Chapter Four: Multiple Integrals

In this chapter we consider the integral of a function of two variables $f(x, y)$ over a region in the plane and the integral of a function of three variables $f(x, y, z)$ over a region in space.

4.1 Double Integrals

Double Integral Over Rectangular Regions (First Form – Fubini's Theorem): If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx\,.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Calculating Double Integrals: Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: 0 \le x \le 2,\ 0 \le y \le 1$ in the xy-plane. If we apply the method of slicing, with slices perpendicular to the x-axis (Figure 4.1), then the volume is

$$\int_{x=0}^{x=2} A(x)\, dx,$$

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy,$$

which is the area under the curve $4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining Equations above, we see that the volume of the entire solid is

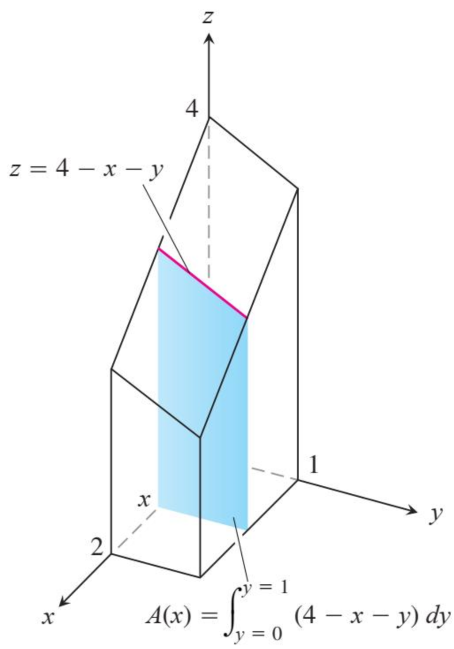


Figure 4.1.

$$\text{Volume} = \int_{x=0}^{x=2} A(x)\,dx = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=1}(4-x-y)\,dy\right)dx = \int_{x=0}^{x=2}\left[4y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=2}\left(\frac{7}{2} - x\right)dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = 5\,.$$

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2\int_0^1 (4-x-y)\,dy\,dx\,.$$

The expression on the right, called an iterated or repeated integral, says that the volume is obtained by integrating $4-x-y$ with respect to y from $y=0$ to $y=1$ holding x fixed, and then integrating the resulting expression in x with respect to x from $x=0$ to $x=2$. The limits of integration 0 and 1 are associated with y, so they are placed on the integral closest to dy. The other limits of integration, 0 and 2, are associated with the variable x, so they are placed on the outside integral symbol that is paired with dx.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis (Figure 4.2)? As a function of y, the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2}(4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2}$$

$$= 6 - 2y\,.$$

The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\,dy = \int_{y=0}^{y=1}(6-2y)\,dy = [6y - y^2]_0^1$$

$$= 5\,,$$

in agreement with our earlier calculation.

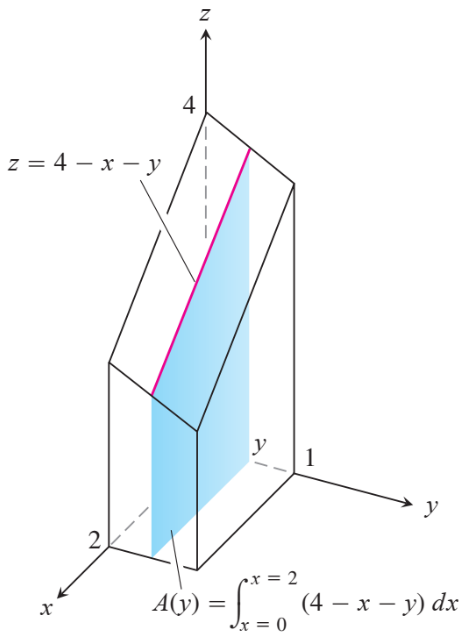


Figure 4.2.

Again, we may give a formula for the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1\int_0^2 (4-x-y)\,dx\,dy\,.$$

The expression on the right says we can find the volume by integrating $4 - x - y$ with respect to x from $x = 0$ to $x = 2$ and integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral, the order of integration is first x and then y, the reverse of the order in the previous case.

Fubini's Theorem also says that we may calculate the double integral by integrating in either order, as we see in Example 1. When we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

Example1: Calculate $\iint_R f(x,y)\, dA$ for

$$f(x,y) = 1 - 6x^2y, \qquad R: 0 \le x \le 2, \qquad -1 \le y \le 1\,.$$

Solution: By Fubini's Theorem,

$$\iint_R f(x,y)\, dA = \int_{-1}^{1}\int_{0}^{2} (1 - 6x^2y)\, dx\, dy = \int_{-1}^{1} [x - 2x^3y]_{x=0}^{x=2}\, dy = \int_{-1}^{1} (2 - 16y)\, dy$$

$$= [2y - 8y^2]_{-1}^{1} = 4\,.$$

Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1} (1 - 6x^2y)\, dy\, dx = \int_{0}^{2} [y - 3x^2y^2]_{y=-1}^{y=1}\, dx = \int_{0}^{2} [(1 - 3x^2) - (-1 - 3x^2)]\, dx = \int_{0}^{2} 2\, dx$$

$$= 4\,.$$

Double Integrals over Bounded Nonrectangular Regions (Stronger Form – Fubini's Theorem): Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b, g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_{a}^{b}\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d, h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\, dA = \int_{c}^{d}\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

Example2: Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y\,.$$

Solution: See Figure 4.3. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 4.3b).

Hence,

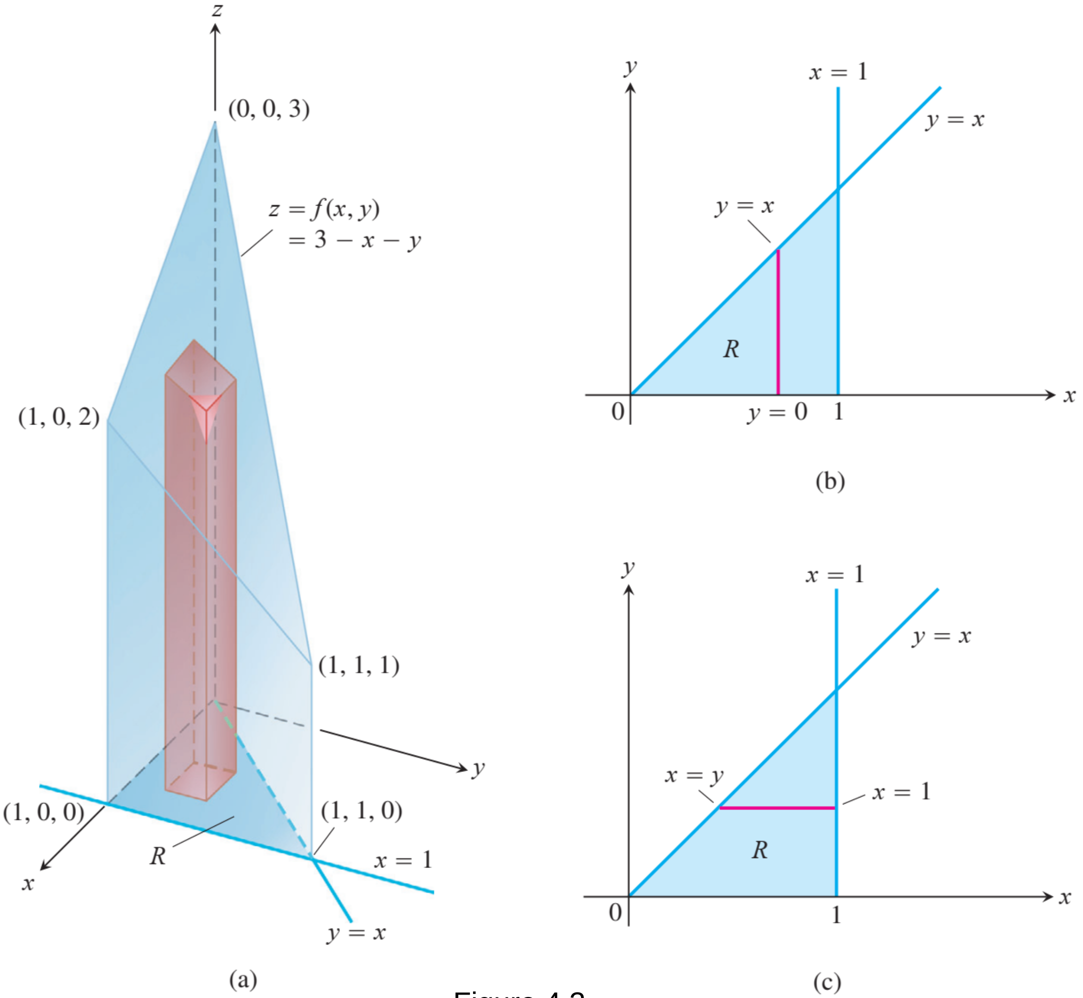


Figure 4.3.

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1}$$

$V = 1.$

When the order of integration is reversed (Figure 4.3c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy = \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal.

Example3: Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution: The region of integration is shown in Figure 4.4. If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(\left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} \right) dx = \int_0^1 \sin x \;dx$$

$$= -\cos(1) + 1 \approx 0.46\,.$$

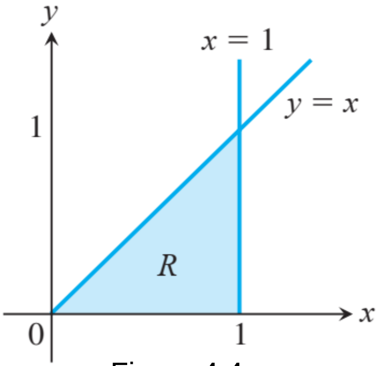


Figure 4.4.

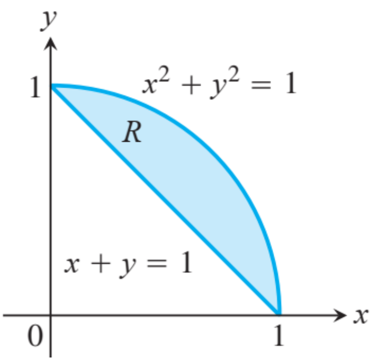
If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy\,, \qquad \text{we run into a problem.}$$

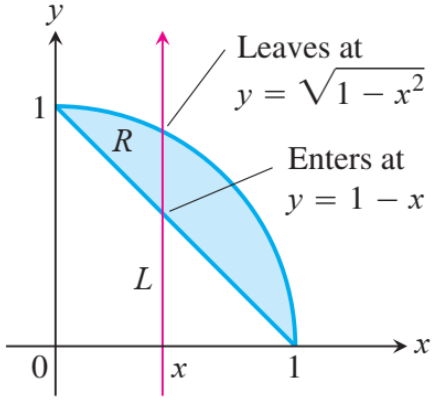
Finding Limits of Integration: We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating integrating $\iint_R f(x, y)\, dA$, first with respect to y and then with respect to x, do the following:

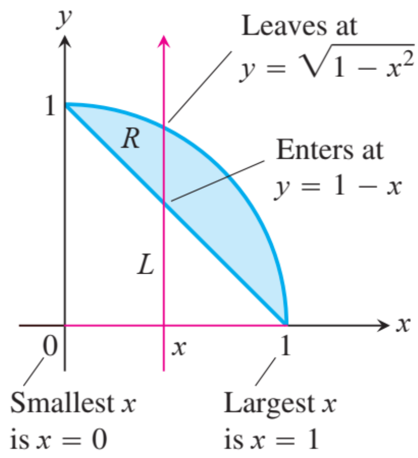
1. Sketch: Sketch the region of integration and label the bounding curves.



2. Find the y-limits of integration: Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).

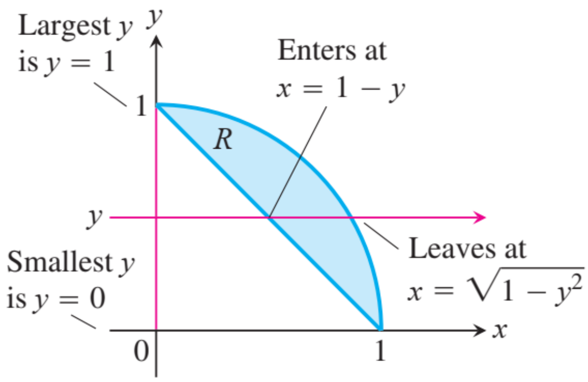


3. Find the x-limits of integration: Choose x-limits that include all the vertical lines through R. The integral shown here is



$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx\,.$$

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is



$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy\,.$$

Example4: Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

and write an equivalent integral with the order of integration reversed.

Solution: The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 4.5a).

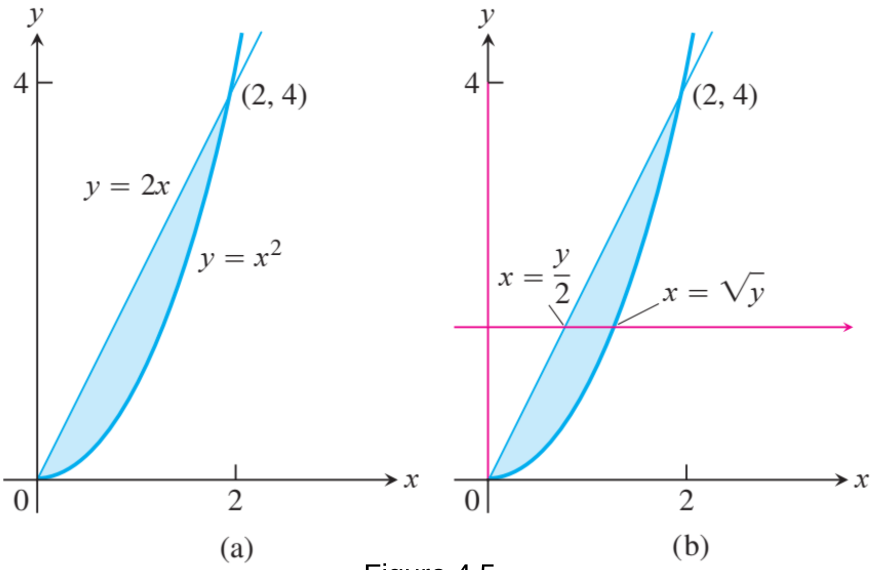


Figure 4.5.

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure 4.5b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy\,.$$

The common value of these integrals is 8.

Properties of Double Integrals: Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{(any number } c\text{)}$

2. *Sum and Difference:*
$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0 \qquad \text{if} \qquad f(x, y) \geq 0 \text{ on } R$

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \qquad \text{if} \qquad f(x, y) \geq g(x, y) \text{ on } R$

4. *Additivity:* $\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

if R is the union of two nonoverlapping regions R_1 and R_2.

4.2 Area

In this section, we show how to use double integrals to calculate the areas of bounded regions in the plane.

Areas of Bounded Regions in the Plane: The area of a closed, bounded plane region R is

$$A = \iint_R dA\,.$$

Example1: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution: We sketch the region (Figure 4.6), noting where the two curves intersect, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$A = \frac{1}{6}.$$

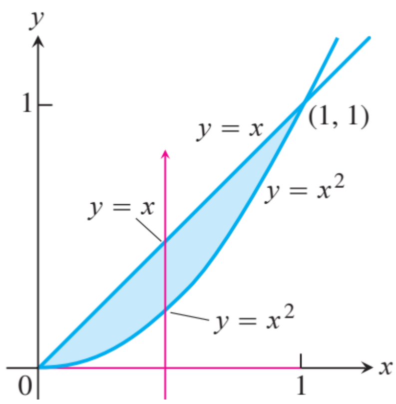


Figure 4.6.

Example2: Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution: If we divide R into the regions R_1 and R_2 shown in Figure 4.7a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy\,.$$

On the other hand, reversing the order of integration (Figure 4.7b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx$$

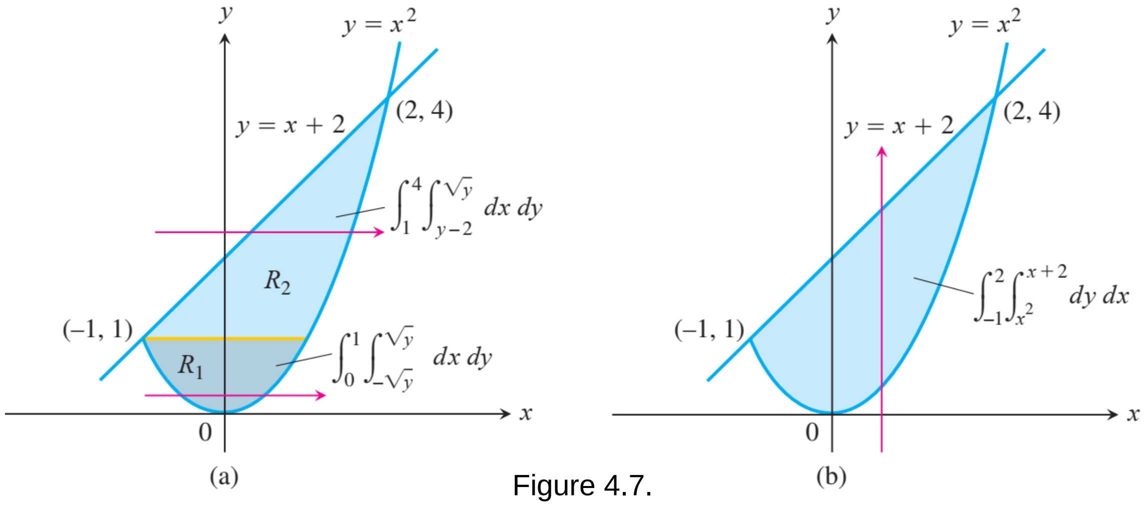


Figure 4.7.

This second result, which requires only one integral, is simpler. The area is

$$A = \int_{-1}^{2} [y]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

4.3 Double Integrals in Polar Form

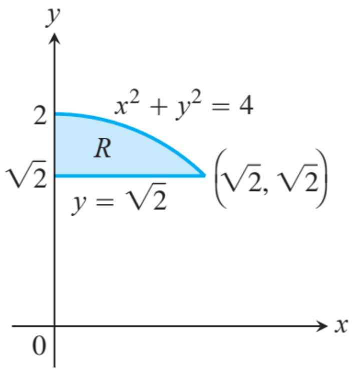
Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates: Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$.

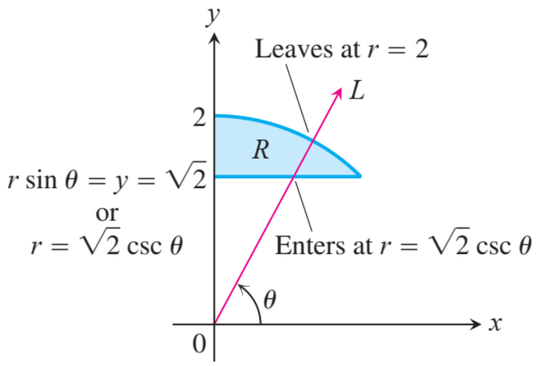
$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta\, .$$

Finding Limits of Integration: The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ take the following steps.

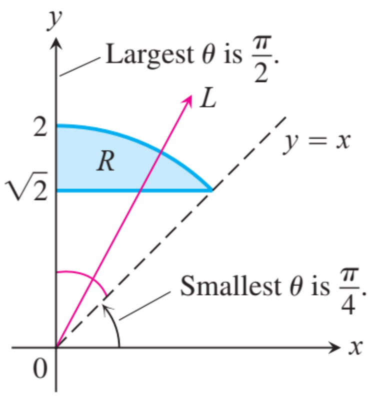
1. Sketch: Sketch the region and label the bounding curves.



2. Find the r-limits of integration: Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.



3. Find the θ-limits of integration: Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.



The integral is

$$\int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\, r\, dr\, d\theta \, .$$

Example1: Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution:

1. We first sketch the region and label the bounding curves (Figure 4.8).

2. Next we find the r-limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

3. Finally we find the θ-limits of integration. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta\,.$$

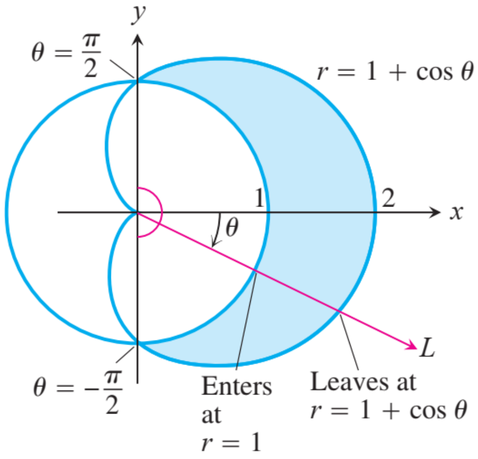


Figure 4.8.

If $f(r,\theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area in Polar Coordinates: The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta\,.$$

Example2: Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution: We graph the lemniscate to determine the limits of integration (Figure 4.9) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\ d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4\,.$$

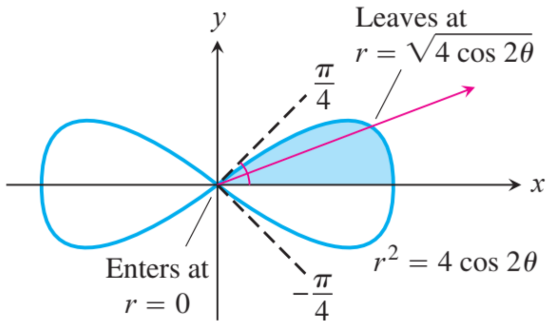


Figure 4.9.

Changing Cartesian Integrals into Polar Integrals: The procedure for changing a Cartesian integral $\iint_R f(x,y)\,dx\,dy$ into a polar integral has two steps. Firstly, substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R. The Cartesian integral then becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

where G denotes the region of integration in polar coordinates.

Example3: Find the polar moment of inertia about the origin of a thin plate bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution: We sketch the plate to determine the limits of integration (Figure 4.10). In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx\,.$$

Integration with respect to y gives

$$\int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

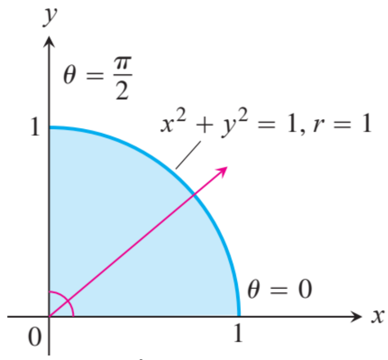


Figure 4.10.

It will be much easier if we change the original integral to polar coordinates. Substituting $x = r\cos\theta, y = r\sin\theta$ and replacing $dx\,dy$ by $r\,dr\,d\theta$, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2} \int_0^1 (r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{\pi}{8}.$$

Example4: Evaluate

$$\iint_R e^{x^2+y^2}\,dy\,dx,$$

where R is the semi-circular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ (Figure 4.11).

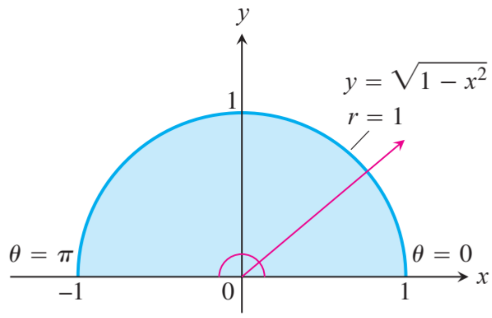


Figure 4.11.

Solution: In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\iint\limits_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1)\,.$$

4.4 Triple Integrals in Rectangular Coordinates

We use triple integrals to calculate the volumes of three-dimensional shapes.

Triple Integrals: If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball, then the integral of F over D may be defined in the following way.

$$\iiint\limits_D F(x, y, z)\,dV = \iiint\limits_D F(x, y, z)\,dx\,dy\,dz\,.$$

Volume of a Region in Space: If F is the constant function whose value is 1, then the volume of a close, bonded region D in space is

$$V = \iiint\limits_D dV\,.$$

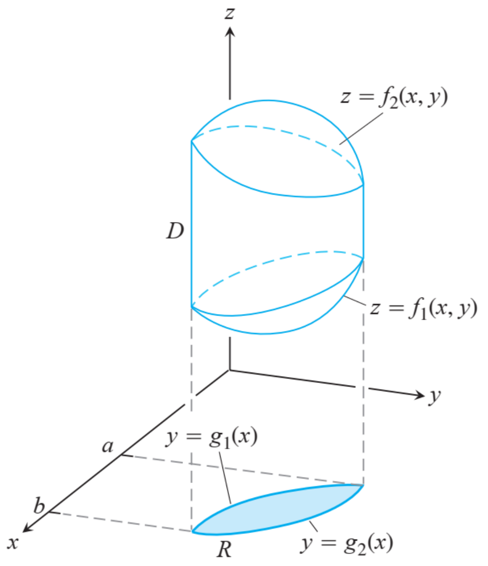
Finding Limits of Integration: We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 4.1) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

To evaluate

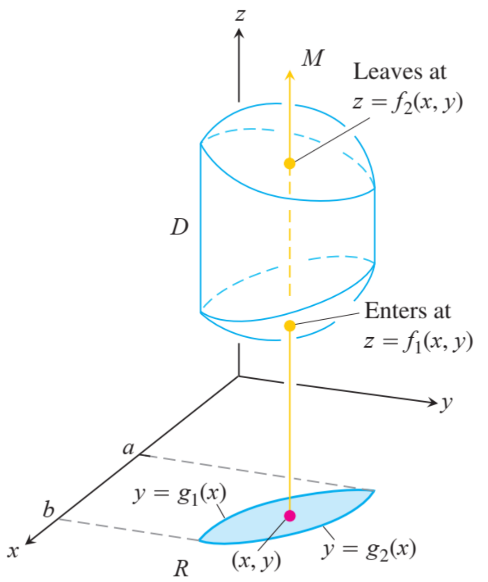
$$\iiint\limits_D F(x, y, z)\,dV$$

over a region D, integrate first with respect to z, then with respect to y, finally with x.

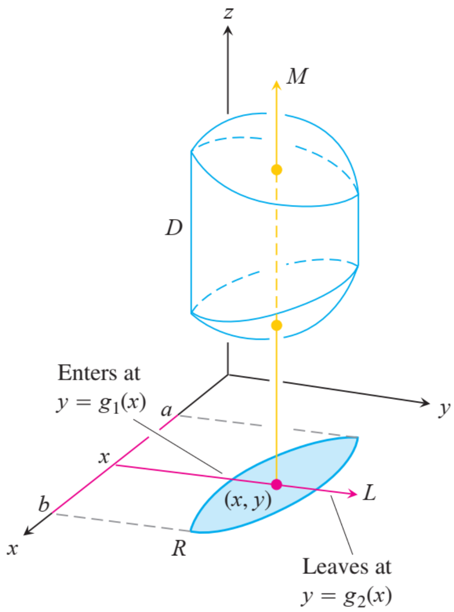
1. Sketch: Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. Find the z-limits of integration: Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. Find the y-limits of integration: Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. Find the x-limits of integration: Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx\,.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

Example1: Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution: The volume is

$$V = \iiint_D dz\, dy\, dx\,,$$

To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 4.12) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4, z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

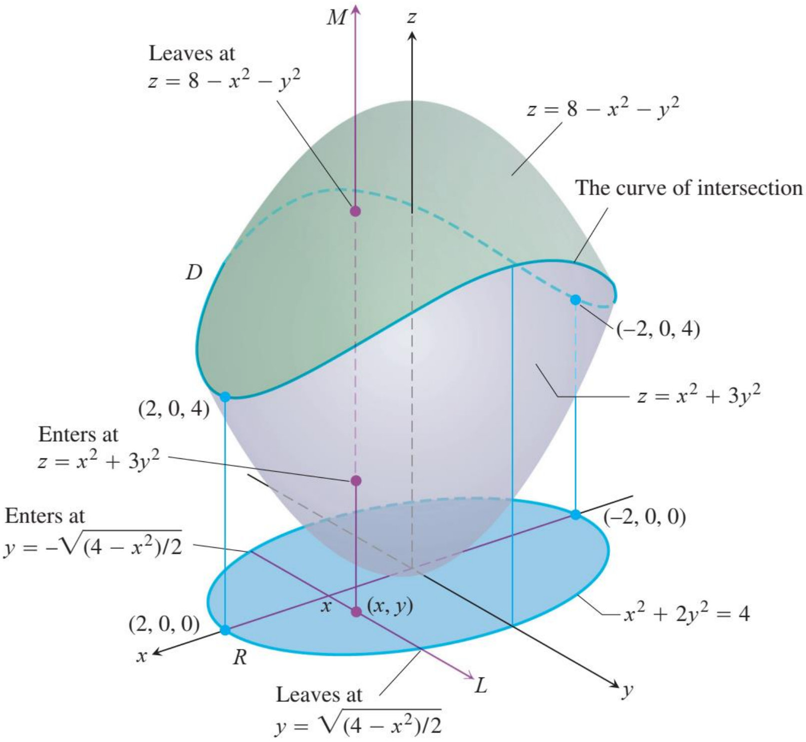


Figure 4.12.

$$V = \iiint_D dz\,dy\,dx = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx = \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{\frac{3}{2}}\right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2}\right)^{\frac{3}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{\frac{3}{2}}\right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\,dx = 8\pi\sqrt{2}\,.$$

After integration with the substitution $x = 2\sin u$.

Example2: Each of the following integrals gives the volume of the solid shown in Figure 4.13.

(a) $\int_0^1 \int_0^{1-z} \int_0^2 dx\,dy\,dz$ **(b)** $\int_0^1 \int_0^{1-y} \int_0^2 dx\,dz\,dy$

(c) $\int_0^1 \int_0^2 \int_0^{1-z} dy\,dx\,dz$ **(d)** $\int_0^2 \int_0^1 \int_0^{1-z} dy\,dz\,dx$

(e) $\int_0^1 \int_0^2 \int_0^{1-y} dz\,dx\,dy$ **(f)** $\int_0^2 \int_0^1 \int_0^{1-y} dz\,dy\,dx$

We work out the integrals in parts (b) and (c):

$$V=\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy=\int_0^1\int_0^{1-y} 2\,dz\,dy=\int_0^1 [2z]_0^{1-y}dy$$

$$=\int_0^1 2(1-y)dy=[2y-y^2]_0^1=1.$$

Also,

$$V=\int_0^1\int_0^2\int_0^{1-z} dy\,dx\,dz=\int_0^1\int_0^2 (1-z)\,dx\,dz$$

$$=\int_0^1 [x-xz]_0^2 dz=\int_0^1 (2-2z)dz=[2z-z^2]_0^1=1.$$

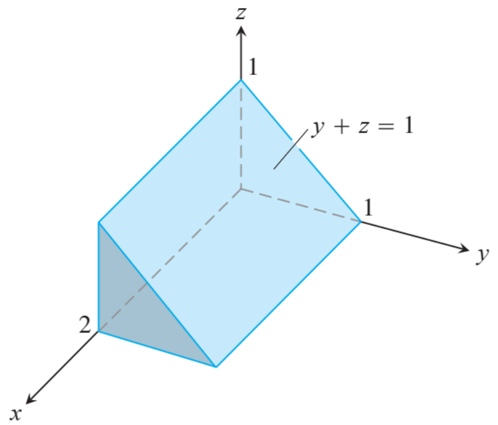


Figure 4.13.

The integrals in parts (a), (d), (e), and (f) also give $V=1$.

Properties of Triple integrals: If $F=F(x,y,z)$ and $G=G(x,y,z)$ are continuous, then

1. *Constant Multiple:* $\iiint\limits_D kF\,dV=k\iiint\limits_D F\,dV$ (any number k)

2. *Sum and Difference:* $\iiint\limits_D (F\pm G)\,dV=\iiint\limits_D F\,dV\pm\iiint\limits_D G\,dV$

3. *Domination:*
   (a) $\iiint\limits_D F\,dV\geq 0$ if $F\geq 0$ on D
   (b) $\iiint\limits_D F\,dV\geq\iiint\limits_D G\,dV$ if $F\geq G$ on D

4. *Additivity:* $\iiint\limits_D F\,dV=\iiint\limits_{D_1} E\,dV+\iiint\limits_{D_2} F\,dV$

if D is the union of two nonoverlapping regions and D_1 and D_2.

4.5 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied in Section 4.3.

Integration in Cylindrical Coordinates: We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z) as shown in Figure 4.14.

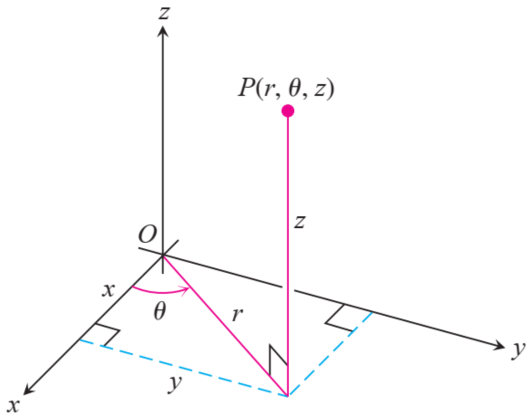


Figure 4.14.

Cylindrical Coordinates: Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane

2. z is the rectangular vertical coordinate.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical Coordinates:

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad z = z, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Figure 4.15). The z-axis is given by $r = 0$. The equation $\theta = \theta_o$ describes the plane that contains the z-axis and makes an angle θ_o with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_o$ describes a plane perpendicular to the z-axis.

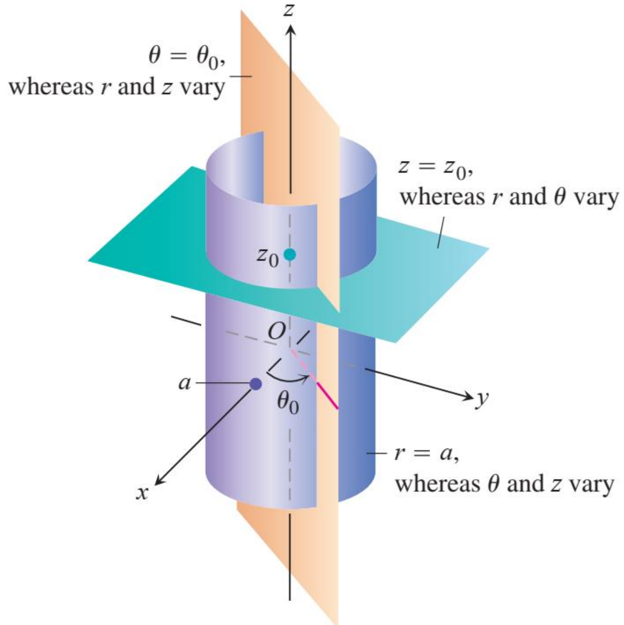


Figure 4.15.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$r = 4$. Cylinder, radius 4, axis the z-axis

$\theta = \dfrac{\pi}{3}$. Plane containing the z-axis

$z = 2$. Plane perpendicular to the z-axis

The triple integral of a function f over D is

$$\iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta\,.$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.

Example1: Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution: The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y-1)^2 = 1 \;\Rightarrow\; x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r\sin\theta = 0 \;\Rightarrow\; r = 2\sin\theta\,.$$

The region is sketched in Figure 4.16.

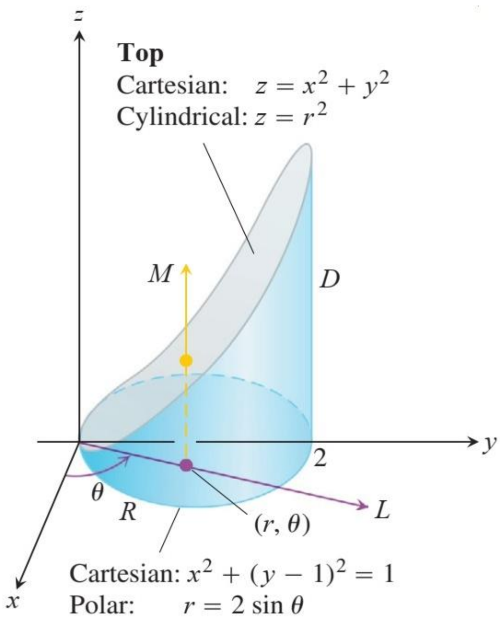


Figure 4.16.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r,θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next, we find the r-limits of integration. A ray L through (r,θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally, we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

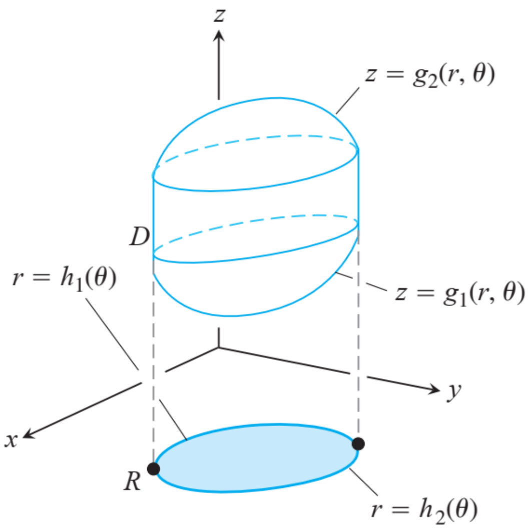
$$\iiint_D f(r,\theta,z)\, dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\, dz\, r\, dr\, d\theta\,.$$

How to Integrate in Cylindrical Coordinates: To evaluate

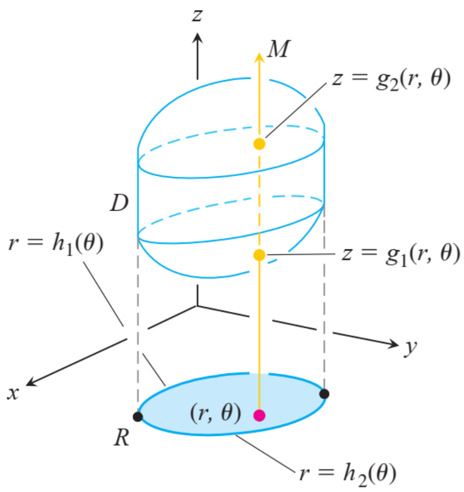
$$\iiint_D f(r,\theta,z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

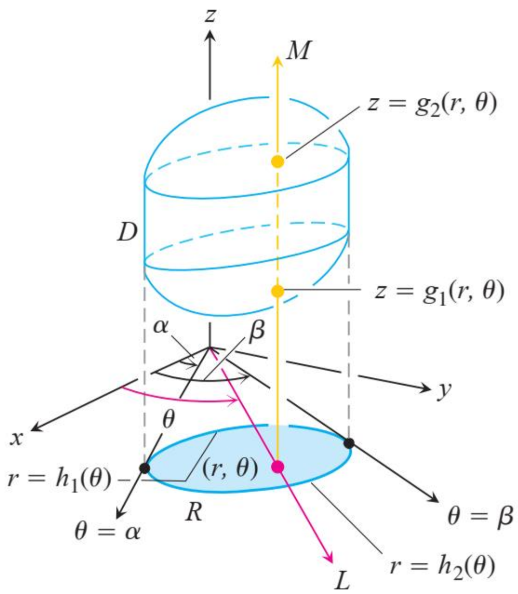
1. Sketch: Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. Find the z-limits of integration: Draw a line M through a typical point (r,θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r,\theta)$ and leaves at $z = g_2(r,\theta)$ These are the z-limits of integration.



3. Find the r-limits of integration: Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4. Find the θ-limits of integration: As L sweeps across R, the angle it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example2: Find limits of the integrations of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution: We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 4.17). Its base R is the disk $0 \le r \le 2$ in the xy-plane.

To find the limits of integration, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$.

$$\int_0^{2\pi}\int_0^{2}\int_0^{r^2} f(r,\theta,z)\, dz\, r\, dr\, d\theta.$$

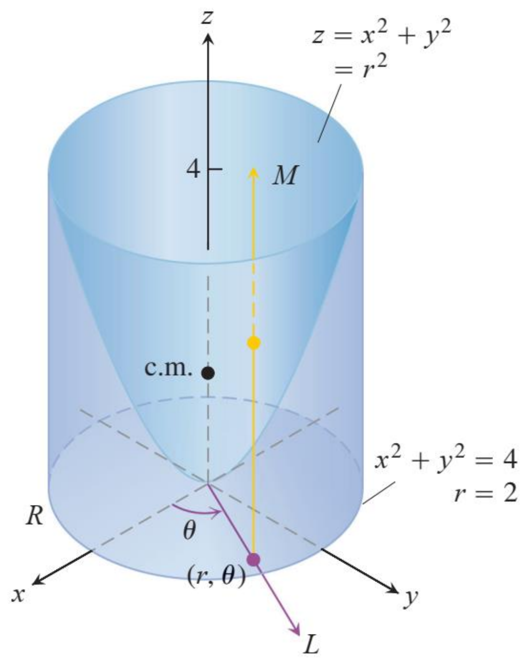


Figure 4.17.

Spherical Coordinates and Integration: Spherical coordinates locate points in space with two angles and one distance, as shown in Figure 4.18. The first coordinate, $\rho = |\overrightarrow{OP}|$ is the point's distance from the origin. Unlike r, the variable ρ is never negative. The second coordinate, ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0,\pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

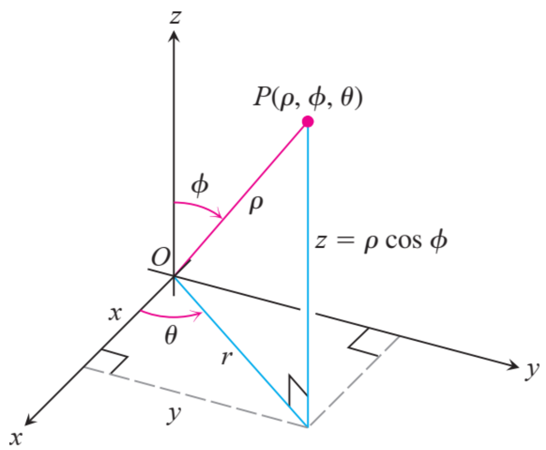


Figure 4.18.

Spherical Coordinates: Spherical coordinates represents a point P in space by ordered triples in which

1. ρ is the distance from P to the origin.

2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.

3. θ is the angle from cylindrical coordinates.

The equation $\rho = a$ describes the sphere of radius a centred at the origin (Figure 4.19). The equation $\phi = \phi_o$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_o is greater than $\pi/2$ the cone $\phi = \phi_o$ opens downward. The equation $\theta = \theta_o$

describes the half-plane that contains the z-axis and makes an angle θ_o with the positive x-axis.

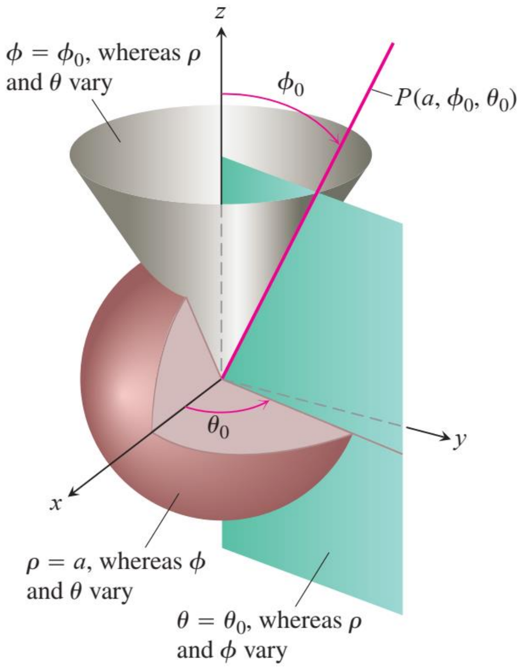


Figure 4.19.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates:

$$r = \rho \sin\phi\,, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$

$$z = \rho\cos\phi\,, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}\,.$$

Example3: Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

Solution: See Figure 4.20.

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\rho^2\sin^2\phi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2\underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi.$$

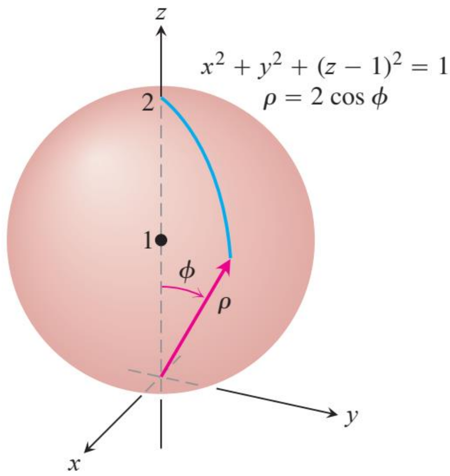


Figure 4.20.

Example4: Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ (Figure 4.21).

Solution: $z = \sqrt{x^2 + y^2}$

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$$

$$\rho\cos\phi = \rho\sin\phi$$

$$\cos\phi = \sin\phi$$

$$\phi = \frac{\pi}{4}.$$

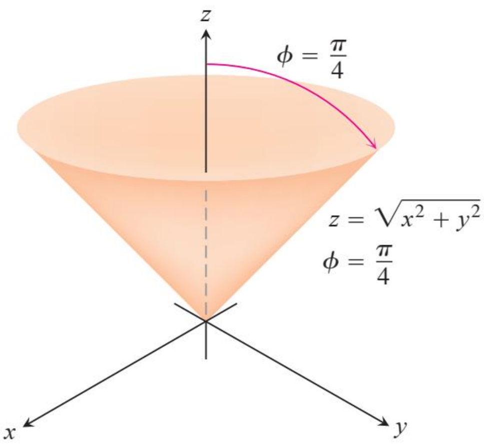


Figure 4.21.

Spherical coordinates are good for describing spheres centred at origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$\rho = 4$	Sphere, radius 4, centre at origin
$\phi = \dfrac{\pi}{3}$	Cone opening up from the origin, making an angle of $\pi/3$ radians with positive z-axis
$\theta = \dfrac{\pi}{3}.$	Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with positive x-axis

The triple integral of a function F over D is

$$\iiint\limits_D F(\rho,\phi,\theta)\, dV = \iiint\limits_D F(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

In spherical coordinates, we have

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

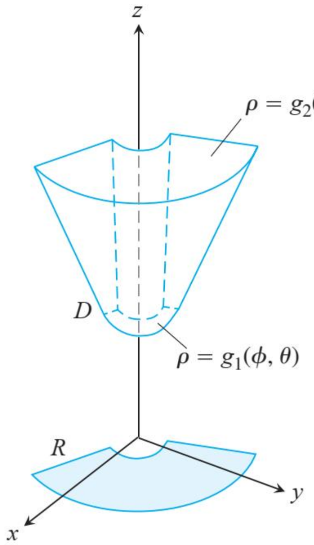
To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is shown below. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

How to Integrate in Spherical Coordinates: To evaluate

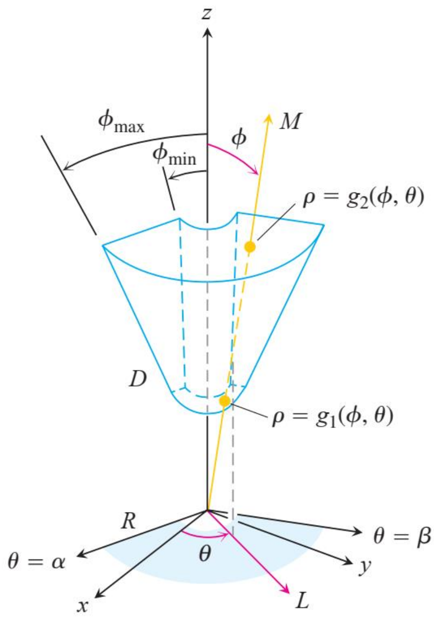
$$\iiint\limits_D F(\rho,\phi,\theta)\, dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ then with respect to ϕ and finally with respect to θ take the following steps.

1. Sketch: Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2. Find the ρ-limits of integration: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



3. Find the ϕ-limits of integration: For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are ϕ-limits the of integration.

4. Find the θ-limits of integration: The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi_{\min}}^{\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example5: Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

Solution: The volume is $\iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure 4.22).

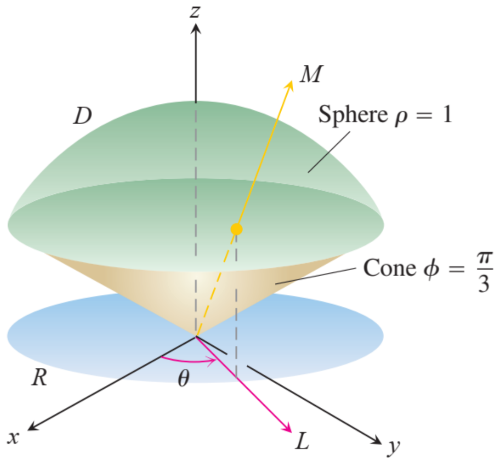


Figure 4.22.

The ρ-limits of integration. We draw a ray M from the origin through D making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ-limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

The θ-limits of integration. The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

Coordinate Conversion Formulas:

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Solved Problems:

4.1 Double Integrals

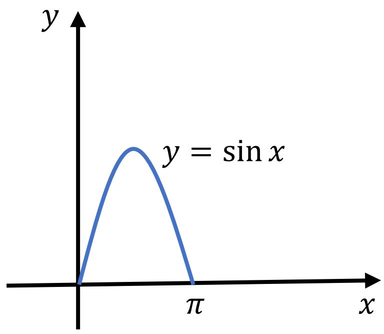
Prob1: Sketch the region of integration and evaluate the integral

$$\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx.$$

Solution: $\displaystyle\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx$

$$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}.$$



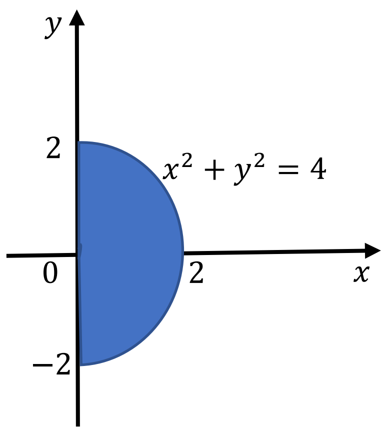
Prob2: Integrate f over the given region. $f(x,y) = x/y$ over the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1, x = 2$.

Solution: $\displaystyle\int_1^{2}\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^{2}[x\ln y]_x^{2x}dx = (\ln 2)\int_1^{2} x\,dx = \frac{3}{2}\ln 2\,.$

Prob3: Sketch the region of integration and write an equivalent double integral with the order of integration reversed

$$\int_0^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\,dy\,dx\,.$$

Solution: $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} 6x\,dx\,dy\,.$

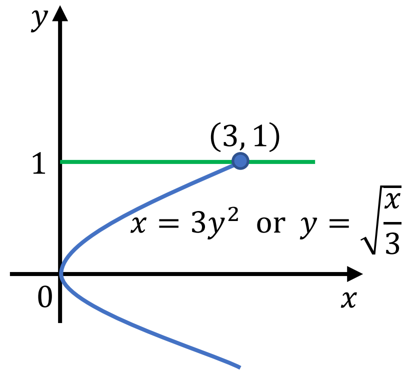


Prob4: Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{3}\int_{\sqrt{x/3}}^{1} e^{y^3}\,dy\,dx\,.$$

Solution: $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy$

$$= \int_0^1 3y^2 e^{y^3}\, dy = \left[e^{y^3}\right]_0^1 = e - 1.$$



Prob5: Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by lines $y = x, x = 0$, and $x + y = 2$ in the xy-plane.

Solution: $V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx$

$$= \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1 = \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}.$$

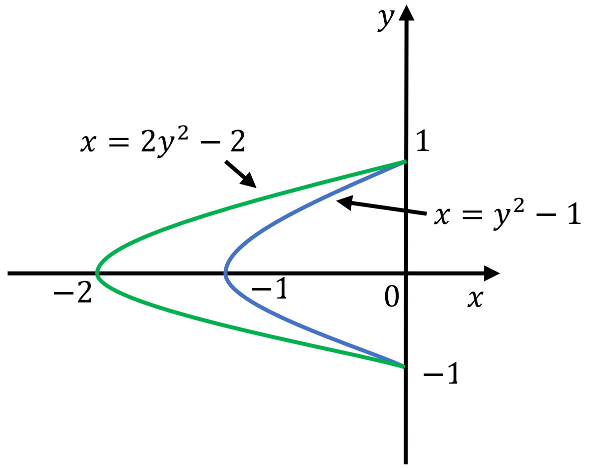
4.2 Area

Prob1: Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral. The parabola $x = y^2 - 1$ and $x = 2y^2 - 2$.

Solution: $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\, dy$

$$= \int_{-1}^1 (y^2 - 1 - 2y^2 + 2)\, dy$$

$$= \int_{-1}^1 (1 - y^2)\, dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}.$$



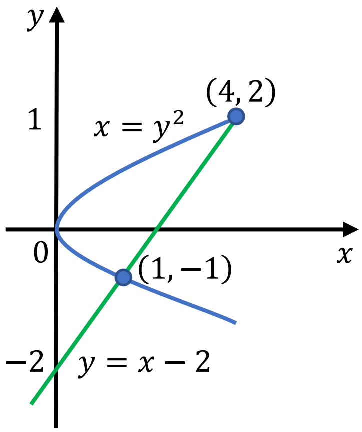
Prob2: Sketch the region, label bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

$$\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy\,.$$

Solution:

$$\int_{-1}^{2}\int_{y^2}^{y+2} dx\,dy = \int_{-1}^{2}(y+2-y^2)\,dy = \left[\frac{y^2}{2}+2y-\frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left(2+4+\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right) = 5-\frac{1}{2}=\frac{9}{2}.$$



4.3 Double Integrals in Polar Form

Prob1: Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx\,.$$

Solution: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = 4\int_{0}^{\pi/2}\int_{0}^{1}\frac{2r}{(1+r^2)^2}\,dr\,d\theta = 4\int_{0}^{\pi/2}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta$

$$= 2\int_{0}^{\pi/2} d\theta = \pi\,.$$

Prob2: Find the area of the region cut from the first quadrant by the cardioid $r = 1+\sin\theta$.

Solution: $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_{0}^{\pi/2}\left(\frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{8}+1\,.$

4.4 Triple Integrals in Rectangular Coordinates

Prob1: Let D be the region bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\,dx\,dy$ and $dz\,dy\,dx$ that give the volume of D.

Solution: The projection of D onto the xy-plane has the boundary

$$x^2+y^2 = 2y \quad\Rightarrow\quad x^2+(y-1)^2 = 1\,,$$

which is a circle. Therefore, the two integrals are:

$$\int_{0}^{2}\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}}\int_{x^2+y^2}^{2y} dz\,dx\,dy \quad \text{and} \quad \int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_{x^2+y^2}^{2y} dz\,dy\,dx\,.$$

Prob2: Evaluate the integral

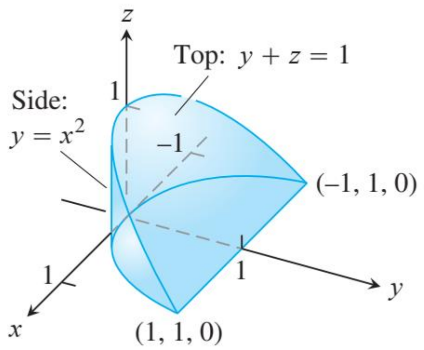
$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\, dz\, dy\, dx\,.$$

Solution: $\displaystyle\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x^2} x(1-x^2-y)\, dy\, dx$

$$= \int_0^1 x\left[(1-x^2)^2 - \frac{1}{2}(1-x^2)\right] dx = \int_0^1 \frac{1}{2}x(1-x^2)^2\, dx$$

$$= \left[-\frac{1}{12}(1-x^2)^3\right]_0^1 = \frac{1}{12}.$$

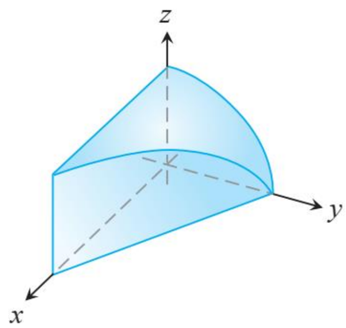
Prob3: Write the integral as iterated integral in the order **(a)** $dz\,dy\,dx$ **(b)** $dx\,dz\,dy$ of the region shown in Figure below.



Solution: **(a)** $\displaystyle\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} dz\, dy\, dx\,.$

(b) $\displaystyle\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy\,.$

Prob4: Find the volume of the region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$.



Solution:

$$\int_0^4 \int_0^{\left(\sqrt{16-y^2}\right)/2} \int_0^{4-y} dx\, dz\, dy = \int_0^4 \int_0^{\left(\sqrt{16-y^2}\right)/2} (4-y)\, dz\, dy$$

$$= \int_0^4 \frac{\sqrt{16-y^2}}{2}(4-y)\, dy = \int_0^4 2\sqrt{16-y^2}\, dy - \frac{1}{2}\int_0^4 y\sqrt{16-y^2}\, dy$$

$$= \left[y\sqrt{16-y^2} + 16\sin^{-1}\frac{y}{4}\right]_0^4 + \left[\frac{1}{6}(16-y^2)^{3/2}\right]_0^4$$

$$= 16\left(\frac{\pi}{2}\right) - \frac{1}{6}(16)^{\frac{3}{2}} = 8\pi - \frac{32}{3}.$$

Prob5: Evaluate the integral by changing the order of integration in an appropriate way

$$\int_0^1\int_0^1\int_{x^2}^1 12\,xze^{zy^2}dy\,dx\,dz\,.$$

Solution: $\displaystyle\int_0^1\int_0^1\int_{x^2}^1 12\,xze^{zy^2}dy\,dx\,dz=\int_0^1\int_0^1\int_0^{\sqrt{y}} 12\,xze^{zy^2}dx\,dy\,dz=\int_0^1\int_0^1 6yze^{zy^2}dy\,dz$

$$=\int_0^1\left[3e^{zy^2}\right]_0^1 dz=3\int_0^1(e^z-z)dz=3[e^z-1]_0^1=3e-6.$$

4.5 Triple Integrals in Cylindrical and Spherical Coordinates

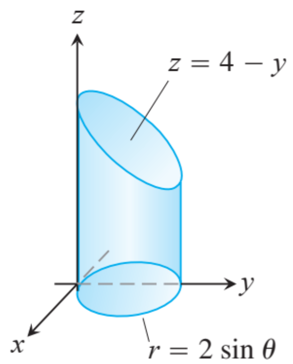
Prob1: Evaluate the cylindrical coordinate integral

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta\,.$$

Solution: $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[r(2-r^2)^{\frac{1}{2}}-r^2\right]dr\,d\theta=\int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{\frac{3}{2}}-\frac{r^3}{3}\right]_0^1 d\theta$

$$=\int_0^{2\pi}\left(\frac{2^{\frac{2}{3}}}{3}-\frac{2}{3}\right)d\theta=\frac{4\pi\left(\sqrt{2}-1\right)}{3}.$$

Prob2: Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,r\,dr\,d\theta$ over the given region D. D is the right circular cylinder whose base is circle $r=2\sin\theta$ in the xy-plane and whose top lies in the plane $z=4-y$.



Solution: $\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta\,.$

Prob3: Evaluate the spherical coordinate integral

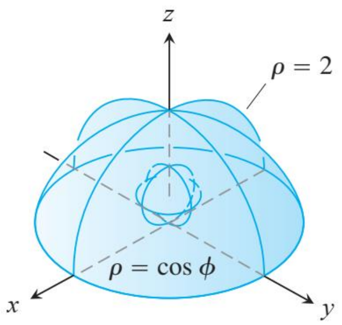
$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\,.$$

Solution: $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\;d\phi\,d\theta$

$$=\frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi}+\frac{3}{4}\int_0^{\pi}\sin^2\phi\;d\phi\right)d\theta$$

$$= 2\int_0^{\pi}\int_0^{\pi} \sin^2\phi \; d\phi \, d\theta = \int_0^{\pi} \left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi} d\theta = \int_0^{\pi} \pi \, d\theta = \pi^2 .$$

Prob4: Find the spherical coordinate limits for the integral that calculates the volume of the given solid and then evaluate the integral. The solid between the sphere $\rho = \cos\phi$ and hemisphere $\rho = 2$, $z \geq 0$.



Solution: $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} (8 - \cos^3\phi)\sin\phi \; d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi} \left[-8\cos\phi + \frac{\cos^4\phi}{4}\right]_0^{\pi/2} d\theta = \frac{1}{3}\int_0^{2\pi} \left(8 - \frac{1}{4}\right) d\theta$$

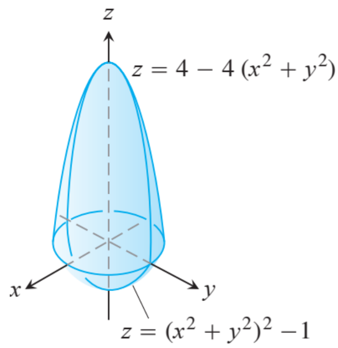
$$= \left(\frac{31}{12}\right)(2\pi) = \frac{31\pi}{6}.$$

Prob5: Find the volume of the portion of the solid sphere $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

Solution: $\displaystyle V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3} \frac{a^3}{3}\sin\phi \; d\phi \, d\theta = \frac{a^3}{3}\int_0^{2\pi} [-\cos\phi]_{\pi/3}^{2\pi/3} \, d\theta$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\right) d\theta = \frac{2\pi a^3}{3}.$$

Prob6: Find the volume of the solid



Solution: $\displaystyle V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta$

$$= 4\int_0^{\pi/2}\int_0^{1} (5r - 4r^3 - r^5) \, dr \, d\theta = 4\int_0^{\pi/2} \left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta$$

$$= 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}.$$