1. Fundamental electric constant

The most fundamental unit of charge is the magnitude of the charge of an electron or a proton, which is denoted by e

$$e = 1.602176634 \times 10^{-19}\ \text{C}$$

ε_0 is the permittivity

$$\varepsilon_0 = 8.854187817 \times 10^{-12}\ \text{C}^2/\text{Nm}^2$$

$$k = \frac{1}{4\pi\varepsilon_0} = \frac{c^2}{10^7} = 8.98755 \times 10^9\ \text{Nm}^2/\text{C}^2$$

2. Electric field

The electric force $\boldsymbol{F}_0$ on a test charge (q_0) is exerted by the electric field ($\boldsymbol{E}$) created by other charged body. We define the electric field $\boldsymbol{E}$ at a point as the electric force $\boldsymbol{F}_0$ experienced by a test charge q_0 at the point,

$$\boldsymbol{E} = \frac{\boldsymbol{F}_0}{q_0}$$

When we use the unit positive charge $q_0 = 1$ C, $\boldsymbol{E}$ is equivalent to $\boldsymbol{F}_0$.

3. Electric field lines

An electric field line is an imaginary line or curve drawn through a region of space so that its tangent at any point is in the direction of the electric field vector ($\boldsymbol{E}$) at that point. The electric field lines show the direction of $\boldsymbol{E}$ at each point, and their spacing gives a general idea of the magnitude of $\boldsymbol{E}$ at each point. When E is strong, we draw lines bunched closely together; where $\boldsymbol{E}$ is weaker, they are father apart. At any particular point, the electric field has a unique direction, so only one field line can pass through each point of the field. In other words, *field lines never intersect.*

In summary,

(1) The electric field vector is tangent to the electric field line at each point.

(2) The number of electric field lines per unit area through a surface that is perpendicular to the lines is proportional to the magnitude of the electric field in that region.

4. Electric field due to a point charge

Find an electric field due to a point charge q at any point a distance r from the point charge. From the Coulomb's law, the force acting on a test charge q_0 is

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\boldsymbol{e}_r$$

The electric field $\boldsymbol{E}$ is defined by

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\boldsymbol{e}_r$$

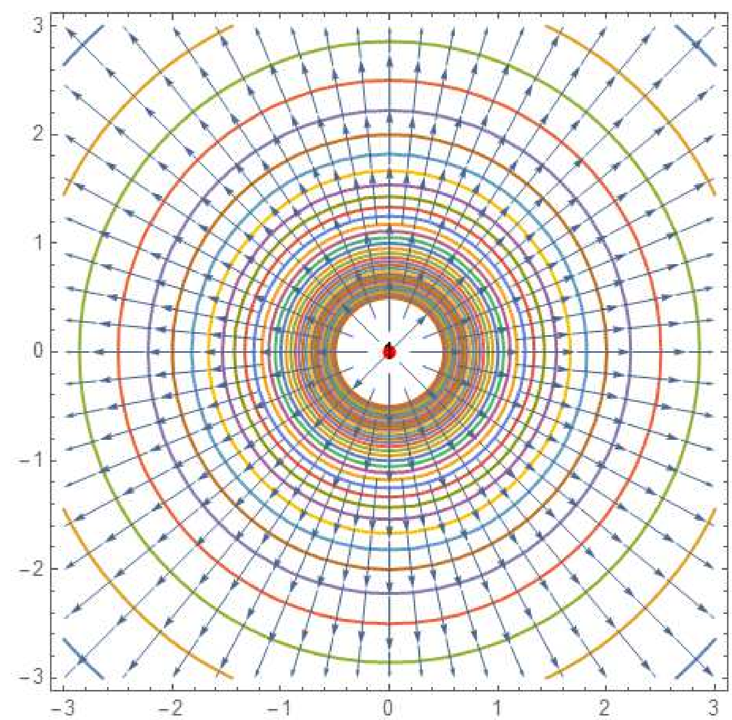
The unit of the electric field is N/C.

$$\frac{N}{C} = \frac{(J/m)}{A \cdot s} = \frac{W \cdot s/m}{A \cdot s} = \frac{W}{A \cdot m} = \frac{V \cdot A}{A \cdot m} = \frac{V}{m}$$

where $C = A \cdot s$. In other word, we have N/C = V/m as a unit of the electric field.

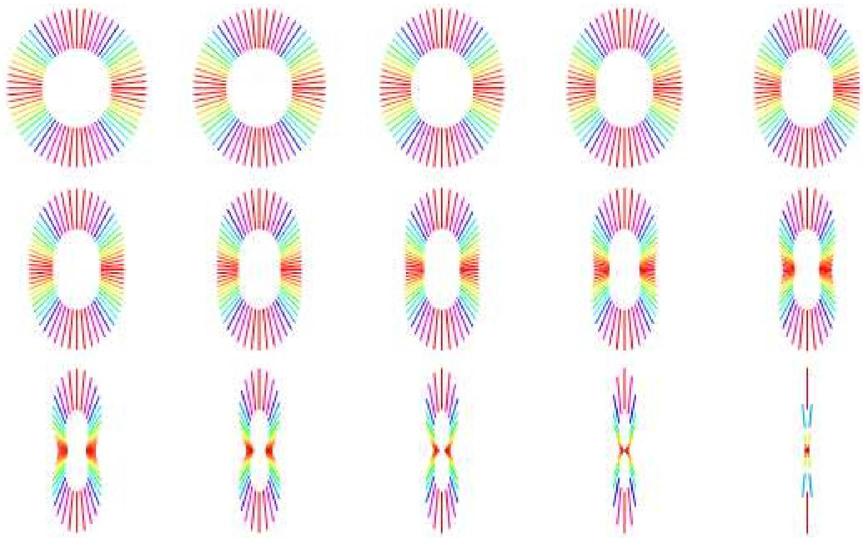
(a) Electric field lines due to a positive charge

The arrows show the direction of the electric field at each point. The same electric potential (V) is denoted by contour surfaces. The direction of electric field is perpendicular to the contour surfaces of constant V.



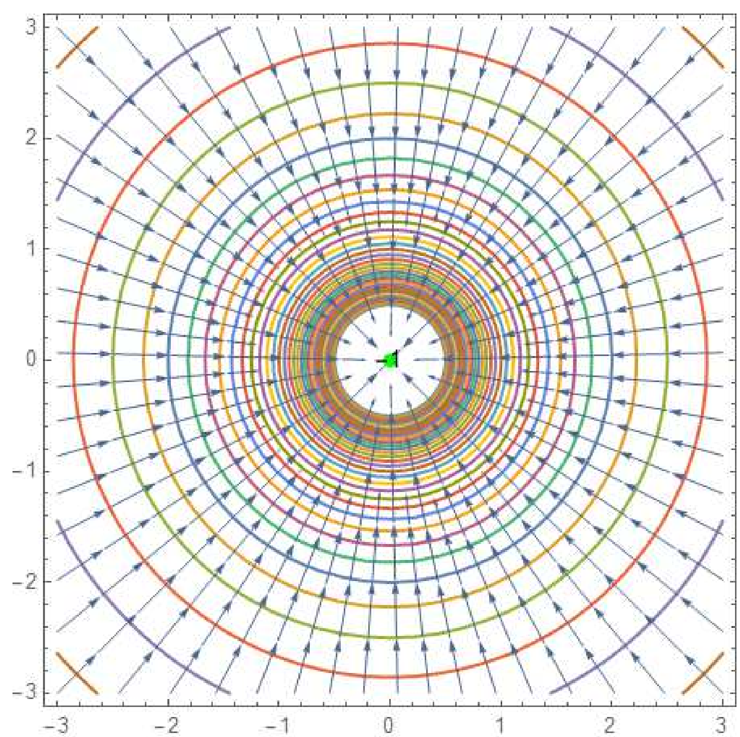
((**Note**))

The electric field of a moving charge along the x axis with the velocity v: $v/c = 0.29 – 0.99$, where $\Delta(v/c) = 0.05$ (he special relativity of electric field). c is the velocity of light.



(b) Electric field lines due to a negative charge

The arrows show the direction of the electric field at each point for the negative charge.

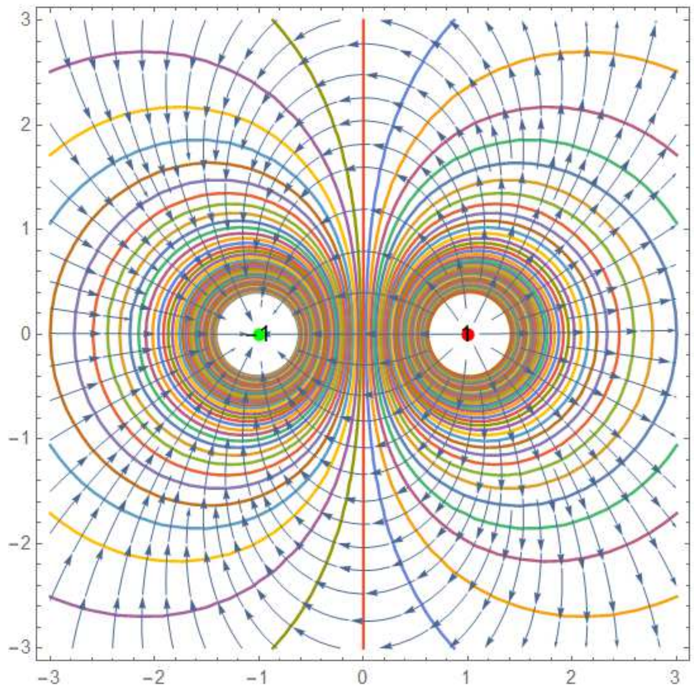


5. The superposition principle

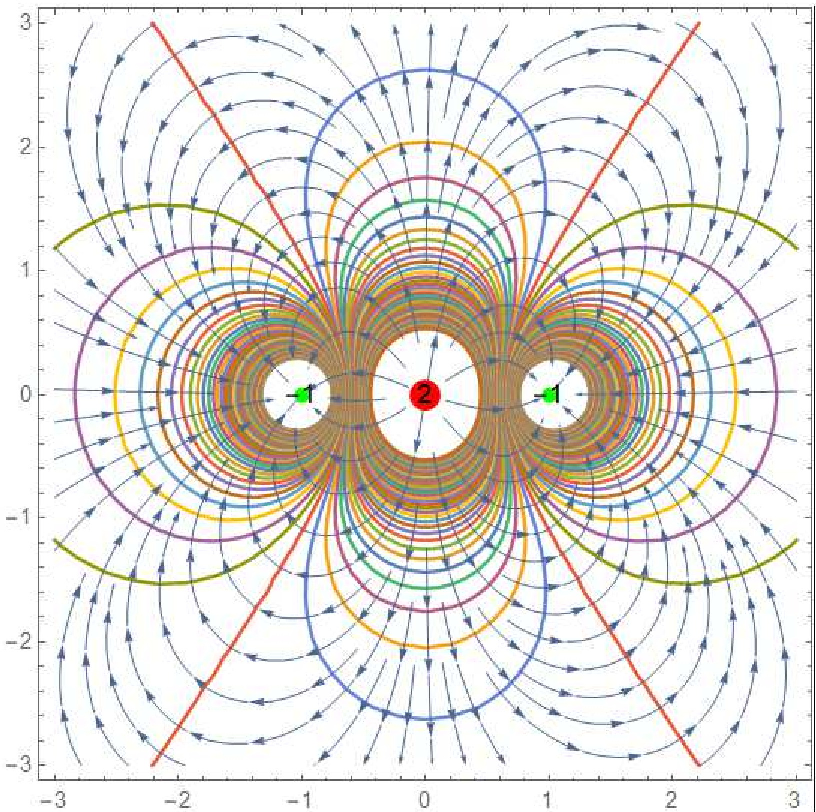
To find the field caused by a charge distribution, we imagine the distribution to be made of many point charges. At any given point P, each point charge produces its own electric field $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$, ….. The total electric field $\boldsymbol{E}$ at the point P is given by the vector sum of these electric fields,

$$\boldsymbol{E} = \sum_{i=1}^{n} \boldsymbol{E}_i$$

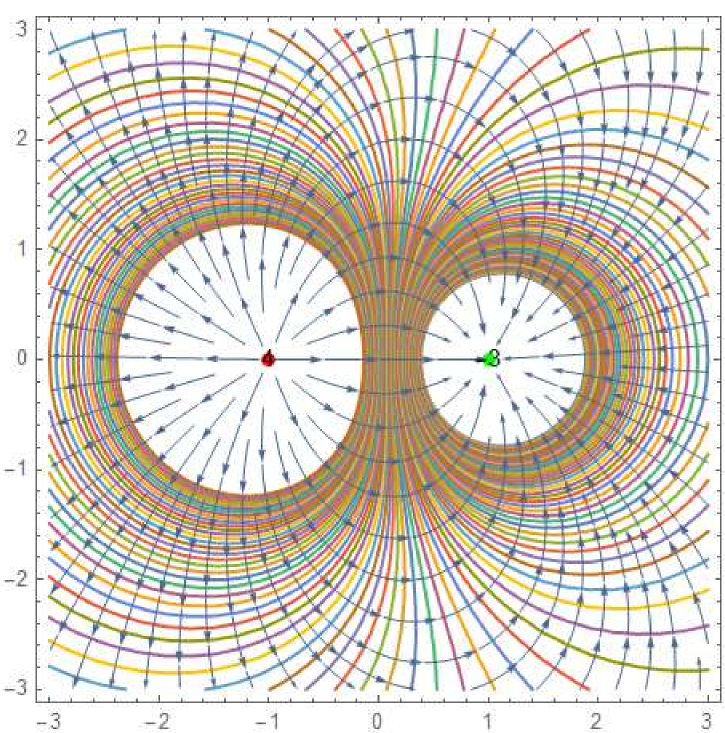
(a) Electric field lines due to a negative charge (-q) and positive charge (q) located on the x axis.



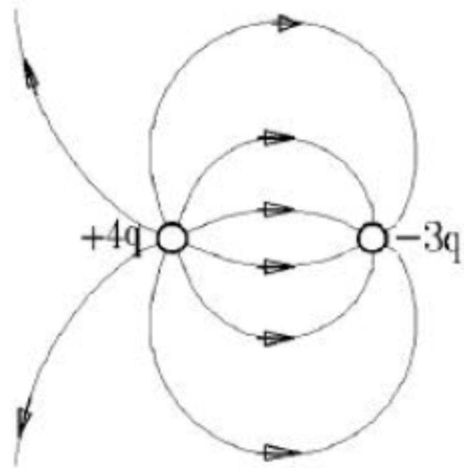
(b) Electric field lines due to a negative charge (-q) lying in the negative x direction, a positive charge ($2q$) at the center, and a negative charge (-q) lying in the positive x direction.



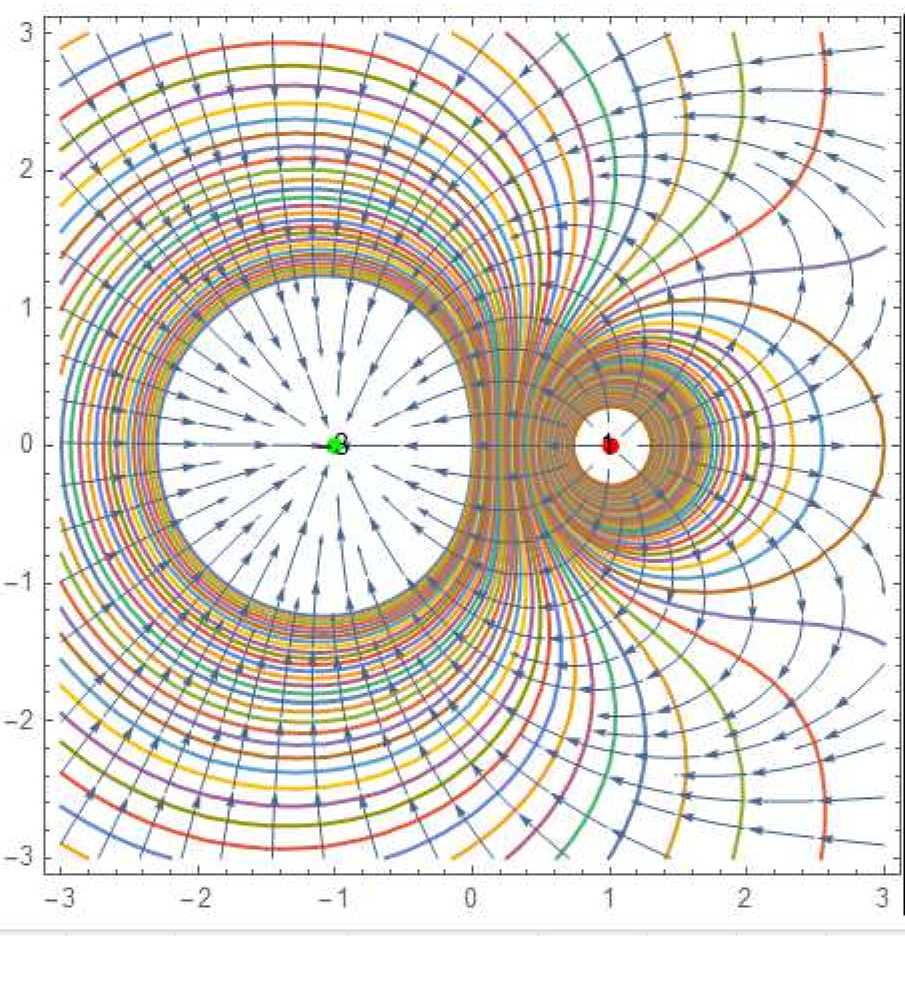
(c) Two charges $+4q$ and $-3q$ are separated by a small distance.



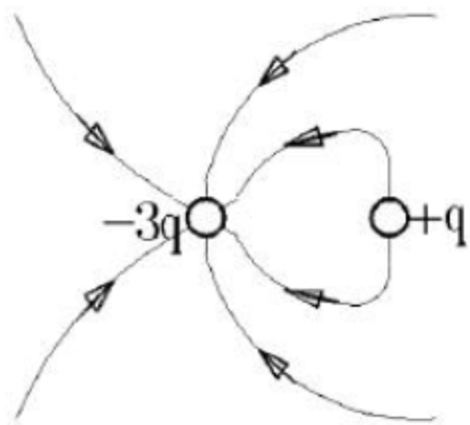
A sketch of the electric field lines is shown. We assign two lines per charge q.



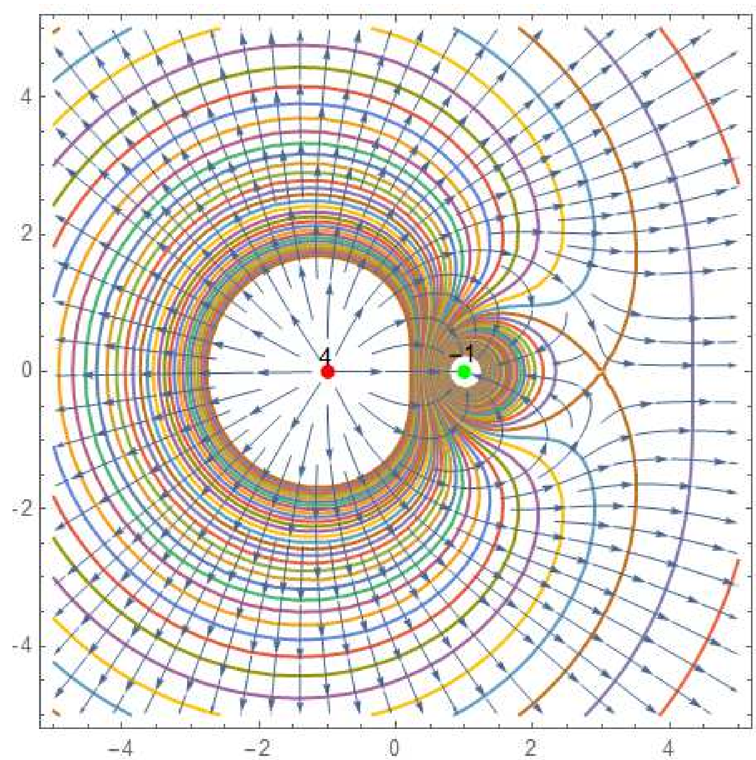
(d) Two charges -3q and q are separated by a small distance.



A sketch of the electric field lines is shown. We assign two lines per charge q.



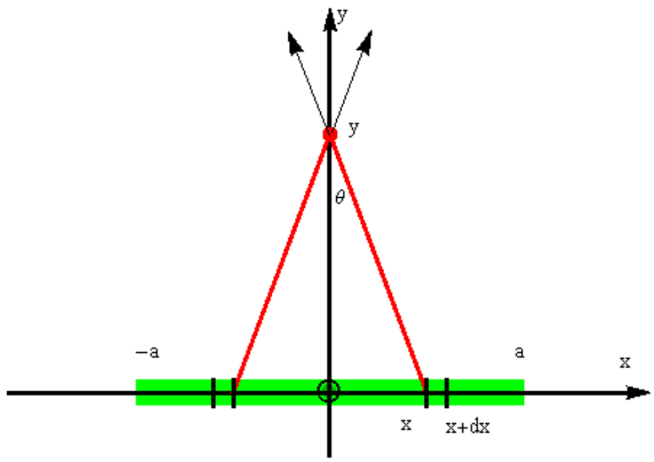
(e) Two charges $4q$ and $-q$ are separated by a small distance.



Note that $E_x = 0$ and $E_y = 0$ at the point (3,0).

6. Electric field of a line of charge

Positive charge Q is distributed uniformly along a line with length $2a$, lying along the x axis between $x = -a$ and $x = a$. Find the electric field at point P on the y axis at a distance y. The line charge density is given by λ.



From the symmetry the direction of the electric field is along the y axis.

$$
\begin{aligned}
dE_y &= 2\cos\theta \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{r^2} \\
&= 2\frac{1}{4\pi\varepsilon_0}\frac{y}{\sqrt{x^2+y^2}}\frac{\lambda dx}{\sqrt{x^2+y^2}^{\,2}} \\
&= \frac{Qy}{4\pi\varepsilon_0 a}\frac{dx}{(x^2+y^2)^{3/2}}
\end{aligned}
$$

where

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\cos\theta &= \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \\
\lambda &= \frac{Q}{2a}
\end{aligned}
$$

Then we have

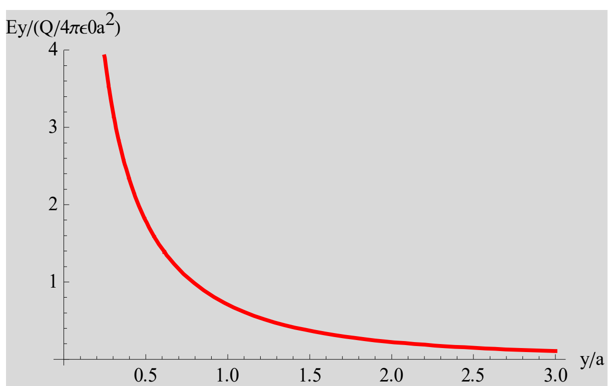
$$
\begin{aligned}
E_y &= \frac{Qy}{4\pi\varepsilon_0 a}\int_0^a \frac{dx}{(x^2+y^2)^{3/2}} \\
&= \frac{Qy}{4\pi\varepsilon_0 a}\frac{a}{y^2\sqrt{y^2+a^2}} \\
&= \frac{Q}{4\pi\varepsilon_0}\frac{1}{y\sqrt{y^2+a^2}} \\
&= \frac{Q}{4\pi\varepsilon_0 a^2}\frac{1}{\frac{y}{a}\sqrt{\left(\frac{y}{a}\right)^2+1}}
\end{aligned}
$$

or

$$
\frac{E_y}{\frac{Q}{4\pi\varepsilon_0 a^2}} = \frac{1}{\xi\sqrt{\xi^2+1}},
$$

where $\xi = \dfrac{y}{a}$. Here we make a plot of $E_y/(Q/4\pi\varepsilon_0 a^2)$ vs $\xi = y/a$.

Out[7]=



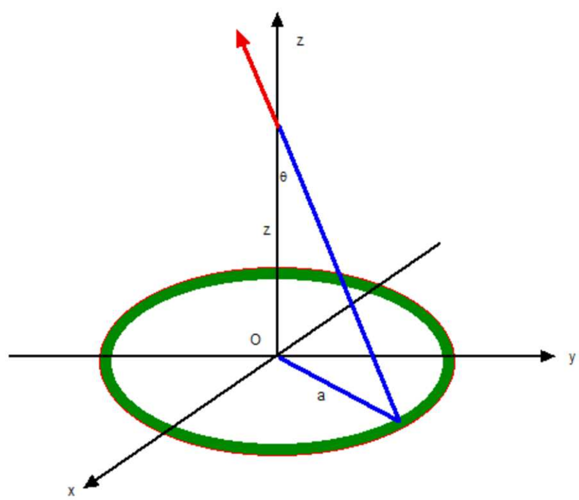
((**Mathematica**))

```
Clear["Global`*"];
Integrate[1/(x^2 + y^2)^(3/2), {x, 0, a},
 Assumptions → {a > 0, y > 0}]
```

$$\frac{a}{y^2 \sqrt{a^2 + y^2}}$$

7. Electric field of a ring charge

A ring-shaped conductor with radius *a*, carriers a total charge *Q* uniformly distributed around it. Find the electric field at a point P that lies on the axis of the ring at a distance *x* from its center.



From the symmetry, the direction of the electric field is along the *z* axis.

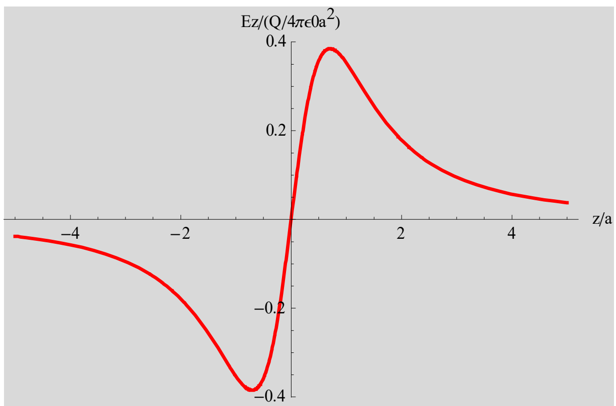
$$\begin{aligned} E_z &= \cos\theta \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{z}{\sqrt{z^2+a^2}}\frac{Q}{\sqrt{z^2+a^2}^{\,2}} \\ &= \frac{Q}{4\pi\varepsilon_0}\frac{z}{(z^2+a^2)^{3/2}} \\ &= \frac{Q}{4\pi\varepsilon_0 a^2}\frac{z/a}{(\frac{z^2}{a^2}+1)^{3/2}} \end{aligned}$$

or

$$\frac{\dfrac{E_z}{\dfrac{Q}{4\pi\varepsilon_0 a^2}}}{} = \frac{\xi}{(\xi^2+1)^{3/2}},$$

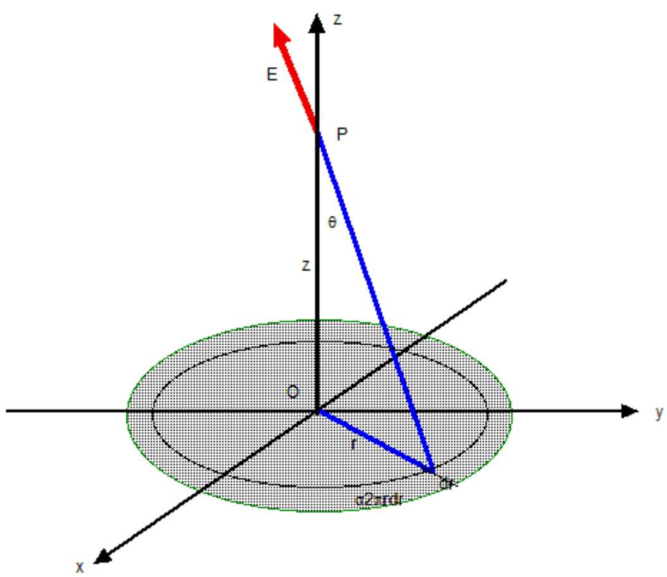
where $\xi = \dfrac{z}{a}$. Here we make a plot of $E_z/(Q/4\pi\varepsilon_0 a^2)$ vs $\xi = z/a$.

Out[9]=



8. Electric field of a uniformly charged disk

Find the electric field caused by a disk of radius R with a uniform positive surface charge density σ, at a point along the axis of the disk a distance z from its center. Assume that z is positive.



From the symmetry the direction of the electric field is along the z axis.

$$dE_z = \cos\theta \frac{1}{4\pi\varepsilon_0} \frac{\sigma(2\pi r)dr}{(z^2 + r^2)}$$
$$= \frac{1}{4\pi\varepsilon_0} \frac{z}{\sqrt{z^2 + r^2}} \frac{\sigma(2\pi r)dr}{(z^2 + r^2)}$$
$$= \frac{\sigma z}{2\varepsilon_0} \frac{rdr}{(z^2 + r^2)^{3/2}}$$

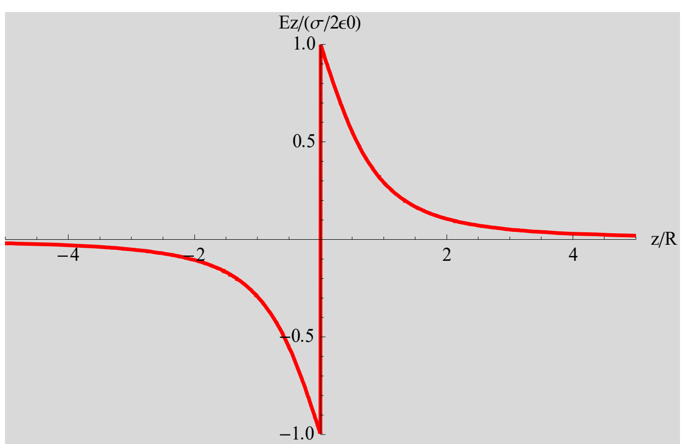
$$E_z = \frac{\sigma z}{2\varepsilon_0} \int_0^R \frac{rdr}{(z^2 + r^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}(1 - \frac{z}{\sqrt{z^2 + R^2}})$$

or

$$E_z = \frac{\sigma}{2\varepsilon_0}(1 - \frac{z/R}{\sqrt{\frac{z^2}{R^2} + 1}})$$

$$\frac{E_z}{\frac{\sigma}{2\varepsilon_0}} = (1 - \frac{\xi}{\sqrt{\xi^2 + 1}}),$$

where $\xi = z/R$. We make a plot of $E_z/(\sigma/2\varepsilon_0)$ vs $\xi = z/R$



Note that at $z = 0$, the following boundary condition is satisfied;

$$E_z(z = 0+) - E_z(z = 0-) = \frac{\sigma}{\varepsilon_0}.$$

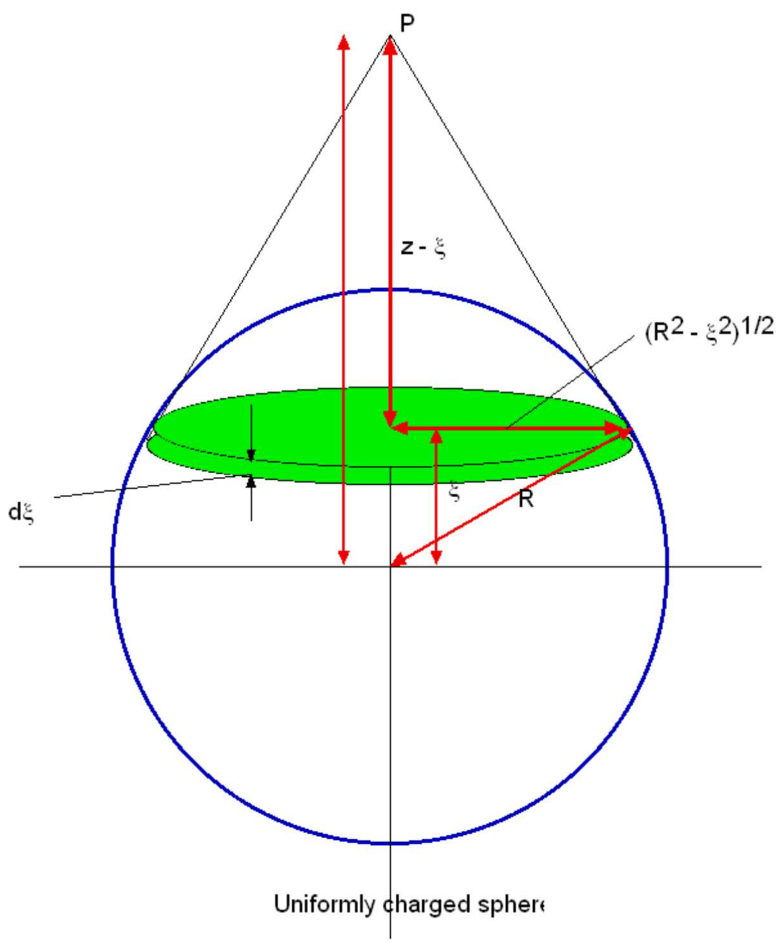
((**Mathematica**))

```
Clear["Global`*"];
Integrate[r/(z^2 + r^2)^(3/2), {r, 0, R},
 Assumptions → {R > 0, z > 0}]
```

$$\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}}$$

9. Electric field of a uniformly charged sphere

Find the electric field caused by a sphere of radius R with a uniform positive charge density ρ, at a point, a distance z from its center ($z>R$). Assume that z is positive.



Uniformly charged sphere

We use the result of Sec. 9.

$$dE_z = \frac{Q}{2\varepsilon_0(\pi R^2)}(1 - \frac{z}{\sqrt{z^2 + R^2}}),$$

where $\sigma = \frac{Q}{\pi R^2}$ and Q is the total surface charge.

Here we use

$$\begin{aligned}
&R \to \sqrt{R^2 - \xi^2} \\
&z \to (z - \xi) \\
&Q \to \pi(\sqrt{R^2 - \xi^2})^2 d\xi\rho
\end{aligned}$$

Then we have

$$\begin{aligned}
dE_z &= \frac{\pi(\sqrt{R^2 - \xi^2})^2 d\xi\rho}{2\varepsilon_0 \pi(\sqrt{R^2 - \xi^2})^2}(1 - \frac{z - \xi}{\sqrt{(z - \xi)^2 + R^2 - \xi^2}}) \\
&= \frac{\rho d\xi}{2\varepsilon_0}(1 - \frac{z - \xi}{\sqrt{z^2 - 2\xi z + R^2}})
\end{aligned}$$

where

$$Q_{sphere} = \rho\frac{4\pi R^3}{3}.$$

The resultant electric field is

$$\begin{aligned}
E_z &= \int_{-R}^{R} \frac{\rho d\xi}{2\varepsilon_0}(1 - \frac{z - \xi}{\sqrt{z^2 - 2\xi z + R^2}}) \\
&= \frac{\rho}{2\varepsilon_0}\frac{2R^3}{3z^2} \\
&= \frac{1}{2\varepsilon_0}\frac{2R^3}{3z^2}\frac{Q_{sphere}}{\frac{4\pi R^3}{3}} \\
&= \frac{Q_{sphere}}{4\pi\varepsilon_0 z^2}
\end{aligned}$$

((**Mathematica**))

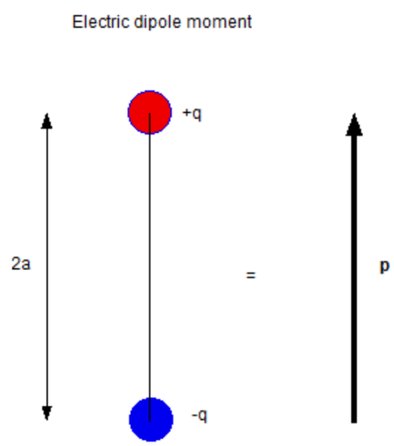
$$\int_{-R}^{R}\left(1-\frac{z-\xi}{\sqrt{z^2-2\,\xi\,z+R^2}}\right)d\xi\ //$$

```
Simplify[#, {z > R, R > 0}] &
```

$$\frac{2\,R^3}{3\,z^2}$$

10. Electric dipole moment

10.1 Definition of electric moment



In physics, the electric dipole moment (or electric dipole for short) is a measure of the polarity of a system of electric charges. In the above figure, $|\boldsymbol{r}| = 2a = d$ In the simple case of two point charges, one with charge $+q$ and one with charge $-q$, the electric dipole moment is:

$$\boldsymbol{p} = q\boldsymbol{r}$$

where $\boldsymbol{r}$ is the displacement vector pointing from the negative charge to the positive charge ($r = 2a = d$). This implies that the electric dipole moment vector points from the negative charge to the positive charge.

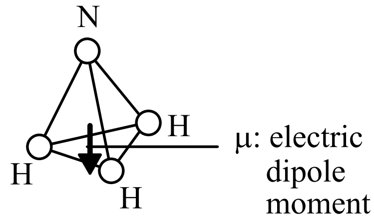
The dimension of electric dipole moment is

[p] = Cm

10.2 Example

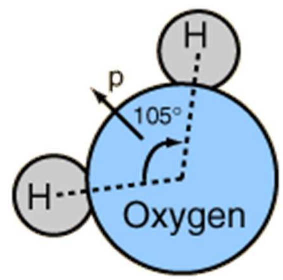
1. **NH_3 (ammonia)**

Ammonia has an electric dipole moment.



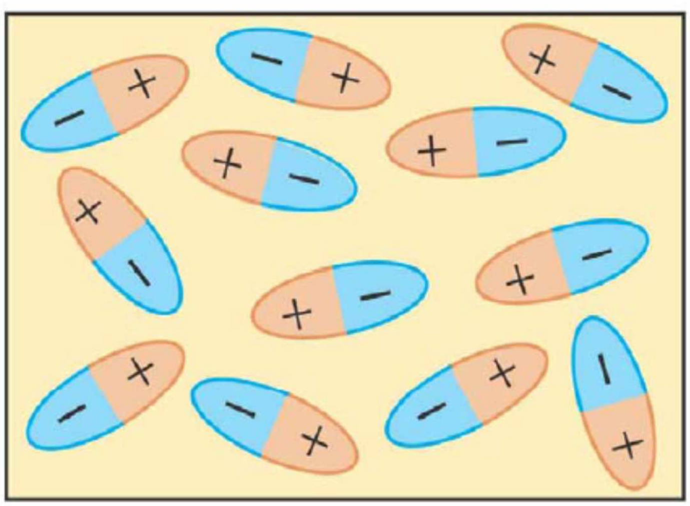
2. Water (H_2O)

The water molecule H_2O has a permanent polarization resulting from its nonlinear geometry



3. Polar molecules

The polar molecules are randomly oriented in the absence of an external electric field.



11. Electric field due to the electric dipole moment

11.1 Case-1 [Problem 22-19 (SP-22)]

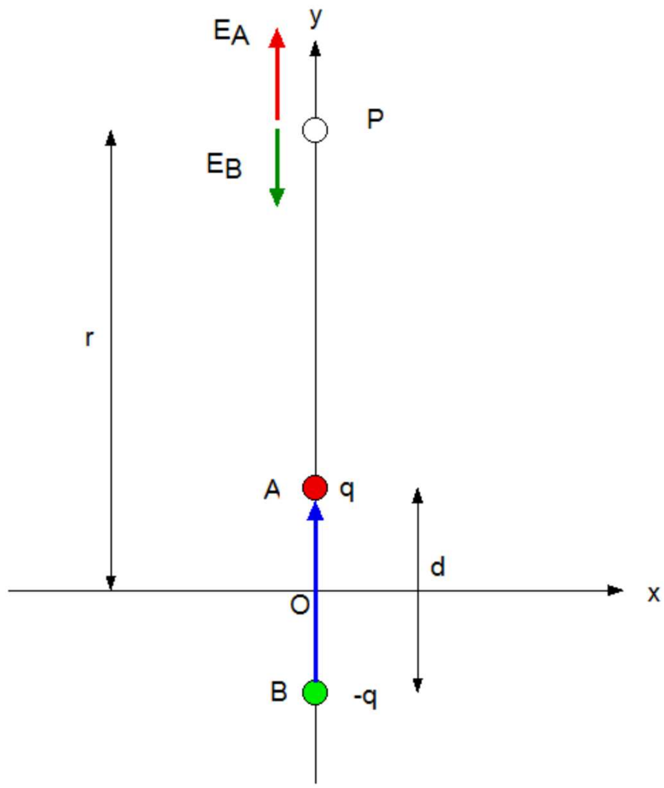


Figure shows an electric dipole. What are the (a) magnitude and (b) direction (relative to the positive direction of the *y* axis) of the dipole's electric field at point P, located at distance $r \gg d$?

$$E_A = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-\frac{d}{2})^2}$$

$$E_B = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r+\frac{d}{2})^2}$$

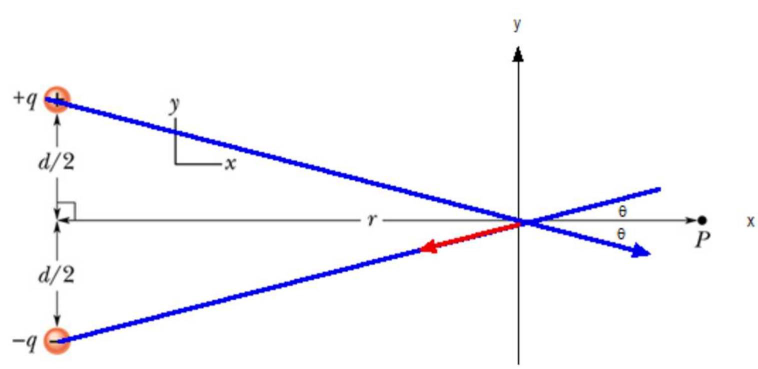
The resultant electric field along the *y* axis is

$$
\begin{aligned}
E_y &= E_A - E_B \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-\frac{d}{2})^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{(r+\frac{d}{2})^2} \\
&= \frac{q}{4\pi\varepsilon_0 r^2}[\frac{1}{(1-\frac{d}{2r})^2} - \frac{1}{(r+\frac{d}{2})^2}] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}[(1-\frac{d}{2r})^{-2} - (1+\frac{d}{2r})^{-2}] \\
&= \frac{q}{4\pi\varepsilon_0 r^2}[(1+\frac{d}{r}) - (1-\frac{d}{r})] \\
&= \frac{qd}{2\pi\varepsilon_0 r^3} = \frac{p}{2\pi\varepsilon_0 r^3}
\end{aligned}
$$

where $p = qd$.

11.2 Case-2

Figure shows an electric dipole. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the dipole's electric field at point P, located at distance $r>>d$?



The electric dipole moment is $p = qd$. From the symmetry, we have

$$E_y = \frac{-2q}{4\pi\varepsilon_0(r^2+\frac{d^2}{4})}\sin\theta, \qquad E_x = 0.$$

with

$$\sin\theta = \frac{\frac{d}{2}}{\sqrt{r^2 + \frac{d^2}{4}}}$$

$$E_y = \frac{-2q}{4\pi\varepsilon_0(r^2 + \frac{d^2}{4})} \frac{\frac{d}{2}}{\sqrt{r^2 + \frac{d^2}{4}}} = \frac{-qd}{4\pi\varepsilon_0(r^2 + \frac{d^2}{4})^{3/2}}$$

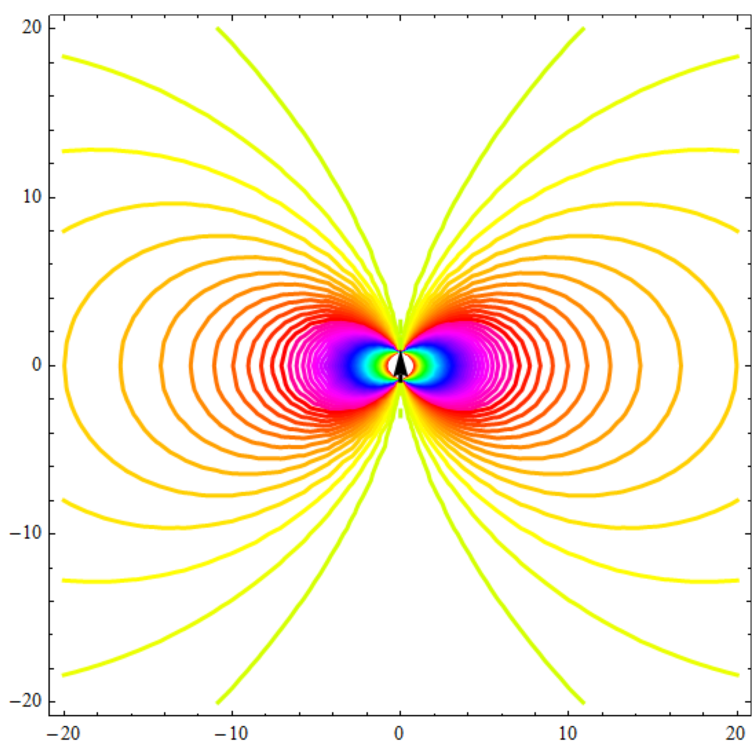
$$= \frac{-qd}{4\pi\varepsilon_0 r^3 (1 + \frac{d^2}{4r^2})^{3/2}}$$

$$= \frac{-qd}{4\pi\varepsilon_0 r^3}(1 + \frac{d^2}{4r^2})^{-3/2}$$

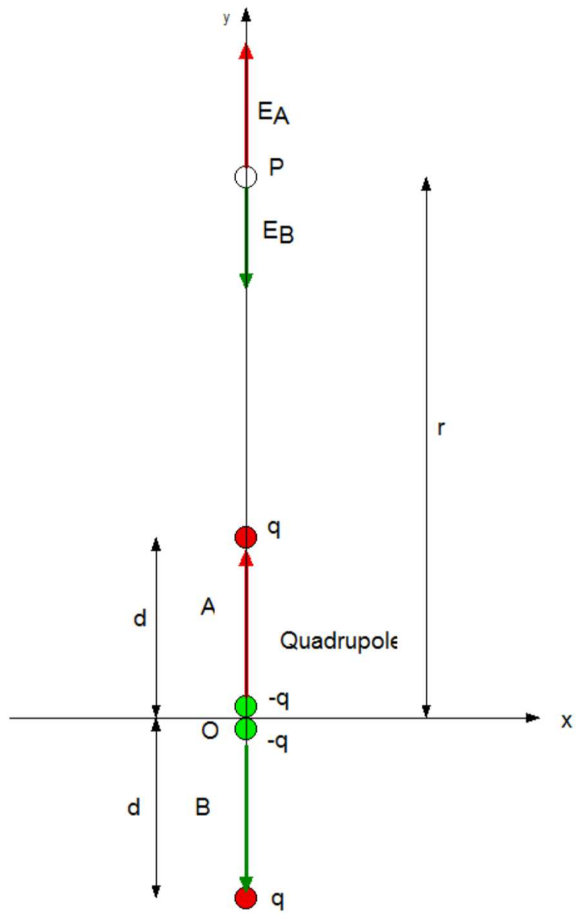
$$\approx \frac{-qd}{4\pi\varepsilon_0 r^3}(1 - \frac{3}{2}\frac{d^2}{4r^2})$$

$$\approx \frac{-qd}{4\pi\varepsilon_0 r^3} = \frac{-p}{4\pi\varepsilon_0 r^3}$$

The electric field line of the electric dipole moment [q = 1 at the point (0, 1) and q = -1 at the point (0, -1)] is schematically shown as follows.



11.3 The electric field line of electric quadrupole (SP 22-21)



This figure shows an electric quadrupole. It consists of two dipoles with the dipole moments that are equal in magnitude but opposite in direction. The value of E on the axis of the quadrupole for a point P a distance r from its center (assumed $r>>d$) is given by

$$E = \frac{3Q}{4\pi\varepsilon_0 r^4},$$

in which Q (= $2qd^2$) is known as the quadrupole moment of the charge distribution.

$$E_A = \frac{qd}{2\pi\varepsilon_0}\frac{1}{(r-\frac{d}{2})^3} = \frac{qd}{2\pi\varepsilon_0 r^3}\frac{1}{(1-\frac{d}{2r})^3}$$

$$E_B = -\frac{qd}{2\pi\varepsilon_0}\frac{1}{(r+\frac{d}{2})^3} = -\frac{qd}{2\pi\varepsilon_0 z^3}\frac{1}{(1+\frac{d}{2r})^3}$$

$$E_{tot} = E_A + E_B = \frac{qd}{2\pi\varepsilon_0 r^3}[\frac{1}{(1-\frac{d}{2r})^3} - \frac{1}{(1+\frac{d}{2r})^3}]$$

$$= \frac{qd}{2\pi\varepsilon_0 r^3}[6\left(\frac{d}{2r}\right) + 20\left(\frac{d}{2r}\right)^3 + ...]$$

$$\approx \frac{qd}{2\pi\varepsilon_0 r^3}\frac{3d}{r} + \frac{qd}{2\pi\varepsilon_0 r^3}20\frac{d^3}{8r^3}$$

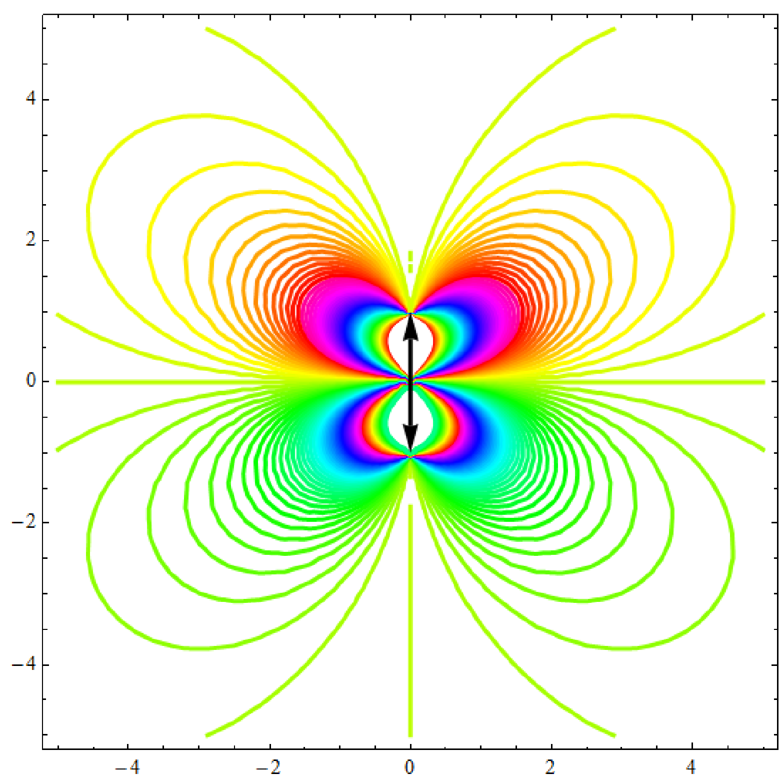
$$= \frac{3qd^2}{2\pi\varepsilon_0 r^4} + \frac{5qd^4}{4\pi\varepsilon_0 r^6}$$

We define Q as

$$Q = 2qd^2.$$

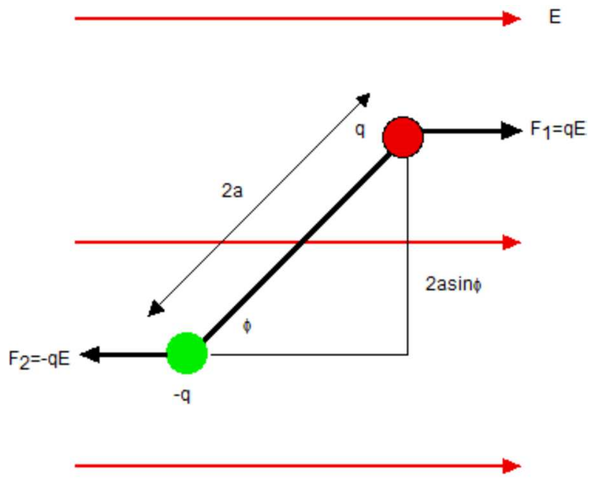
Then we have

$$E_{tot} = \frac{3\frac{Q}{2}}{2\pi\varepsilon_0 r^4} = \frac{3Q}{4\pi\varepsilon_0 r^4}$$



12. Torque

We consider the behavior of the electric dipole moment in the presence of an electric field,



An electric dipole is a pair of point charges with equal magnitude and opposite sign (a positive charge q and a negative charge $-q$) separated by a distance d (= $2a$). The electric dipole moment is defined by

$$p = 2aq\,.$$

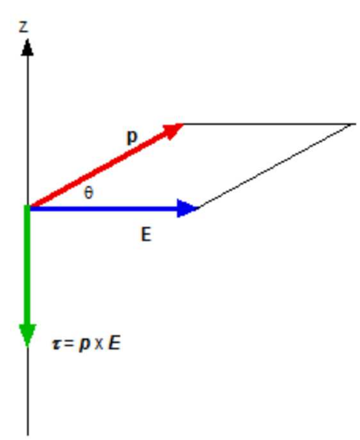
The unit of p is C m. The magnitude of the torque τ exerted by the field $\boldsymbol{E}$ is

$$\tau = (qE)(2a\sin\phi) = pE\sin\phi$$

In this example, the torque is directed toward the $-z$ axis (**into the paper, clock-wise**). Note that the direction of $\boldsymbol{p}$ x $\boldsymbol{E}$ is also the $-z$ axis. Then we have

$$\boldsymbol{\tau} = \boldsymbol{p}\times\boldsymbol{E}\,,$$

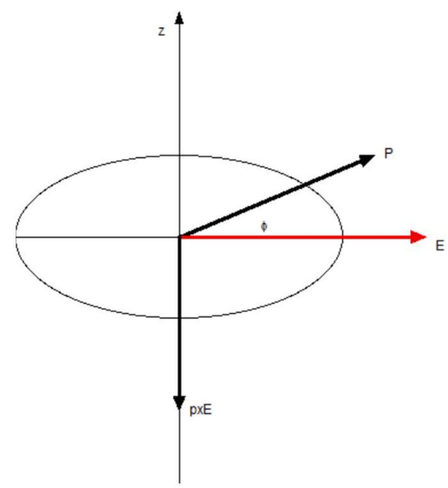
in a vector form.



Or more directly (mathematically), we can show

$$\boldsymbol{\tau} = (\boldsymbol{r} \times q\boldsymbol{E}) + (-\boldsymbol{r}) \times (-q\boldsymbol{E}) = (2q\boldsymbol{r}) \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E} .$$

13. Potential energy of electric dipole



The work done on the electric dipole moment $\boldsymbol{p}$ by the electric field $\boldsymbol{E}$ is given by

$$W = -\int \tau d\varphi = -pE\int^{\theta} \sin\varphi d\varphi = pE\cos\theta = \boldsymbol{p}\cdot\boldsymbol{E} .$$

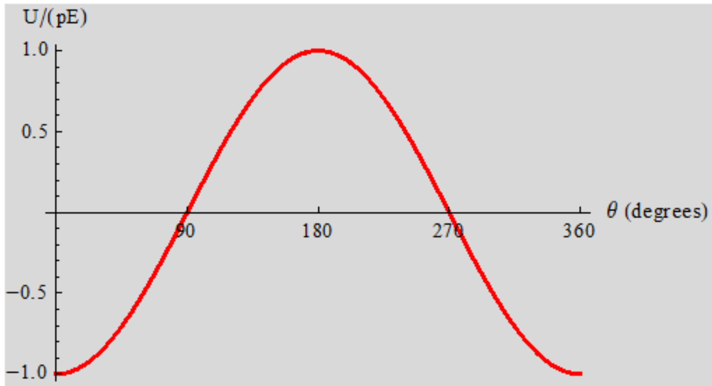
Note that (-τ) is the torque in the counterclockwise direction (+z direction). This torque (-τ) is in the direction of increasing ϕ. Since $\Delta U = -\Delta W$, we have the expression for the potential energy U (in units of J)

$$U = -\boldsymbol{p}\cdot\boldsymbol{E} .$$

((**Note-1**))
Work-energy theorem; $\Delta K = W = \boldsymbol{F}\cdot d\boldsymbol{r} = -dU$ or $\boldsymbol{F} = -\nabla U$.

((**Note-2**))
The kinetic energy work theorem; $\Delta K = W = -\Delta U$; or $\Delta(K+U) = 0$ (energy conservation law).

((**Note-3**))
The potential energy has its minimum value where $\boldsymbol{p}$ and $\boldsymbol{E}$ are parallel (θ = 0). The potential energy has its maximum value where $\boldsymbol{p}$ and $\boldsymbol{E}$ are antiparallel (θ = π)



((**Note-4**)) Using the electric potential (Chapter 24),

$$U = -qV(x, y, z) + qV(x + \Delta x, y + \Delta y, z + \Delta z)$$
$$= \frac{\partial V}{\partial x}(q\Delta x) + \frac{\partial V}{\partial y}(q\Delta y) + \frac{\partial V}{\partial z}(q\Delta z)$$

Here we have

$$\boldsymbol{p} = (q\Delta x, q\Delta y, q\Delta z)$$
$$\boldsymbol{E} = -\nabla V$$

(We will discuss this notation of the electric field in Chapter 23).
Then we have

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}$$

14. Force on the electric dipole moment in a nonhomogeneous electric field

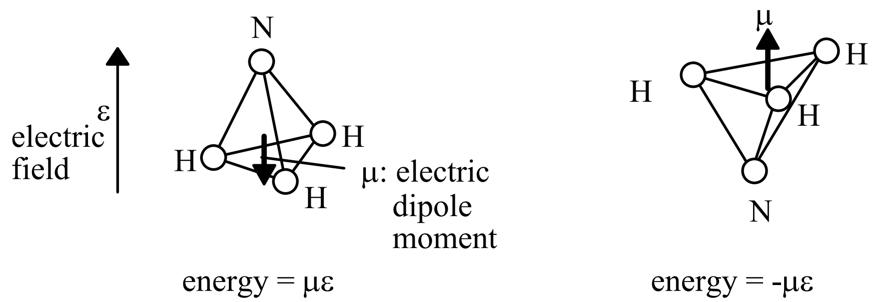
The force exerted on the electric dipole in the presence of nonhomogeneous electric field is given by

$$\boldsymbol{F} = -\nabla U = \nabla(\boldsymbol{p} \cdot \boldsymbol{E}) = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}$$

((Note)) In general case

$$\nabla(\boldsymbol{p} \cdot \boldsymbol{E}) = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E} + (\boldsymbol{E} \cdot \nabla)\boldsymbol{p} + \boldsymbol{p} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{E} \times (\nabla \times \boldsymbol{p})$$

((Example)) Ammonia maser
Application of electric field on the ammonia



Let us consider NH_3 in a region where the electric field ε is weak but has a strong gradient along the axis of molecules (*x* direction)).

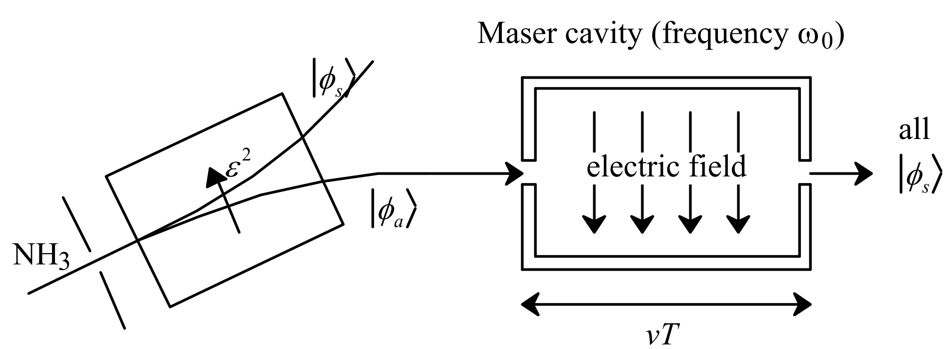
The molecules in the state $|\varphi_s\rangle$ are subjected to a force parallel to the *x* axis:

$$F_s = -\frac{dU_s}{dx}$$

Similarly, the molecules in the state $|\varphi_a\rangle$ are subjected to an opposite force:

$$F_a = -\frac{dU_a}{dx}$$

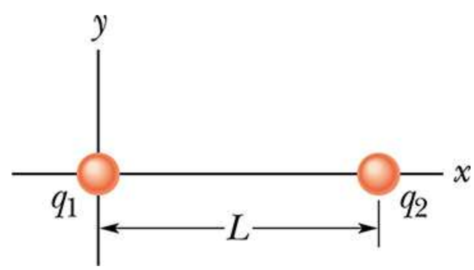
This is the basis of the method which is used in the ammonia maser to sort the molecules and select those in the higher energy state.



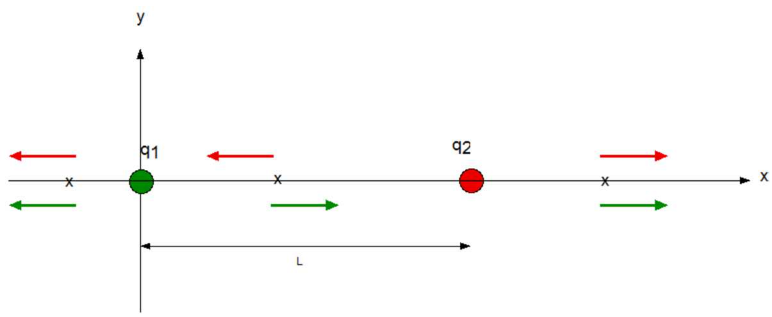
15. Typical examples

15.1 Problem 22-8 (SP-22)

In Fig., particle 1 of charge q_1 = -5.00 q and particle 2 of charge q_2 = +2.00 q are fixed to an axis. (a) As a multiple of distance L, at what coordinate on the axis is the net electric field of the particles zero? (b) Sketch the net electric field lines.



((**Solution**))



$q_1 = -5\,q$
$q_2 = 2\,q$

For $x>L$

$$E_x = \frac{q}{4\pi\varepsilon_0 L^2}[\frac{-5}{\xi^2} + \frac{2}{(\xi-1)^2}]$$

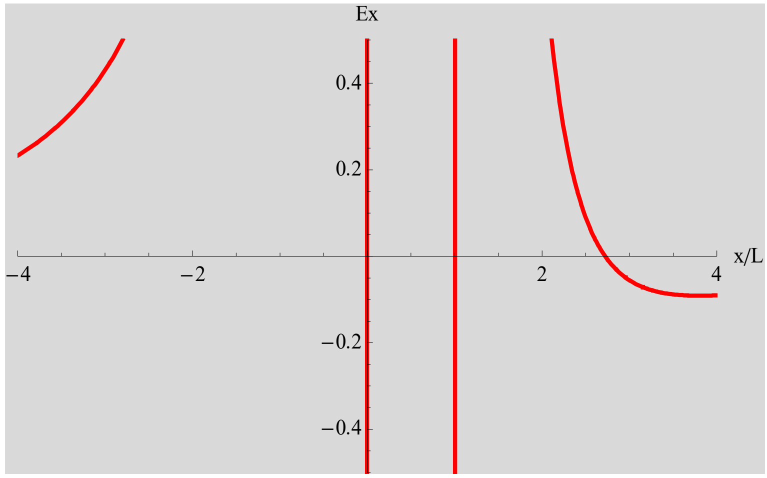
For $0<x<L$

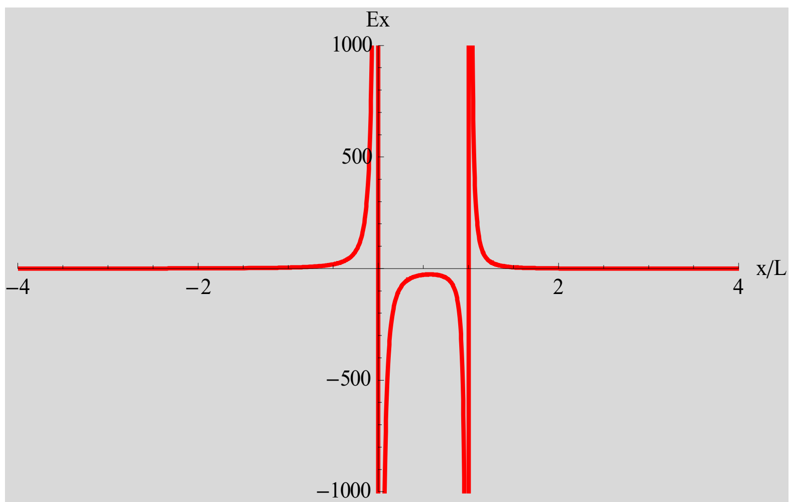
$$E_x = \frac{q}{4\pi\varepsilon_0 L^2}[\frac{-5}{\xi^2} - \frac{2}{(\xi-1)^2}]$$

For $x<0$

$$E_x = \frac{q}{4\pi\varepsilon_0 L^2}[\frac{5}{\xi^2} - \frac{2}{(\xi-1)^2}]$$

where $\xi = x/L$.

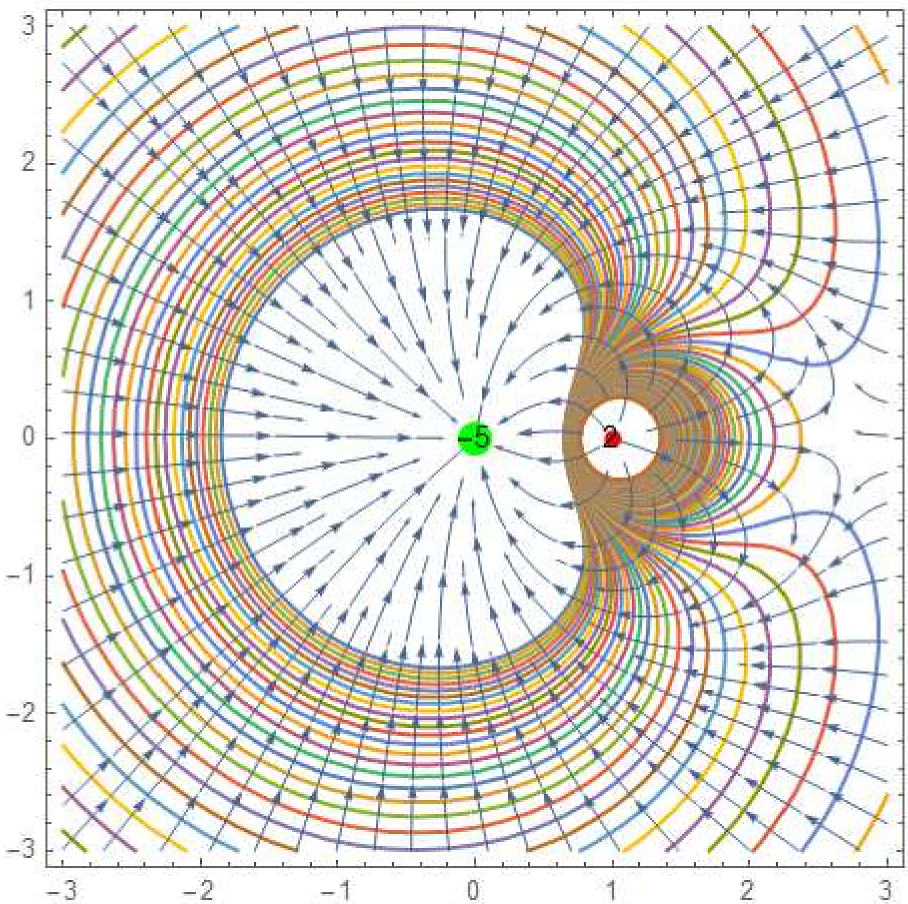




The electric field E_x become zero when

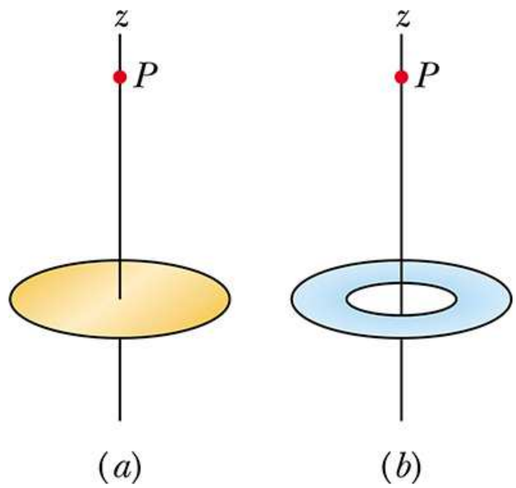
$$\xi = x/L = 2.72076.$$

(b)

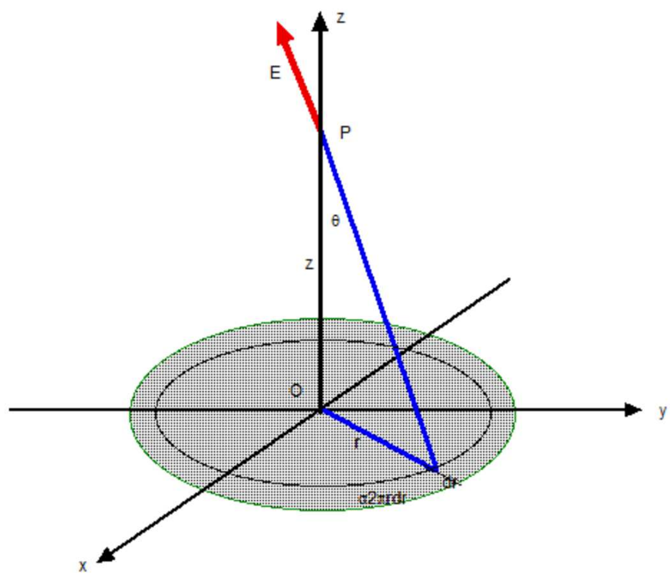


15.2 Problem 22-37 (SP-22)

Suppose you design an apparatus in which a uniformly charged disk of radius R is to produce an electric field. The field magnitude is most important along the central perpendicular axis of the disk, at a point P at distance 2.00 R from the disk (Fig. a). Cost analysis suggests that you switch to a ring of the same outer radius R but with inner radius $R/2.00$ (Fig. b). Assume that the ring will have the same surface charge density as the original disk. If you switch to the ring, by what percentage will you decrease the electric field magnitude at P?



((**Solution**))
(a)



From the symmetry the direction of the electric field is along the *z* axis.

$$
\begin{aligned}
dE_z &= \cos\theta \frac{1}{4\pi\varepsilon_0} \frac{\sigma(2\pi r)dr}{(z^2 + r^2)} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{z}{\sqrt{z^2 + r^2}} \frac{\sigma(2\pi r)dr}{(z^2 + r^2)} \\
&= \frac{\sigma z}{2\varepsilon_0} \frac{rdr}{(z^2 + r^2)^{3/2}}
\end{aligned}
$$

$$
E_z^{(1)} = \frac{\sigma z}{2\varepsilon_0} \int_0^R \frac{rdr}{(z^2 + r^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}(1 - \frac{z}{\sqrt{z^2 + R^2}})
$$

When $z = 2R$,

$$E_z^{(1)} = \frac{\sigma}{2\varepsilon_0}(1 - \frac{z}{\sqrt{z^2 + R^2}}) = \frac{\sigma}{2\varepsilon_0}(1 - \frac{2}{\sqrt{5}}) = 0.106 \times \frac{\sigma}{2\varepsilon_0}$$

(b) We use the principle of superposition: disk (radius R) with the surface charge density σ and the disk (radius $R/2$) with the surface charge density ($-\sigma$).

$$E_z^{(2)} = \frac{\sigma}{2\varepsilon_0}(1 - \frac{z}{\sqrt{z^2 + R^2}}) + (-\frac{\sigma}{2\varepsilon_0})(1 - \frac{z}{\sqrt{z^2 + \left(\frac{R}{2}\right)^2}})$$

$$= \frac{\sigma}{2\varepsilon_0}(-\frac{z}{\sqrt{z^2 + R^2}} + \frac{z}{\sqrt{z^2 + \left(\frac{R}{2}\right)^2}})$$

When $z = 2R$,

$$E_z^{(2)} = \frac{\sigma}{2\varepsilon_0}(-\frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}}) = \frac{\sigma}{2\varepsilon_0} \times 0.0761$$

Then we have

$$\frac{E_z(2)}{E_z(1)} = \frac{0.0761}{0.106} \times 100 = 71.8\%$$

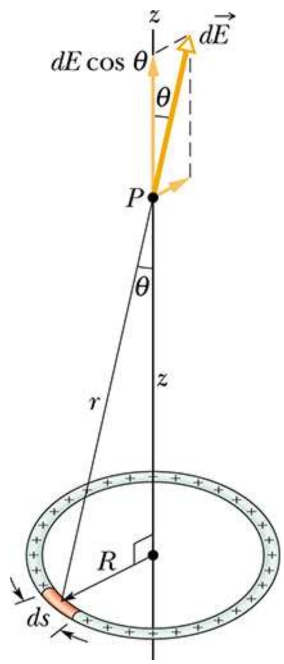
which means the decrease of 28.2%.

15.3 Problem 22-76 (SP-22)

An electron is constrained to the central axis of the ring of charge of radius R in Fig., with $z \ll R$. Show that the electrostatic force on the electron can cause it to oscillate through the ring center with an angular frequency

$$\omega = \sqrt{\frac{eq}{4\pi\varepsilon_0 m R^3}}$$

where q is the ring's charge and m is the electron's mass



((**Solution**))

$$m\ddot{z} = F_z = (-e)E_z = (-e)\frac{q}{4\pi\varepsilon_0(R^2+z^2)}\cos\theta$$

$$= -\frac{eqz}{4\pi\varepsilon_0 R^3(1+\frac{z^2}{R^2})^{3/2}}$$

$$\approx -\frac{eqz}{4\pi\varepsilon_0 R^3}$$

or

$$\ddot{z} = -\frac{eqz}{4\pi\varepsilon_0 m R^3} = -\omega^2 z \qquad \text{(simple harmonics)}$$

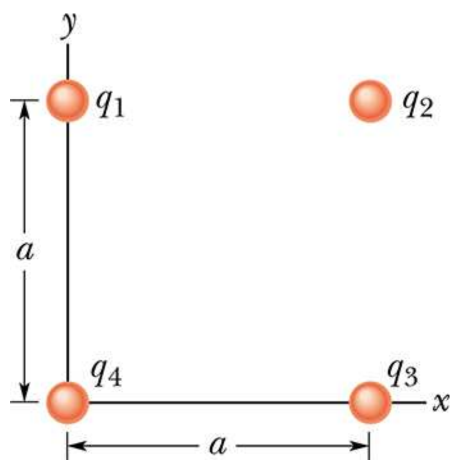
with

$$\omega = \sqrt{\frac{eq}{4\pi\varepsilon_0 m R^3}}$$

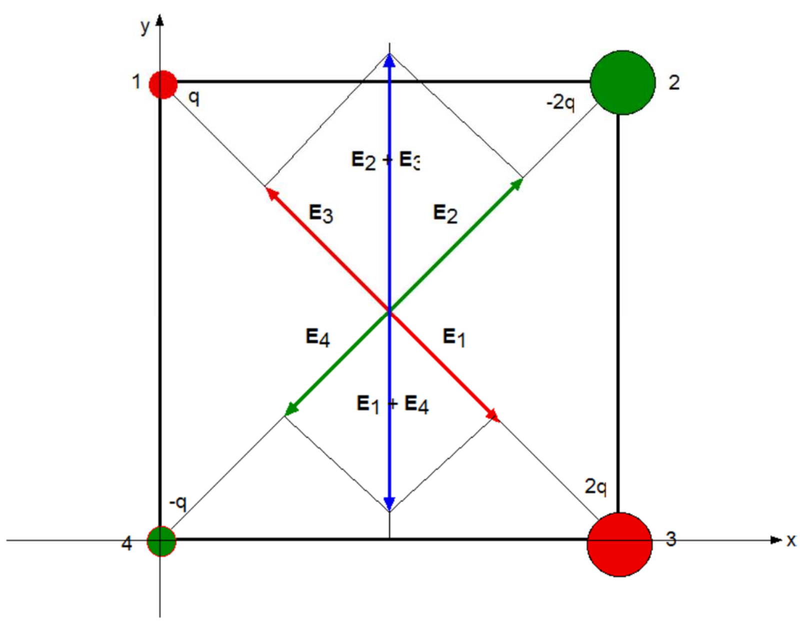
16. Hint of HW-22

16.1 Problem 22-9 (HW-22)

In Fig., the four particles form a square of edge length a = 5.00 cm and have charges; q_1 = +10.0 nC, q_2 = -20.0 nC, q_3 = +20.0 nC, and q_4 = -10.0 nC. In unit vector notation, what net electric field do the particles produce at the square's center?

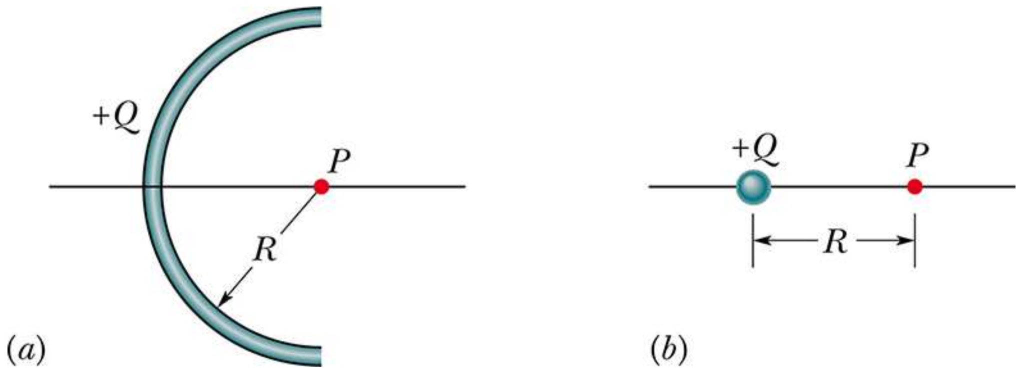


((**Solution**))
$q_1 = 10$ nC = q
q_2=-20nC = -2q
q_3 = 20.0 nC = 2q
q_4 = -10.0 nC = -q

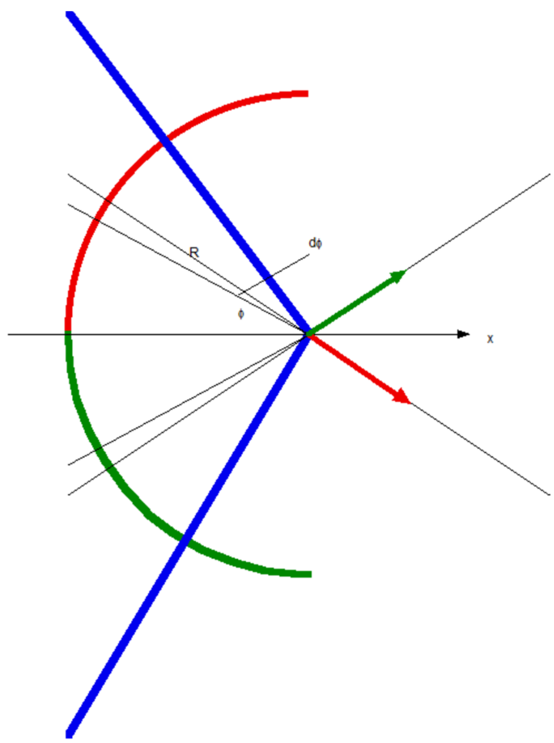


16.2 Problem 22-31 (HW-22)

Figure shows a nonconducting rod with a uniformly dustributed charge $+Q$. The rod forms a half circle with radius R and produces an electric field of magnitude E_{arc} at its center of curvature P. If the arc is collapsed to a point at distance R from P, by what factor is the magnitude of the electric field at P multiplied?



((**Solution**))



We consider an electric field due to the arc part.
Line density

$$\lambda = \frac{Q}{R\theta}$$

$$dE_x = \frac{\lambda(Rd\phi)}{4\pi\varepsilon_0 R^2} 2\cos\phi = \frac{\frac{Q}{R\theta}(Rd\phi)}{4\pi\varepsilon_0 R^2} 2\cos\phi$$

$$= \frac{\frac{Q}{\theta}}{2\pi\varepsilon_0 R^2} \cos\phi d\phi$$

Appendix

NIST Physical constant

SI Units

cal=4.19 J, 1 atm=101.3 kPa, g=Acceleration due to gravity (m/s^2)
G=gravitational constant ($N\ m^2/kg^2$),
u=atomic mass constant
NA=Avogadro number
R=Gas constant (J/mol K),
me=mass of electron (kg)
qe=Charge of electron (C),
ge=electron g factor
kB=Boltzmann constant (J/K)
rB=Bohr radius (m),
μN=Nuclear magneton (J/T)
λc=Compton wavelength (m)
c=velocity of light (m/s),
μ0=Magnetic constant
ϵ0=electric constant (C^2/Nm^2)
mn=mass of neutron (kg)
mp=mass of proton (kg),
h=Planck constant
$\hbar$=Dirac constant,

Planck mass=mpl = $\sqrt{\frac{\hbar c}{G}}$,

Planck time = τpl = $\left(\frac{\hbar G}{c^5}\right)^{1/2}$,

Planck length=lpl = $\left(\frac{\hbar G}{c^3}\right)^{1/2}$,

σSB=Stefan-Boltzmann constant ($W/m^2\ K^4$),
z0 =impedance of free space (Ω),
Φ0=magnetic flux quantum ($T\ m^2$)
Rk=von Klitzing constant (Ω),
Mea = 5.9736 x 10^{24} kg; Mass of the earth,
Rea=6372.797 km, radius of the earth,
Msun=mass of sun (kg) =Solar mass
Rsun=radius of Sun (m)=Solar radius
Mmoon=Mass of moon,
light year=a distance light travels in a vacuum in one year=9.4605 x 10^{15} m,
Parsec (pc) = a unit of distance = 3.26 light yeras = 30.857 x 10^{15} m,
AU = astronomical unit = average distance between the Earth and the Sun = 1.49597870 x10^{11} m

```
Physconst = {cal → 4.19, atm → 101.3, g → 9.80665,
  G → 6.6742867 10^-11, NA → 6.02214179 10^23, R → 8.314472,
  me → 9.1093821545 10^-31, u → 1.660538782 10^-27,
  eV → 1.602176487 10^-19, qe → 1.602176487 10^-19,
  ge → 2.0023193043622, kB → 1.3806504 10^-23,
  rB → 0.52917720859 10^-10, μB → 927.40091523 10^-26,
  μN → 5.05078324 10^-27, λc → 2.4263102175 10^-12,
  c → 2.99792458 10^8, μ0 → 12.566370614 10^-7,
  ϵ0 → 8.854187817 10^-12, mn → 1.674927211 10^-27,
  mp → 1.672621637 10^-27, h → 6.62606896 10^-34,
  ħ → 1.05457162853 10^-34, σSB → 5.670400 10^-8,
  z0 → 376.730313461, Φ0 → 2.06783366752 10^-15,
  Rk → 25812.80755718, Mea → 5.9736 10^24, Rea → 6.372 10^6,
  Msun → 1.988435 10^30, Rsun → 6.9599 10^8,
  Mmoon → 7.3483 10^22, ly → 9.4605 10^15, pc → 30.857 10^15,
  AU → 1.49597870 10^11, mile → 1.609344 10^3, hour → 3600}
```

$$\frac{c^2}{10^7} \;/.\; \text{Physconst}$$

8.98755×10^{9}

$$qe \;/.\; \text{Physconst}$$

1.60218×10^{-19}

$$\frac{qe^2}{4\,\pi\,\epsilon 0} \;/.\; \text{Physconst}$$

2.30708×10^{-28}

$$\frac{qe^2}{4\,\pi\,\epsilon 0}\,\frac{1}{rB^2} \;/.\; \text{Physconst}$$

8.23872×10^{-8}

$$\frac{1}{4\,\pi\,\epsilon 0\,c^2} \;/.\; \text{Physconst}$$

$1.\times10^{-7}$