} \cos 2x = 3e^{-2(0)} \cos 2\pi = (3)(1)(1) = 3\,.$

Prob4: At what point (x, y) in the plane is the function continuous?

$$f(x, y) = \sin(x + y)\,.$$

Solution: All (x, y).

Prob5: At what point (x, y, z) in space is the function continuous?

$$f(x, y, z) = \sqrt{x^2 + y^2 - 1}\,.$$

Solution: All (x, y, z) except the interior of cylinder $x^2 + y^2 = 1$.

3.3 Partial Derivatives:

Prob1: Find $\partial f/\partial x$ and $\partial f/\partial y$ of the function $f(x, y) = e^{xy} \ln y$.

Solution:

$$\frac{\partial f}{\partial x} = e^{xy} \frac{\partial}{\partial x}(xy) \ln y = y\, e^{xy} \ln y\,, \qquad \frac{\partial f}{\partial y} = e^{xy} \frac{\partial}{\partial y}(xy) \ln y + e^{xy} \frac{1}{y} = x e^{xy} \ln y + \frac{e^{xy}}{y}.$$

Prob2: Find f_x, f_y, and f_z of the function $f(x, y, z) = \sinh(xy - z^2)$.

Solution: $f_x = y \cosh(xy - z^2)\,,\ f_y = x \cosh(xy - z^2)\,,\ f_z = -2z \cosh(xy - z^2)$.

Prob3: Find the partial derivative of the function with respect to each variable

$$f(t, \alpha) = \cos(2\pi t - \alpha).$$

Solution: $\dfrac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)\,, \qquad \dfrac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)\,.$

Prob4: Find all the second-order partial derivatives of the function

$$f(x, y) = x^2 y + \cos y + y \sin x.$$

Solution:

$$\frac{\partial f}{\partial x} = 2xy + y \cos x\,, \qquad \frac{\partial f}{\partial y} = x^2 - \sin y + \sin x\,, \qquad \frac{\partial^2 f}{\partial x^2} = 2y - y \sin x\,, \qquad \frac{\partial^2 f}{\partial y^2} = -\cos y,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x + \cos x\,.$$

Prob5: Verify $w_{xy} = w_{yx}$ for $w = xy + x\sin y + y\sin x$.

Solution: $w_x = y + \sin y + y\cos x\,, \quad w_y = x + x\cos y + \sin x,$

$w_{xy} = 1 + \cos y \;+ \cos x\,, \qquad w_{yx} = 1 + \cos y + \cos x\,.$

Prob6: Find the value of $\partial z/\partial x$ at the point $(1,1,1)$ if the equation $xy + z^3x - 2yz = 0$.

Solution: $y + \left(3z^2\dfrac{\partial z}{\partial x}\right)x + z^3 - 2y\dfrac{\partial z}{\partial x} = 0 \;\;\Rightarrow\;\; (3z^2x - 2y)\dfrac{\partial z}{\partial x} = -y - z^3,$

at $(1,1,1) \;\;\Rightarrow\;\; (3-2)\dfrac{\partial z}{\partial x} = -1 - 1 \;\;\Rightarrow\;\; \dfrac{\partial z}{\partial x} = -2\,.$

3.4 The Chain Rule

Prob1: Express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then evaluate dw/dt at the given value of t.

$$w = 2ye^x - \ln z\,, \qquad x = \ln(t^2+1)\,, \qquad y = \tan^{-1} t\,, \qquad z = e^t; \quad t = 1\,.$$

Solution: $\dfrac{\partial w}{\partial x} = 2ye^x, \;\; \dfrac{\partial w}{\partial y} = 2e^x, \;\; \dfrac{\partial w}{\partial z} = -\dfrac{1}{z}\,, \qquad \dfrac{dx}{dt} = \dfrac{2t}{t^2+1}, \;\; \dfrac{dy}{dt} = \dfrac{1}{t^2+1}, \;\; \dfrac{dz}{dt} = e^t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 2ye^x\frac{2t}{t^2+1} + 2e^x\frac{1}{t^2+1} - \frac{1}{z}e^t$$

$$= \frac{(4t)\tan^{-1} t\,(t^2+1)}{t^2+1} + \frac{2(t^2+1)}{t^2+1} - \frac{e^t}{e^t} = 4t\tan^{-1} t + 1\,;$$

$$w = 2ye^x - \ln z = (2\tan^{-1} t)(t^2+1) - t \;\;\Rightarrow\;\; \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)(t^2+1) + (2\tan^{-1} t)(2t) - 1$$

$$\frac{dw}{dt} = 4t\tan^{-1} t + 1 \;\;\overset{t=1}{\Longrightarrow}\;\; \frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1\,.$$

Prob2: Express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point $(u, y) = (1/2, 1)$

$$w = xy + yz + xz, \qquad x = u + v, \qquad y = u - v, \qquad z = uv\,.$$

Solution: $\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}=(y+z)(1)+(x+z)(1)+(y+x)(v)$

$=x+y+2z+v(x+y)=(u+v)+(u-v)+2(uv)+v(2u)=2u+4uv$;

$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}=(y+z)(1)+(x+z)(-1)+(y+x)(u)=y-x+u(x+y)$

$=(u-v)-(u+v)+u(2u)=-2v+2u^2$;

$w=xy+yz+xz=(u+v)(u-v)+(u-v)(uv)+(u+v)(uv)$

$w=(u^2-v^2)+(u^2v-uv^2)+(u^2v+uv^2)=u^2-v^2+2u^2v \quad \Rightarrow \quad \frac{\partial w}{\partial u}=2u+4uv$ and

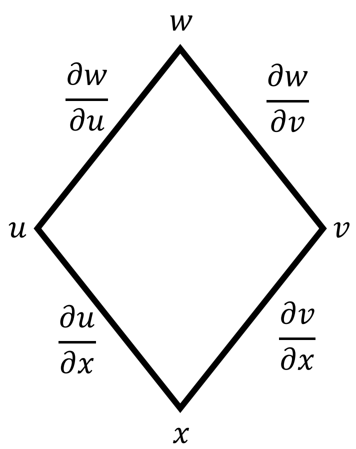
$\frac{\partial w}{\partial v}=-2v+2u^2$.

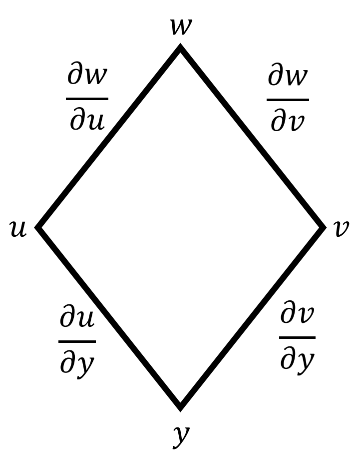
At $\left(\frac{1}{2},1\right)$: $\frac{\partial w}{\partial u}=2\left(\frac{1}{2}\right)+4\left(\frac{1}{2}\right)(1)=3$ and $\frac{\partial w}{\partial v}=-2(1)+2\left(\frac{1}{2}\right)^2=-\frac{3}{2}$.

Prob3: Draw a tree diagram and write a Chain Rule formula for each derivative

$$\frac{\partial w}{\partial x} \text{ and } \frac{\partial w}{\partial y} \text{ for } w=g(u,v), \qquad u=h(x,y), \qquad v=k(x,y)\,.$$

Solution: $\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$, $\qquad \frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$





Prob4: Assuming that the equation $xe^y+\sin xy+y-\ln 2=0$ defines y as a differentiable function of x, find the value of dy/dx at the given point $(0,\ln 2)$.

Solution: Let $F(x,y) = xe^y + \sin xy + y - \ln 2 = 0 \quad \Rightarrow \quad F_x = e^y + y\cos xy$ and

$$F_y = xe^y + x\cos xy + 1 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\cos xy + 1}$$

At $(0, \ln 2)$: $\dfrac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$.

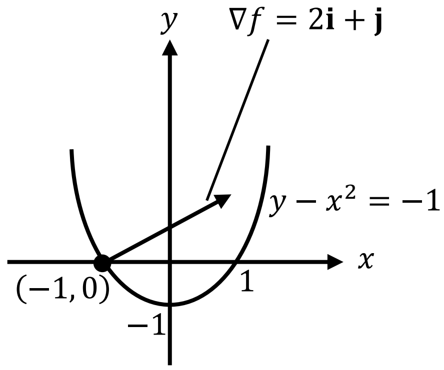
3.5 Directional Derivatives and Gradient Vectors

Prob1: Find the gradient of the function $f(x,y) = y - x^2$ at the given point $(-1, 0)$. Then sketch the gradient together with the level curve that passes through the point.

Solution: $\dfrac{\partial f}{\partial x} = -2x, \quad \Rightarrow \quad \dfrac{\partial f}{\partial x}(-1,0) = 2, \qquad \dfrac{\partial f}{\partial y} = 1$,

$\nabla f = 2\mathbf{i} + \mathbf{j}$; $\quad f(-1,0) = -1$,

$-1 = y - x^2$ is the level curve.



Prob2: Find ∇f at the given point $(1,1,1)$ of the function $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x$.

Solution: $\dfrac{\partial f}{\partial x} = 2x + \dfrac{z}{x} \quad \Rightarrow \quad \dfrac{\partial f}{\partial x}(1,1,1) = 3; \ \dfrac{\partial f}{\partial y} = 2y \quad \Rightarrow \quad \dfrac{\partial f}{\partial y}(1,1,1) = 2;$

$\dfrac{\partial f}{\partial z} = -4z + \ln x \quad \Rightarrow \quad \dfrac{\partial f}{\partial z}(1,1,1) = -4, \qquad$ thus $\nabla f = 3\mathbf{i} + 2\mathbf{i} - 4\mathbf{k}$.

Prob3: Find the derivative of the function at P_o in the direction of $\mathbf{A}$

$$f(x,y,z) = \cos xy + e^{yz} + \ln xz \ , \qquad P_o\left(1, 0, \frac{1}{2}\right) , \qquad \mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \,.$$

Solution: $\mathbf{u} = \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}\,, f_x = -y\sin xy + \dfrac{1}{x} \quad \Rightarrow \quad f_x\left(1, 0, \dfrac{1}{2}\right) = 1;$

$f_y = -x\sin xy + ze^{yz} \quad \Rightarrow \quad f_y\left(1,0,\dfrac{1}{2}\right) = \dfrac{1}{2};\ f_z = ye^{yz} + \dfrac{1}{z} \quad \Rightarrow \quad f_z\left(1,0,\dfrac{1}{2}\right) = 2$,

$\nabla f = \mathbf{i} + \dfrac{1}{2}\mathbf{j} + 2\mathbf{k}\,, \qquad (D_{\mathbf{u}}f)_{P_o} = \nabla f \cdot \mathbf{u} = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{4}{3} = 2$.

Prob4: Find the direction in which the function increases and decreases most rapidly at P_o. Then find the derivative of the function in these directions

$$f(x,y) = x^2 + xy + y^2, \qquad P_o(-1,1).$$

Solution: $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} \;\Rightarrow\; \nabla f(-1,1) = -1\mathbf{i} + \mathbf{j} \;\Rightarrow\; \mathbf{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{-1\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}}$

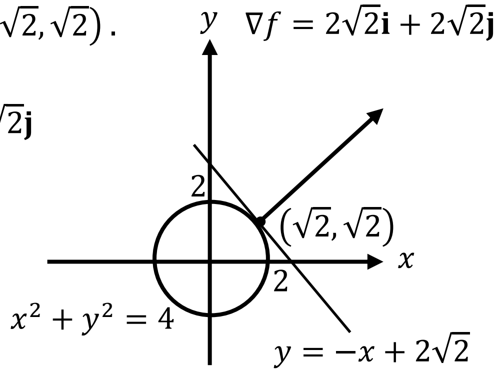
$\mathbf{u} = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction of $\mathbf{u} = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

and decreases most rapidly in the direction of $-\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$.

$(D_{\mathbf{u}}f)_{P_o} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_o} = -|\nabla f| = -\sqrt{2}$.

Prob5: Sketch the curve $f(x,y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line

$$f(x,y) = x^2 + y^2 = 4, \qquad (\sqrt{2}, \sqrt{2}).$$



Solution: $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \;\Rightarrow\; \nabla f(\sqrt{2},\sqrt{2}) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}$

Tangent line: $2\sqrt{2}(x-\sqrt{2}) + 2\sqrt{2}(y-\sqrt{2}) = 0$,

$\sqrt{2}x + \sqrt{2}y = 4$.

3.6 Tangent Planes

Prob1: Find equations for the tangent plane and normal line at the point P_o on the given surface $2z - x^2 = 0$, $P_o(2,0,2)$.

Solution: $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \;\Rightarrow\; \nabla f(2,0,2) = -4\mathbf{i} + 2\mathbf{k}$,

Tangent plane: $-4(x-2) + 2(z-2) = 0 \;\Rightarrow\; -4x + 2z + 4 = 0 \;\Rightarrow\; -2x + z + 2 = 0$;

Normal line: $x = 2 - 4t$, $y = 0$, $z = 2 + 2t$.

Prob2: Find an equation for the plane that is tangent to the given surface at the given point

$$z = 4x^2 + y^2, \qquad (1,1,5).$$

Solution: $z = f(x,y) = 4x^2 + y^2 \;\Rightarrow\; f_x = 8x \;\Rightarrow\; f_x(1,1) = 8,\; f_y = 2y \;\Rightarrow\; f_y(1,1) = 2$

Tangent plane: $8(x-1) + 2(y-1) - (z-5) = 0 \;\Rightarrow\; 8x + 2y - z = 5$.

Prob3: Find parametric equation for the line tangent to the curve of intersection of the surfaces at the given point

Surfaces: $x + y^2 + 2z = 4, x = 1$. Point: $(1, 1, 1)$.

Solution: $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1,1,1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{i}$; $\mathbf{v} = \nabla f \times \nabla g$

$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1,\ y = 1 + 2t,\ z = 1 - 2t$.

3.7 Extreme Values and Saddle Points

Prob1: Find the local maxima, local minima, and saddle points of the function

$$f(x, y) = \frac{1}{x} + xy + \frac{1}{y}.$$

Solution: $f_x = -\frac{1}{x^2} + y = 0$ and $f_y = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1$,

the critical point is $(1, 1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1,1) = 2, f_{yy}(1,1) = 2, f_{xy} = 1$;

$f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} = 2 > 0 \Rightarrow$ local minimum of $f(1,1) = 3$.