1. Electric potential

The electrostatic force is a *conservative force*. This means that the work it does on a particle depends only on the initial and final position of the particle, and not on the path followed. With each conservative force, a potential energy can be associated.

$$\Delta K = W_c = -\Delta U, \qquad \text{(energy-work theorem)}$$

where K is the kinetic energy and W_c is the work due to the electrostatic force (conservative force). The potential energy U associated with a conservative force $\boldsymbol{F}$ is defined in the following manner

$$W_c = \boldsymbol{F} \cdot d\boldsymbol{r} = -dU .$$

To describe the potential energy associated with a charge distribution the concept of the **electric potential** V is introduced. The electric potential V at a given position is defined as the potential energy of a test particle divided by the charge q of this object:

$$U(\boldsymbol{r}) = qV(\boldsymbol{r}).$$

Since $\boldsymbol{F} = q\boldsymbol{E}$, this equation can be rewritten as

$$W_c = q\boldsymbol{E} \cdot d\boldsymbol{r} = -dU = -qdV,$$

or

$$\boldsymbol{E} \cdot d\boldsymbol{r} = -dV,$$

or

$$\boldsymbol{E} = -\frac{\partial}{\partial \boldsymbol{r}} V = -\nabla V$$

where ∇V is the gradient of V.

For the Cartesian coordinate (x, y, z)

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z}$$

For the spherical coordinates [$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$]

$$\nabla = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

For the cylindrical coordinates [$x = \rho\cos\phi, \rho\sin\phi, z = z$]

$$\nabla = \boldsymbol{e}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \boldsymbol{e}_z \frac{\partial}{\partial z}$$

The unit of the electric potential is the **volt** (V) and The unit of the electric field is V/m.

1 V = 1 J/C = 1 Nm/C.

1 N/C = 1 V/m

A unit for the energy commonly used in physics is defined the electron volt (eV).

1 eV = (1.602176487 x 10^{-19} C)(1V) = 1.602176487 x 10^{-19} J

q_e is the charge of electron (negative charge) and is given

$q_e = -e$

with e = 1.602176487 x 10^{-19} C

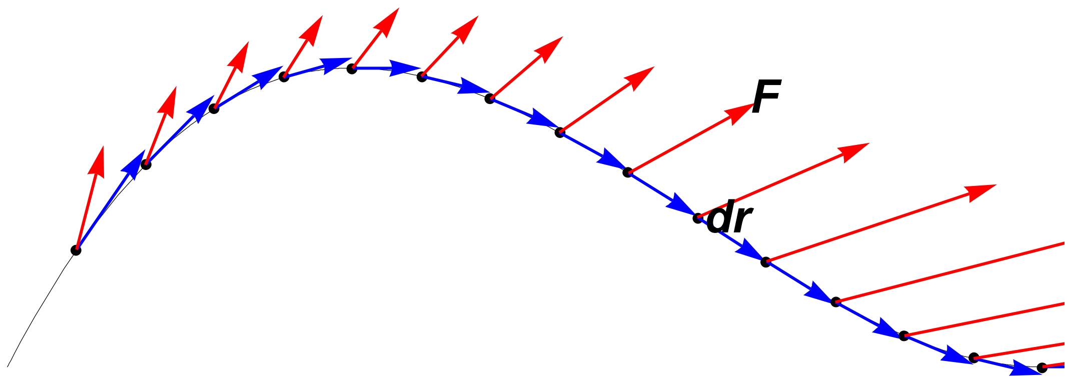
((**Note-1**)) Relation between work done and potential energy (Work-energy theorem)

$$\boldsymbol{F} = q\boldsymbol{E}$$

$$W = \int \boldsymbol{F}\cdot d\boldsymbol{r} = q\int \boldsymbol{E}\cdot d\boldsymbol{r} = -q\int \nabla V\cdot d\boldsymbol{r} = -qV = -U$$

((**Note-2**))

We consider the work to move charge (q>0) from infinity to a finite distance r, where the positive charge Q is located at r = 0.



•

$$\Delta W = \int_{\infty}^{r} (-\boldsymbol{F})\cdot(-d\boldsymbol{r})$$
$$= \int_{\infty}^{r} \boldsymbol{F}\cdot d\boldsymbol{r}$$
$$= \int_{\infty}^{r} \frac{qQ}{4\pi\varepsilon_0 r^2}\boldsymbol{e}_r\cdot\boldsymbol{e}_r dr$$
$$= \int_{\infty}^{r} \frac{qQ}{4\pi\varepsilon_0 r^2} dr$$
$$= -\frac{qQ}{4\pi\varepsilon_0 r}$$

where $\boldsymbol{F}$ is the repulsive force, and is given by

$$\boldsymbol{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\boldsymbol{e}_r .$$

The displacement vector is

$$d\boldsymbol{r} = dr\boldsymbol{e}_r + rd\theta\boldsymbol{e}_\theta + r\sin\theta d\phi\boldsymbol{e}_\phi .$$

The scalar product is

$$\boldsymbol{F}\cdot d\boldsymbol{r} = \boldsymbol{F}\cdot d\boldsymbol{r} = \frac{qQ}{4\pi\varepsilon_0 r^2}\boldsymbol{e}_r\cdot dr\boldsymbol{e}_r = \frac{qQ}{4\pi\varepsilon_0 r^2}dr\,.$$

Using the work energy theorem, we have

$$\Delta W = -U = -\frac{eQ}{4\pi\varepsilon_0 r},$$

or

$$U = \frac{eQ}{4\pi\varepsilon_0 r} = qV \qquad V = \frac{Q}{4\pi\varepsilon_0 r}.$$

Using the electric field $\boldsymbol{E}$,

$$\boldsymbol{F} = q\boldsymbol{E}, \qquad U = qV,$$

we have the final form for the electric potential,

$$V = -\int_{\infty}^{r}\boldsymbol{E}\cdot d\boldsymbol{r}$$

or

$$\boldsymbol{E} = -\nabla V\,.$$

((Note)) **The path integral**

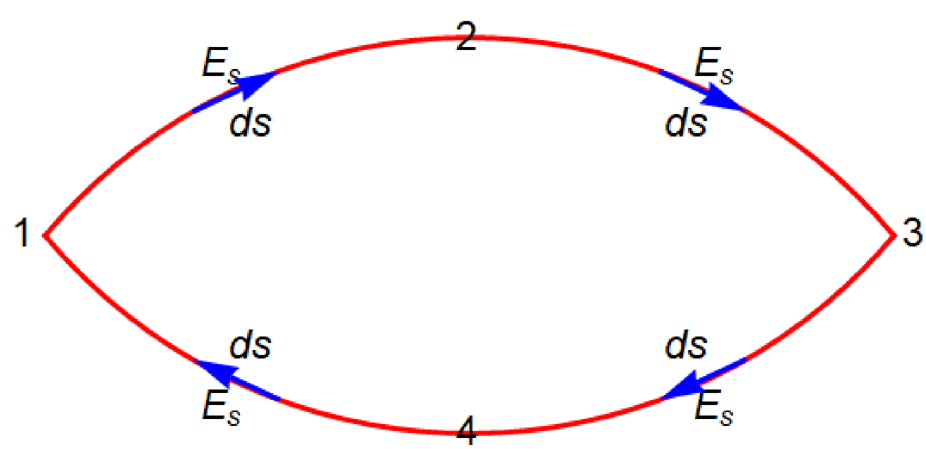


Fig. Path integral $\int \boldsymbol{E}\cdot d\boldsymbol{r} = \int E_s ds$ along the path 1→2→3 and the path 3→4→1

$$\oint \boldsymbol{E}\cdot d\boldsymbol{r} = \int_{1\to 2\to 3} \boldsymbol{E}\cdot d\boldsymbol{r} + \int_{3\to 4\to 1} \boldsymbol{E}\cdot d\boldsymbol{r}$$

We use the Stokes' theorem for the path integral,

$$\oint \boldsymbol{E}\cdot d\boldsymbol{r} = \int (\nabla\times\boldsymbol{E})\cdot d\boldsymbol{r}$$

Suppose that the electric field $\boldsymbol{E}$ is expressed by

$$\boldsymbol{E} = -\nabla V \,.$$

Leading to $\nabla\times\boldsymbol{E} = 0$. Then we have

$$\oint \boldsymbol{E}\cdot d\boldsymbol{r} = 0$$

and

$$\int_{1\to 2\to 3} \boldsymbol{E}\cdot d\boldsymbol{r} = -\int_{3\to 4\to 1} \boldsymbol{E}\cdot d\boldsymbol{r} = \int_{1\to 4\to 3} \boldsymbol{E}\cdot d\boldsymbol{r}$$

Thus the path integral is independent of any path between the points 1 and 3.

2. Potential differences in a uniform electric field

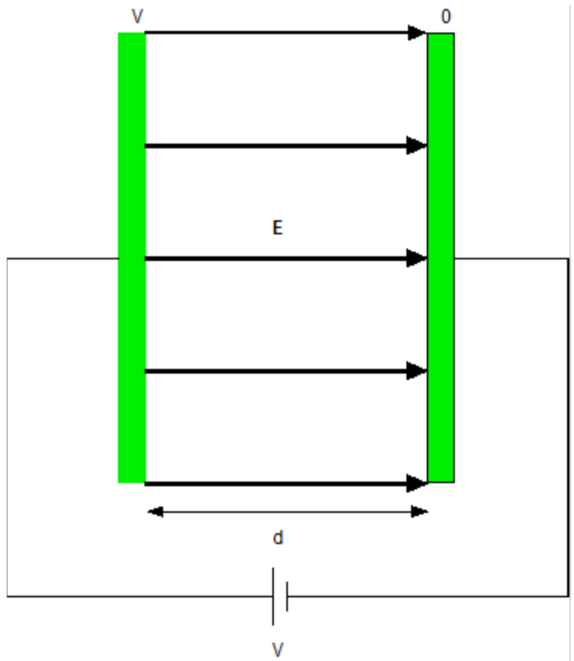
We consider the two parallel conducting plates which are separated by a distance d. A battery V is connected between these two plates. The electric field E is assumed to be uniform.

$$V(d) - V(0) = -V = -E(d-0) = -Ed$$

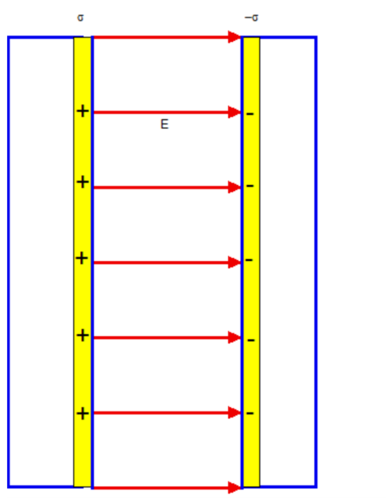
or

$$V = Ed$$

Note that the direction of the electric field is from *the higher electric potential to the lower electric potential.*



or



((**Example**)) Walter Lewin: 8.02x-MIT Physics II Electricity and Magnetism (Chapter 5)

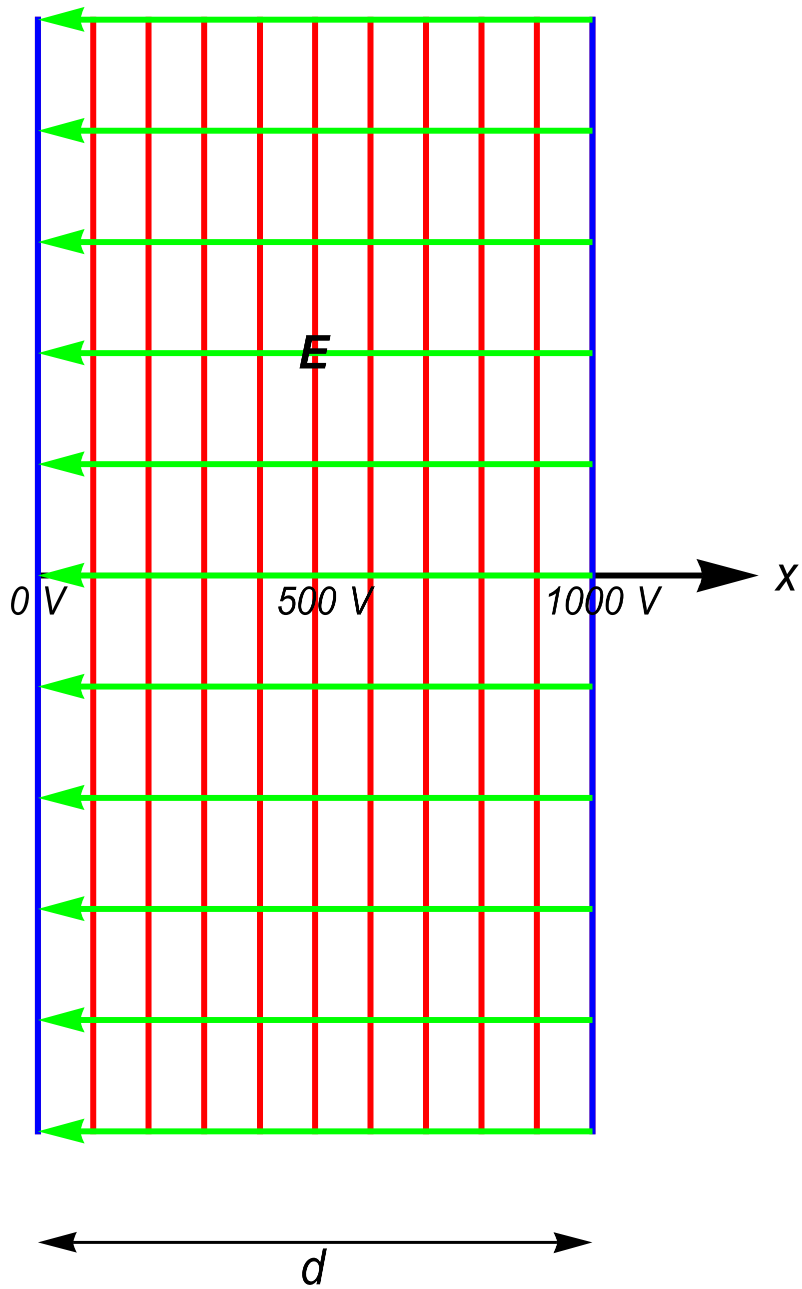
Suppose that the two parallel conducting plates are separated by a distance d. A battery V is connected between these two plates. The electric potential V is linearly proportional to the position x: $V = 0$ at $x = 0$ and $V = 10^3$ V at $x = d = 0.01$ m. So the electric potential V is given by

$$V = 10^5 x \qquad (0 \leq x \leq d).$$

as a function of x. The electric field is constant,

$$E_x = -\frac{dV}{dx} = -10^5 \text{ V/m},$$

The direction of the electric field is from the high voltage point to the low voltage point.



The direction of the electric field (denoted by the green line) is along the negative direction. The red line denotes the constant V with $V = 0 - 1000$ V. The equi-potential line with the constant V is perpendicular to the electric field $\boldsymbol{E}$.

3. Electric potential due to point charges

The electric field due to a point charge q is given by

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\boldsymbol{e}_r$$

$$V(r) - V(\infty) = V(r) = -\int_\infty^r \boldsymbol{E}\cdot d\boldsymbol{r} = -\int_\infty^r \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\boldsymbol{e}_r\cdot\boldsymbol{e}_r dr = -\frac{q}{4\pi\varepsilon_0}\int_\infty^r \frac{1}{r^2}dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

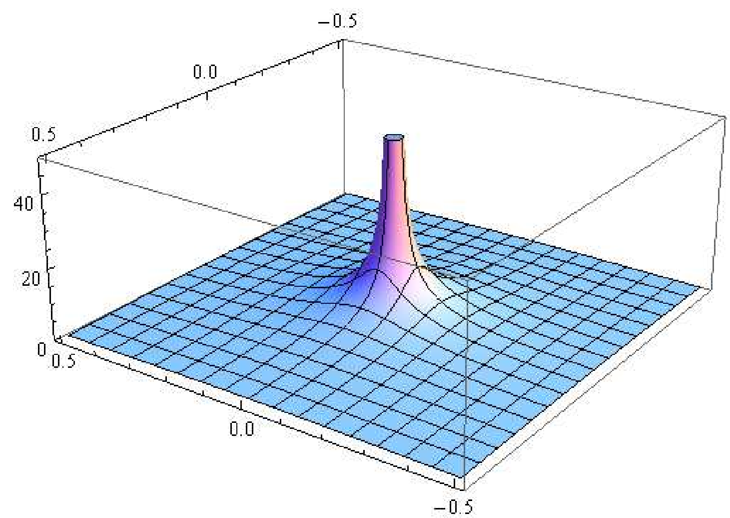
where $V(\infty) = 0$.

or

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \text{(electric potential)}$$

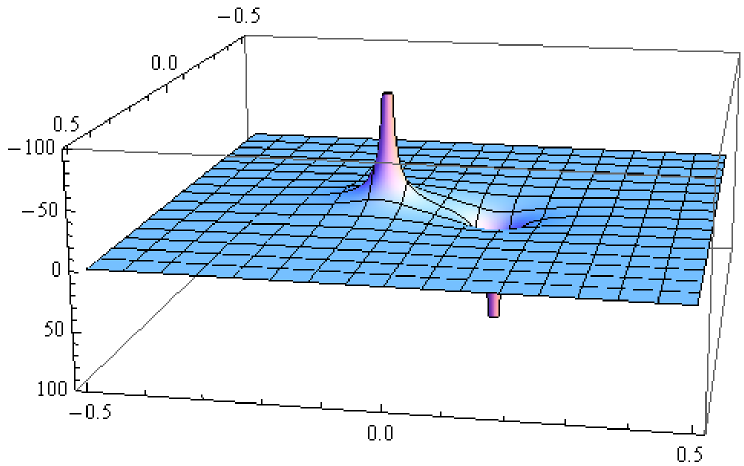
((**Mathematica**))

Plot3D of the electric potential due to a positive charge located at $x = y = 0$



((**Mathematica**))

Plot3D of the electric potential due to a negative charge located at x = -0.1, y = 0 and a negative charge located at x = 0.1 and y = 0.



The electric potential at a point P in space due to two, or more point charges is obtained from the superposition principle.

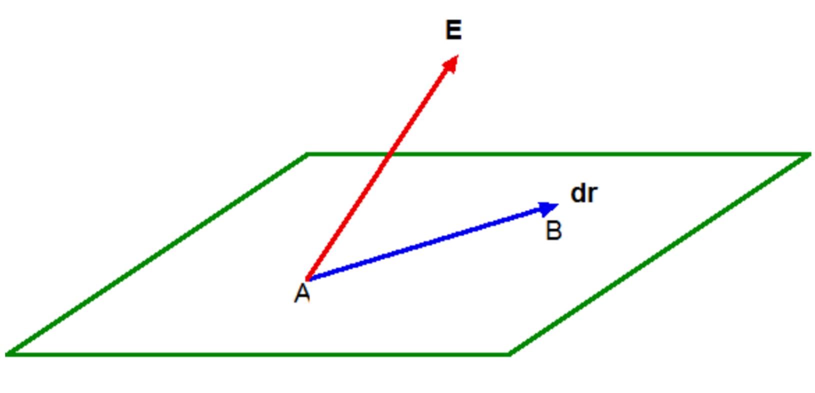
$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

where r_i is the distance from the point P to the charge q_i.

In the case of a continuous charge distribution, the electric potential is the integral of all the contributions from the point-like charge elements

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}')d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

4. Electric field and equipotential surface



Equipotential surface

The electric field lines are everywhere perpendicular to the equipotential surfaces.

If the points A and B are lie on the same equipotential surface, then by definition, we have $V_A = V_B$, or

$$\Delta V = 0.$$

The electric potential difference is described as

$$\Delta V = -\boldsymbol{E} \cdot d\boldsymbol{r},$$

where d$\boldsymbol{r}$ is a displacement (= AB) that lies entirely on an equipotential surface ($\Delta V = 0$). leading to

$$\boldsymbol{E} \cdot d\boldsymbol{r} = 0$$

or, $\boldsymbol{E}$ must be perpendicular to the equipotential surface.

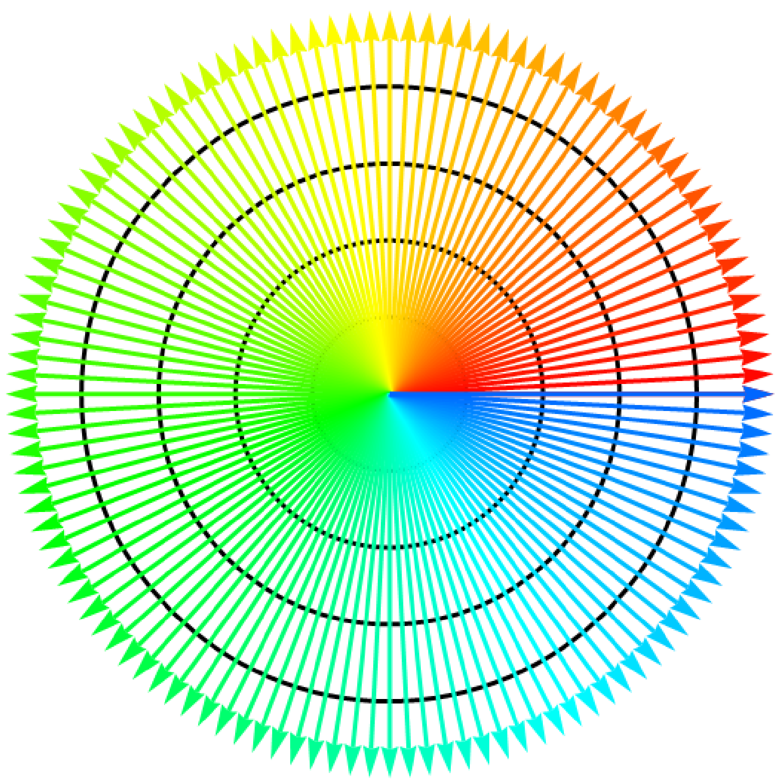
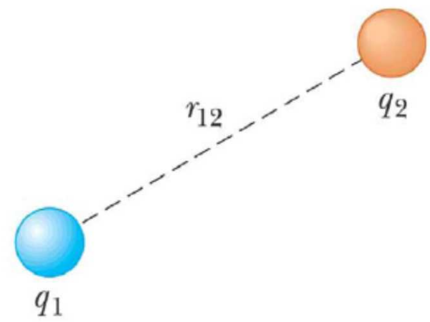


Fig. The electric field line is perpendicular to the equipotential surface.

5. Electric potential energy

5.1 Two charges (proton and electron) Bohr model in a hydrogen atom



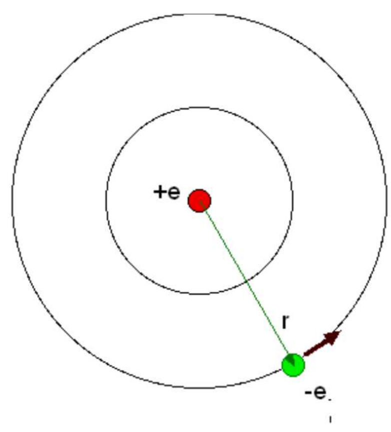
We now consider the potential energy of a system of the two charged particles. If V_2 is the electric potential at a point P due to a charge q_2, the work an external agent must do to bring a second charge q_1 from infinity to the point P without acceleration is q_1V_2. The electric potential energy of a pair of point charges is given by

$$U = q_1V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}$$

The hydrogen atom consists of a proton (a charge e) and an electron (a charge –e). The potential energy is given by

$$U = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

where the electron rotates around a proton, with a radius r. U is negative since the interaction is attractive.



The total energy of the electron is

$$E = K + U = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = -\frac{1}{2}(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r})$$

since $m\frac{v^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$ or $mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$

where $r = r_B$ is the Bohr radius; $r_B = 0.529177 \times 10^{-10}$ m.
E is negative and equal to -13.6 eV.

((**Mathematica**))

```
Physconst = {eV → 1.602176487 10^-19, qe → 1.602176487 10^-19,
  me → 9.1093821545 10^-31, rB → 0.52917720859 10^-10, ϵ0 → 8.854187817 10^-12}

{eV → 1.60218 × 10^-19, qe → 1.60218 × 10^-19,
 me → 9.10938 × 10^-31, rB → 5.29177 × 10^-11, ϵ0 → 8.85419 × 10^-12}
```

$$E1 = \frac{1}{2}\left(-\frac{1}{4\pi\epsilon0}\frac{qe^2}{rB}\right)$$

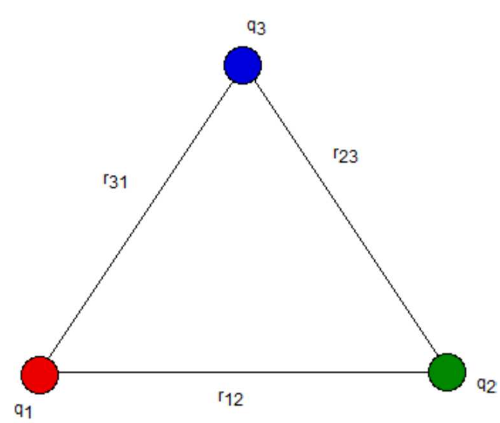
$$-\frac{qe^2}{8\pi\, rB\, \epsilon0}$$

```
E1 / eV /. Physconst

-13.6057
```

5.2 More than two charges

If there are more than two charges present, the potential energy of the system is obtained by finding the energy of each pair of charges and then summing to obtain the total energy. For three charges we obtain

$$U = \frac{1}{4\pi\varepsilon_0}(\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_3q_1}{r_{31}})$$



For q_1, the electric potential V_1 due to q_2 and q_3 and the potential energy U_1 are given by

$$V_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{31}}\right)$$

$$U_1 = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{31}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_3 q_1}{r_{31}}\right)$$

For q_2, the electric potential V_2 due to q_1 and q_3 and the potential energy U_2 are given by

$$V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{12}} + \frac{q_3}{r_{23}}\right)$$

$$U_2 = \frac{q_2}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{12}} + \frac{q_3}{r_{23}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}}\right)$$

For q_3, the electric potential V_3 due to q_1 and q_2 and the potential energy U_3 are given by

$$V_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{31}} + \frac{q_2}{r_{23}}\right)$$

$$U_3 = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{31}} + \frac{q_2}{r_{23}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_3 q_1}{r_{31}} + \frac{q_2 q_3}{r_{23}}\right)$$

Therefore we get the potential energy for the three charges as

$$U = \frac{1}{2}(q_1 V_1 + q_2 V_2 + q_3 V_3) = \frac{1}{2}(U_1 + U_2 + U_3) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{31}}\right)$$

where the factor 1/2 is necessary since we count twice for each pair. In general case,

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V_i$$

for the N-charges system. This equation can be rewritten as

$$U = \frac{1}{2}\int \rho(\boldsymbol{r}) V(\boldsymbol{r}) d^3\boldsymbol{r}$$

for the continuous system.

((**Example**)) Potential energy of sphere with constant charge density ρ.

The electric potential is given by

$$V(r) = \frac{\rho}{2\varepsilon_0}(R^2 - \frac{1}{3}r^2) \qquad \text{(for } r < R\text{)}$$

(see APPENDIX). Then the potential energy is obtained as

$$\begin{aligned}
U &= \frac{1}{2}\rho\int_0^R \frac{\rho}{2\varepsilon_0}(R^2 - \frac{1}{3}r^2)4\pi r^2 dr \\
&= \frac{\pi}{\varepsilon_0}\rho^2\int_0^R (R^2 r^2 - \frac{1}{3}r^4)dr \\
&= \frac{4\pi}{15\varepsilon_0}\rho^2 R^5 \\
&= \frac{3Q^2}{20\pi\varepsilon_0 R} \\
&= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0(\frac{5}{6}R)}
\end{aligned}$$

The energy is proportional to the square of the total charge and inversely proportional to the radius. We can also interpret this equation as saying that the average of $1/r_{ij}$ for all pairs of points in the sphere is $6/(5R)$.

6. Electric potential due to an electric dipole moment

Find the electric potential due to an electric dipole moment far from the dipole.

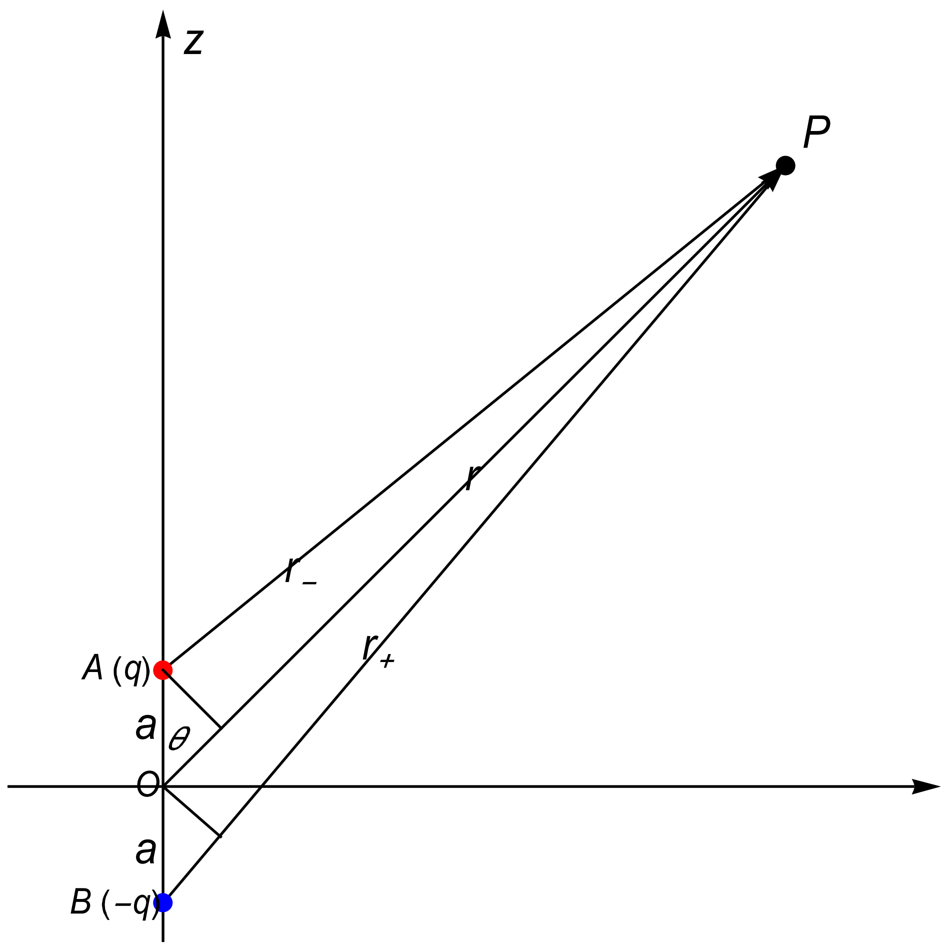


Fig. Finding the electrical potential V at the point P.

(a) Approximation-1

Line BA is on the z axis. The positive charge is at (0, 0, a) and the negative charge is at (0, 0, -a). We consider an electrical potential at the point P, due to the electric dipole moment ($p = 2qa$).

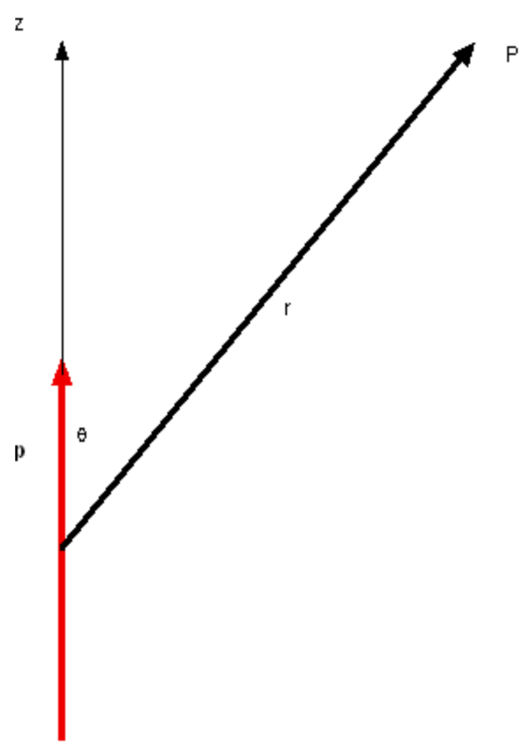
$$r_- = r - a\cos\theta, \qquad r_+ = r + a\cos\theta.$$

where θ is an angle between the vector $\overrightarrow{OP} = \boldsymbol{r}$ and the z axis. The electric potential due to the electric dipole moment $\boldsymbol{p} = (0, 0, p)$ is given by

$$
\begin{aligned}
V &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_-} - \frac{q}{r_+}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r - a\cos\theta} - \frac{q}{r + a\cos\theta}\right) \\
&= \frac{q}{4\pi\varepsilon_0 r}\left(\frac{1}{1 - \frac{a}{r}\cos\theta} - \frac{1}{1 + \frac{a}{r}\cos\theta}\right) \\
&\approx \frac{q}{4\pi\varepsilon_0 r}\left[1 + \frac{a}{r}\cos\theta - \left(1 - \frac{a}{r}\cos\theta\right)\right] \\
&= \frac{p}{4\pi\varepsilon_0 r^2}\cos\theta
\end{aligned}
$$

or

$$
V = \frac{p}{4\pi\varepsilon_0 r^3} r\cos\theta = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = \frac{pz}{4\pi\varepsilon_0 r^3}.
$$



where $\boldsymbol{p} = (0, 0, p)$ and $p\cdot r = \boldsymbol{p}\cdot\boldsymbol{r} = pr\cos\theta$. $p = 2qa$. Using the electric potential given by

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2},$$

the electric field can be expressed by

$$E_x = -\frac{\partial V}{\partial x} = \frac{3pzx}{4\pi\varepsilon_0 r^5} = \frac{3pzx}{4\pi\varepsilon_0 (z^2+x^2)^{5/2}} = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta,$$

$$\begin{aligned} E_z &= -\frac{\partial V}{\partial z} \\ &= \frac{p}{4\pi\varepsilon_0}[\frac{3z^2}{(z^2+x^2)^{5/2}} - \frac{1}{(z^2+x^2)^{3/2}}] \\ &= \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1) \end{aligned}$$

Note that in the polar coordinate

$$\begin{aligned} \boldsymbol{E} &= -\nabla V(r,\theta) \\ &= -\boldsymbol{e}_r \frac{\partial V}{\partial r} - \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} - \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \\ &= -\boldsymbol{e}_r \frac{\partial V}{\partial r} - \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} \end{aligned}$$

leading to

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}, \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

since

$$\frac{\partial V}{\partial \phi} = 0.$$

$$
\begin{aligned}
E_z &= E_r \cos\theta - E_\theta \sin\theta \\
&= \frac{2p\cos^2\theta}{4\pi\varepsilon_0 r^3} - \frac{p\sin^2\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{p(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3} \\
&= \frac{p}{4\pi\varepsilon_0}\frac{(3z^2 - r^2)}{r^5}
\end{aligned}
$$

$$
\begin{aligned}
E_x &= E_r \sin\theta + E_\theta \cos\theta \\
&= \frac{3p\sin\theta\cos\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{p}{4\pi\varepsilon_0}\frac{3zx}{r^5}
\end{aligned}
$$

When $\cos\theta = \pm\frac{1}{\sqrt{3}}$, E_z becomes equal to zero.

In general case, the electric field due to the electric dipole moment can be expressed by

$$
\begin{aligned}
\boldsymbol{E} &= \frac{1}{4\pi\varepsilon_0 r^3}[3(\boldsymbol{p}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}] \\
&= \frac{1}{4\pi\varepsilon_0 r^5}[3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - r^2\boldsymbol{p}]
\end{aligned}
$$

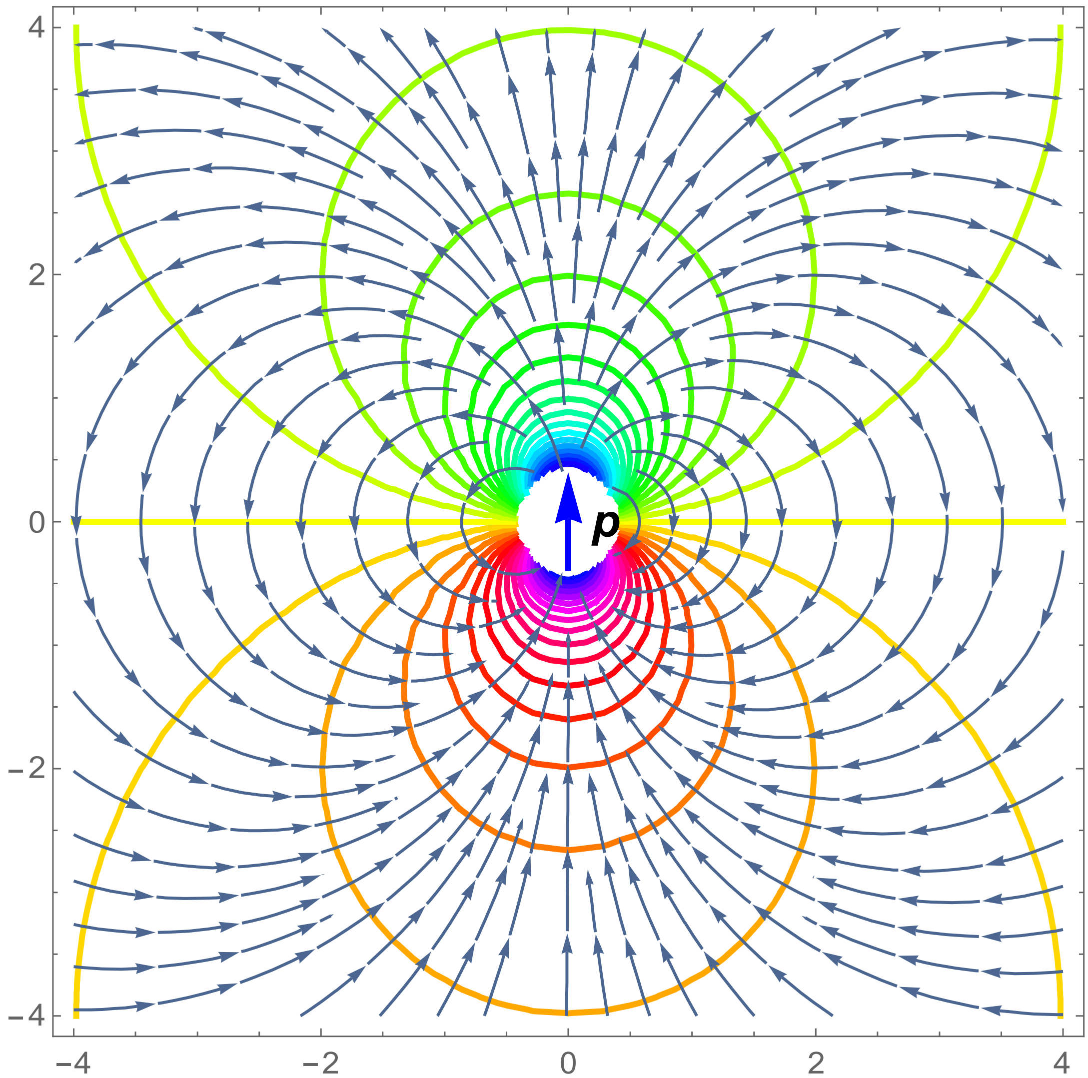


Fig. Electric field of the electric dipole moment $\boldsymbol{p}$ at the origin.

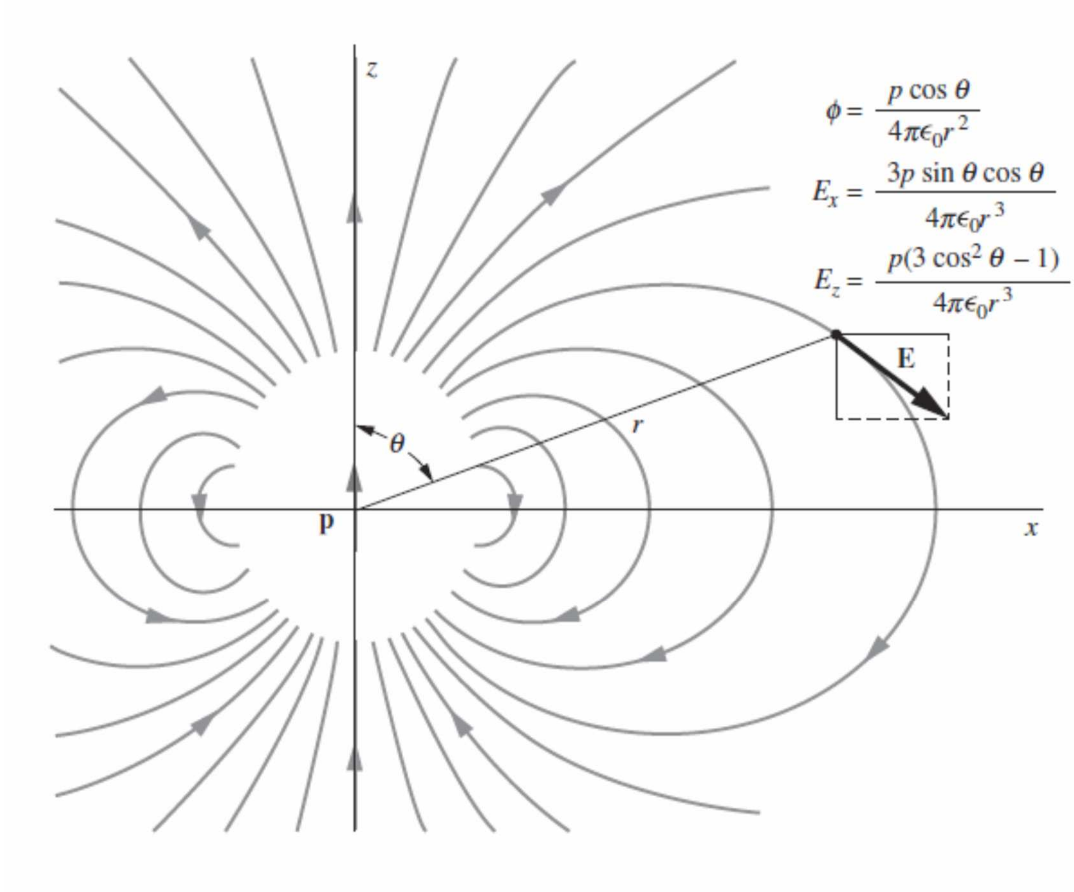


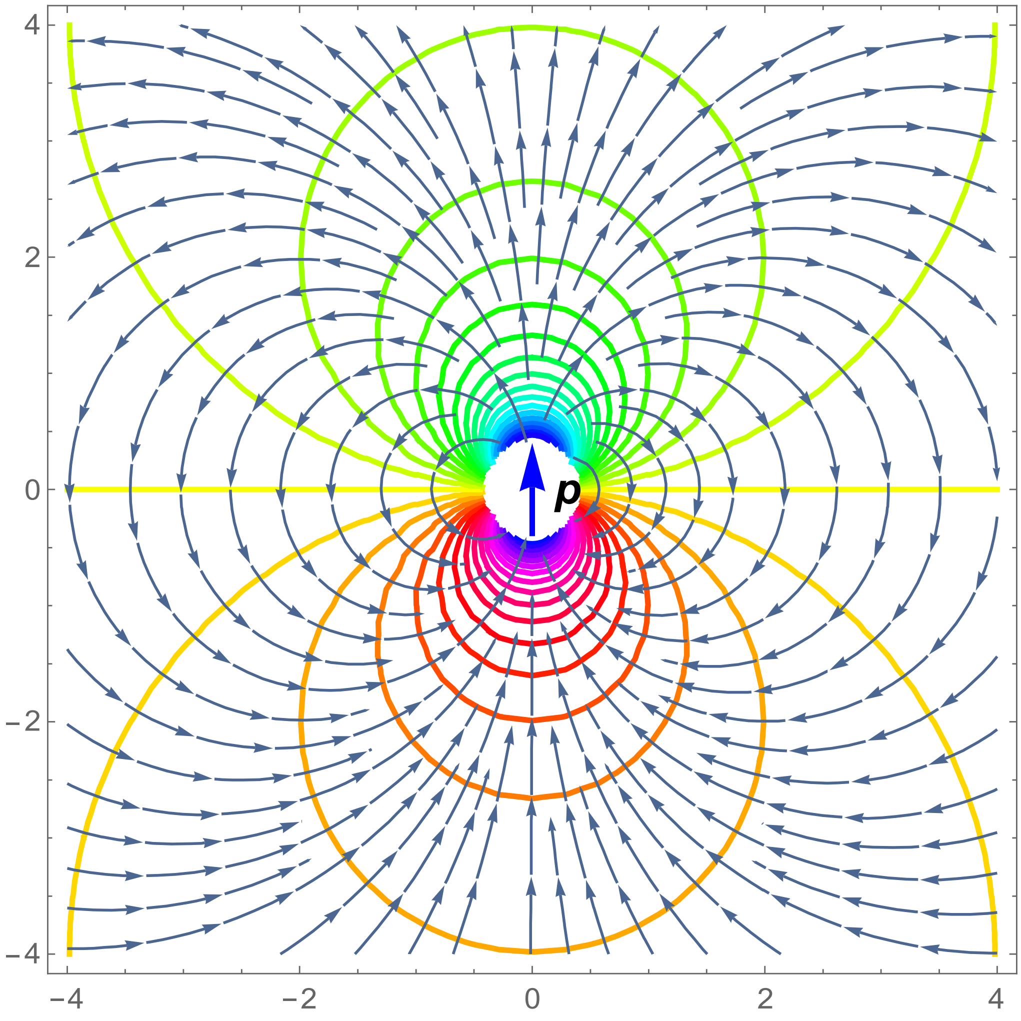
Fig. Electric field due to an electric dipole moment (Purcell and Morin). E.M. Purcell and D.J. Morin, Electricity and Magnetism, 3rd edition (Cambridge, 2013).

((**Mathematica**))

Electric field produced by an electric dipole moment

```
Clear["Global`*"]; V1[r_, θ_] := P/(4 π ϵ0 r) Cos[θ];

r2xRule = {r → Sqrt[x^2 + y^2 + z^2], θ → ArcCos[z/Sqrt[x^2 + y^2 + z^2]],
   ϕ → ArcTan[x, y]};

V2D1 = V1[r, θ] /. r2xRule /. y → 0 // Simplify;

rule1 = { P → 1, ϵ0 → 1};

V2D11 = V2D1 /. rule1;

Ex1 = -D[V2D11, x] // Simplify;
Ez1 = -D[V2D11, z] // Simplify;

g1 = ContourPlot[Evaluate[Table[V2D11 == α, {α, -1, 1, 0.01}]],
   {x, -4, 4}, {z, -4, 4},
   ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
   RegionFunction → Function[{x, z}, x^2 + z^2 > 0.2]];
```

```
g2 = StreamPlot[Evaluate[{Ex1, Ez1}], {x, -4, 4}, {z, -4, 4},
   RegionFunction → Function[{x, z}, x^2 + z^2 > 0.2]];

g3 = Graphics[{Blue, Thick, Arrowheads[0.05],
   Arrow[{{0, -0.4}, {0, 0.4}}],
   Text[Style["p", Italic, Bold, 15, Black], {0.3, 0}]}];
Show[g1, g2, g3, PlotRange → All]
```



(b) Approximation-2: Calculation using the Legendre generating function

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where

$$\frac{1}{r_1} = \frac{1}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1 + (\frac{a}{r})^2 - 2\frac{a}{r}\cos\theta}}$$

$$\frac{1}{r_2} = \frac{1}{\sqrt{a^2 + r^2 + 2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1 + (\frac{a}{r})^2 + 2\frac{a}{r}\cos\theta}}$$

We use the generating function of the Legendre function,

$$g(t,x) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \qquad \text{for } |t| < 1$$
$$= P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + ..$$

where $x = \cos\theta$, $t = a/r$, $P_n(x)$ is a Legendre polynomial.

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}x(5x^2 - 3)$$
$$P_4(x) = \frac{1}{8}x(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}x(63x^4 - 70x^2 + 15)$$

Then we have

$$V = \frac{q}{4\pi\varepsilon_0}(\frac{1}{r_1} - \frac{1}{r_2})$$
$$= \frac{q}{4\pi\varepsilon_0 r}\{[P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + ..] - [P_0(-x) + P_1(-x)t + P_2(-x)t^2 + P_3(-x)t^3 + ..]$$
$$= \frac{q}{4\pi\varepsilon_0 r}\{[P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + ..] - [P_0(x) - P_1(x)t + P_2(x)t^2 - P_3(x)t^3 + ..]$$
$$= \frac{q}{2\pi\varepsilon_0 r}[P_1(\cos\theta)\frac{a}{r} + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + P_5(\cos\theta)\left(\frac{a}{r}\right)^5 + ...]$$

The first term (and the dominant term for $r >> a$) is

$$V = \frac{q}{2\pi\varepsilon_0 r}[P_1(\cos\theta)\frac{a}{r}] = \frac{2aq}{4\pi\varepsilon_0}\frac{P_1(\cos\theta)}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{P_1(\cos\theta)}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

which is the electric dipole potential and $p = 2aq$ is the electric dipole moment.

7. Electric potential of the electric dipole (general case)

The electric potential of the electric dipole is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{r^3}$$

The entire electric potential at point $\boldsymbol{r}$ is obtained by summing the contributions from all parts of electric dipoles at the point $\boldsymbol{r}$',

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-r'|^3}$$

where $\boldsymbol{P}$ is the electric polarization vector. Using the relation

$$\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

we have

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int\boldsymbol{P}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'$$

Note that

$$\nabla'\cdot\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}') + \boldsymbol{P}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

or

$$\boldsymbol{P}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \nabla'\cdot\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}')$$

Thus we get

$$V(\boldsymbol{r}) = -\frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot \boldsymbol{P}(\boldsymbol{r}')d^3\boldsymbol{r}' + \frac{1}{4\pi\varepsilon_0}\int \nabla'\cdot \frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'$$
$$= -\frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot \boldsymbol{P}(\boldsymbol{r}')d^3\boldsymbol{r}' + \frac{1}{4\pi\varepsilon_0}\int \frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot d\boldsymbol{a}'$$
$$= -\frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot \boldsymbol{P}(\boldsymbol{r}')d^3\boldsymbol{r}' + \frac{1}{4\pi\varepsilon_0}\int \frac{\boldsymbol{P}(\boldsymbol{r}')\cdot \boldsymbol{n}'}{|\boldsymbol{r}-\boldsymbol{r}'|}da'$$

The volume charge density is defined by

$$\nabla\cdot \boldsymbol{P}(\boldsymbol{r}) = -\rho_b(\boldsymbol{r}),$$

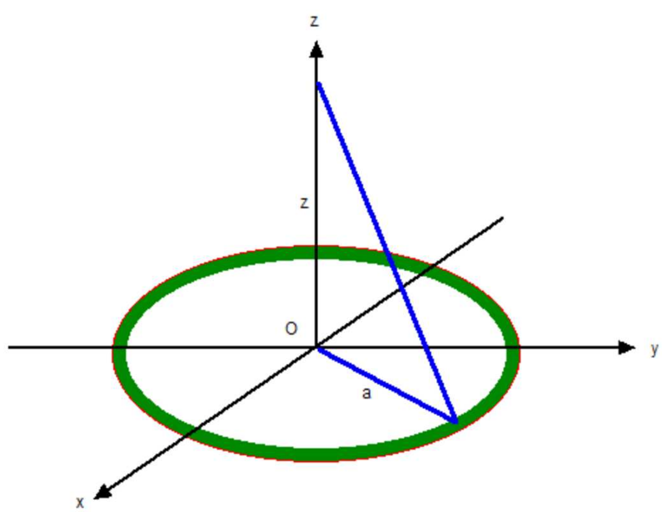
The surface charge density is defined by

$$\boldsymbol{P}(\boldsymbol{r})\cdot \boldsymbol{n} = \sigma_b(\boldsymbol{r})$$

Thus we have

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}' + \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}da'.$$

7. Electric potential due to ring



Charge Q is uniformly distributed on a ring of radius a. Determine the electric potential at a point on the axis of the ring a distance z from the center.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{z^2+a^2}}$$

The electric field ***E*** is directed along the z axis.

$$E_z = -\frac{\partial V}{\partial z} = -\frac{Q}{4\pi\varepsilon_0}\frac{d}{dz}\frac{1}{\sqrt{z^2+a^2}} = \frac{Q}{4\pi\varepsilon_0}\frac{z}{\left(z^2+a^2\right)^{3/2}}$$

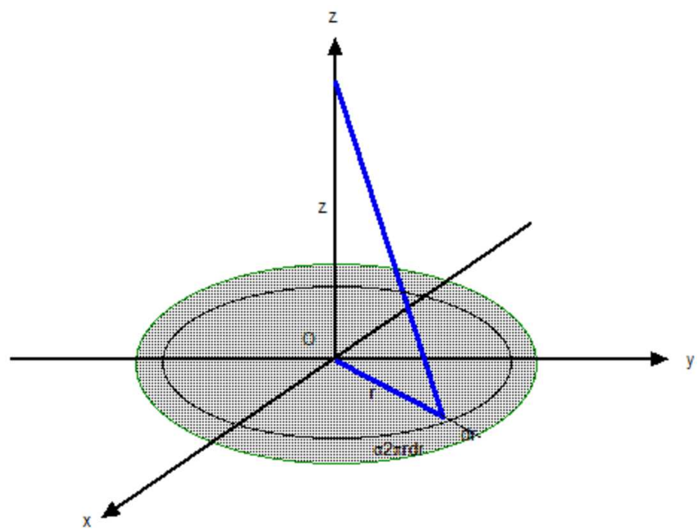
$$E_x = -\frac{\partial V}{\partial x} = 0$$

$$E_y = -\frac{\partial V}{\partial x} = 0$$

8. Electrical potential due to disk

(a) Simple case

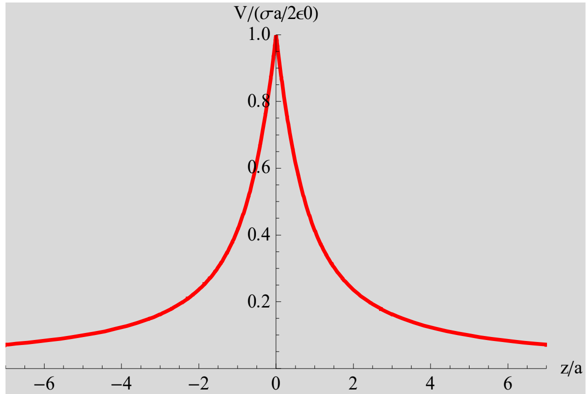
A disk of radius (a) carriers a uniform surface charge density σ. Find the potential on the axis at point A, a distance z from the center.



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi r dr}{\sqrt{z^2+r^2}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{\sigma 2\pi r dr}{\sqrt{z^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r dr}{\sqrt{z^2+r^2}}$$

$$= \frac{\sigma}{2\varepsilon_0}(\sqrt{z^2+a^2} - |z|)$$



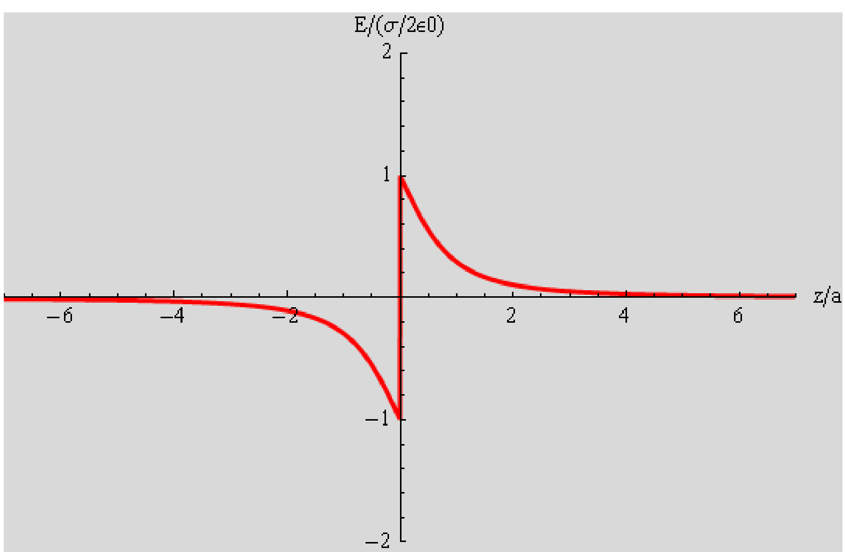
The electric field $\boldsymbol{E}$ is along the z axis.

For $z>0$, we have

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\frac{d}{dz}(\sqrt{z^2+a^2}-z)$$
$$= \frac{\sigma}{2\varepsilon_0}(-\frac{z}{\sqrt{z^2+a^2}}+1)$$

For $z<0$, we have

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\frac{d}{dz}[\sqrt{z^2+a^2}+z]]$$
$$= -\frac{\sigma}{2\varepsilon_0}(\frac{z}{\sqrt{z^2+a^2}}+1)$$



((**Mathematica**))

$$\int_0^a \frac{r}{\sqrt{z^2+r^2}}\,dr\ //$$

```
Simplify[#, {z ∈ Reals, a > 0}] &
```

$$\sqrt{a^2+z^2}-\mathrm{Abs}[z]$$

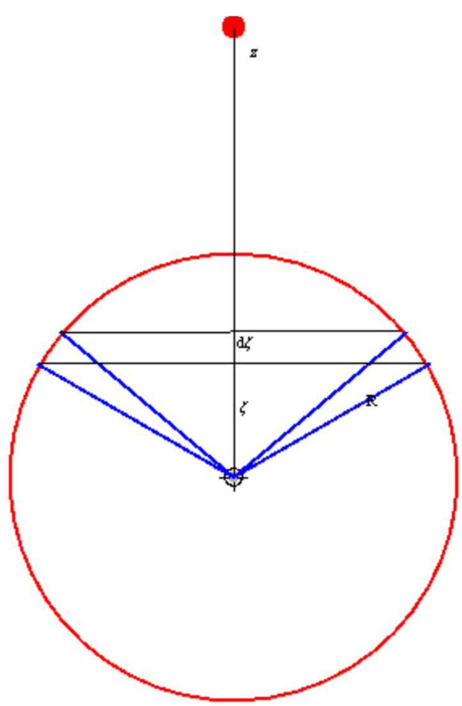
(b) The application to the calculation of the electric potential due to sphere (outside)

We show that the electric potential V (outside the sphere charged with Q) is given by

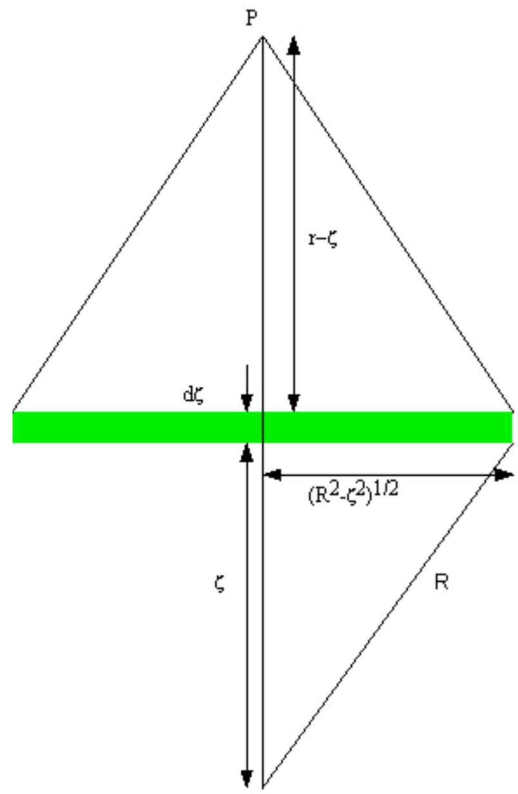
$$V=\frac{Q}{4\pi\varepsilon_0 r}.$$

In the previous calculation, we choose the following replacement in the the calculation of V due to the disk.

$$z \to (r-\zeta),$$
$$a \to \sqrt{R^2-\zeta^2},$$
$$\sigma \to \rho d\zeta$$



or



Then we have an electric potential outside the sphere (radius R),

$$\begin{aligned}
V = \int dV &= \int_{-R}^{R} \frac{\rho d\zeta}{2\varepsilon_0}[\sqrt{(r-\zeta)^2 + R^2 - \zeta^2} - (z-\zeta)] \\
&= \frac{\rho}{2\varepsilon_0}\int_{-R}^{R} d\zeta[\sqrt{(r-\zeta)^2 + R^2 - \zeta^2} - (z-\zeta)] \\
&= \frac{\rho}{2\varepsilon_0}\int_{-R}^{R} d\zeta[\sqrt{r^2 - 2\zeta r + R^2} - (z-\zeta)] \\
&= \frac{\rho}{2\varepsilon_0}\frac{2R^3}{3r} = \frac{\rho}{\varepsilon_0}\frac{R^3}{3r} = \frac{Q}{4\pi\varepsilon_0 r}
\end{aligned}$$

((**Note**))

This is a good lesson for the calculation of V. Through this calculation, we realize that it is much easier for one to calculate the electric field first using Gauss' law. Then the electric potential can be calculated from the integral of electric field.

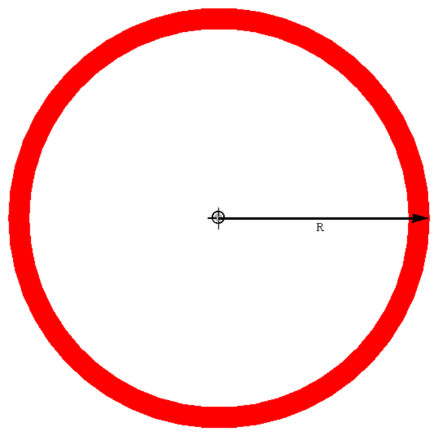
((**Mathemtatica**))

```
∫_{-R}^{R} ((z^2 - 2 z ξ + R^2)^{1/2} - z + ξ) dξ //
 Simplify[#, {z > 0, R > 0, z > R}] &
```

$$\frac{2\,R^3}{3\,z}$$

9. Electric potential due to spherical shell

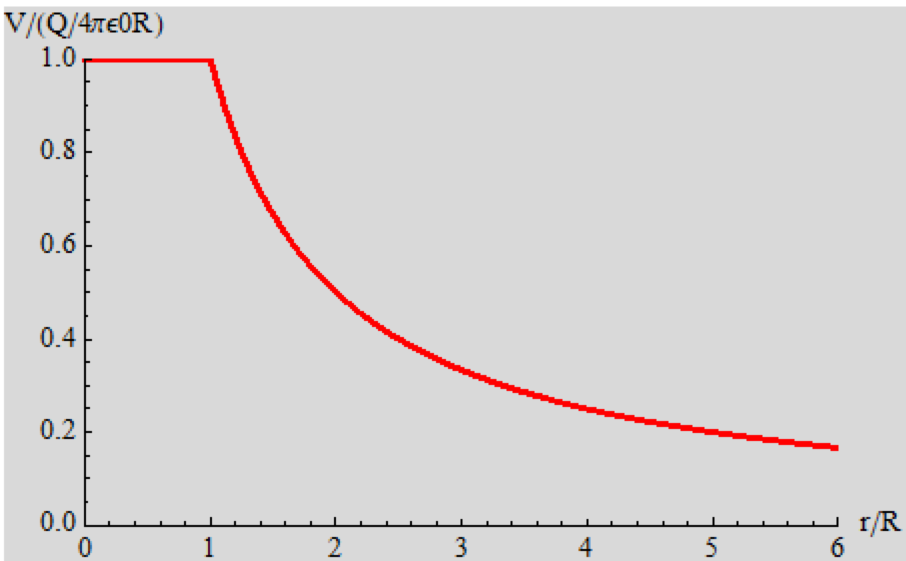
A spherical shell of radius R carries charge Q uniformly distributed over the surface. Determine V inside and outside.



A spherical shell of radius R carries charge Q uniformly distributed over the surface. Determine V inside and outside. For $r>R$, we have $V=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$. For $r<R$, there is no charge. So we have $E = 0$ (Gauss' law). leading to $E=-\frac{\partial V}{\partial r}=0$. V should be constant ($V=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$) inside the shell. Note that V should be continuous at $r = R$ (boundary condition). *The potential inside a shell is equal to the potential at the surface of the shell.*

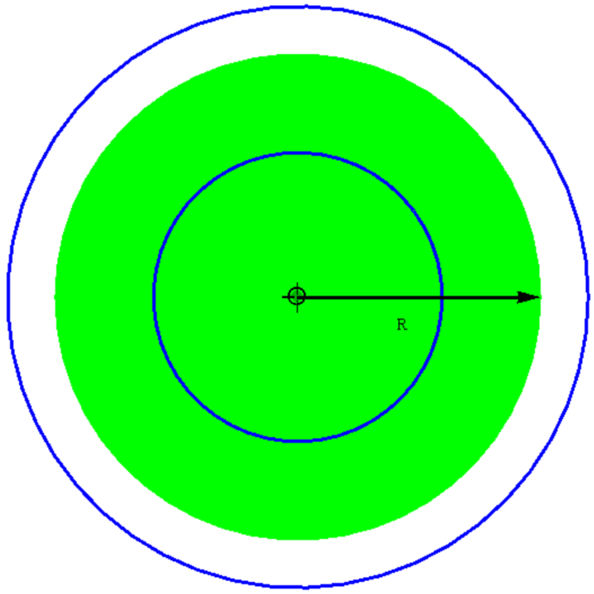
((**Mathematica**))

```
f = If[x < 1, 1, 0] + If[x > 1, 1/x, 0];
Plot[f, {x, 0, 6},
 PlotStyle → {Red, Thick},
 PlotRange → {{0, 6}, {0, 1}},
 AxesLabel → {"r/R", "V/(Q/4πϵ0R)"},
 Background → LightGray]
```



10. Electric potential due to sphere

10.1 Direct calculation of the electric potential (method-1)



An insulating sphere of radius R has uniform volume charge density ρ. Determine the potential everywhere. Later we realize that the method-2 (using the Gauss' law) is much easier than the method-1. The method-2 will be shown after this discussion.

For $r>R$, we have the electric potential given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\frac{4\pi\rho R^3}{3} = \frac{\rho R^2}{3\varepsilon_0}\frac{R}{r}.$$

The electric field is directed along the radial direction.

$$E_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}.$$

For $r<R$, the charge within a sphere (radius r) is

$$Q' = \frac{4\pi r^3}{3}\rho = \frac{4\pi r^3}{3}\frac{Q}{\frac{4\pi R^3}{3}} = \frac{Qr^3}{R^3}.$$

Then the electric potential V_1 arising from the part within r $(0 - r)$

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q'}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\frac{4\pi R^3\rho}{3} = \frac{\rho R^2}{6\varepsilon_0}(2\frac{R}{r}).$$

The electric potential V_2 arising from the charge between r and R, is calculated as a sum of the shell ($r' - r'+dr'$) ($r<r'<R$). We note that the potential inside a shell is equal to the potential at the surface of the shell.

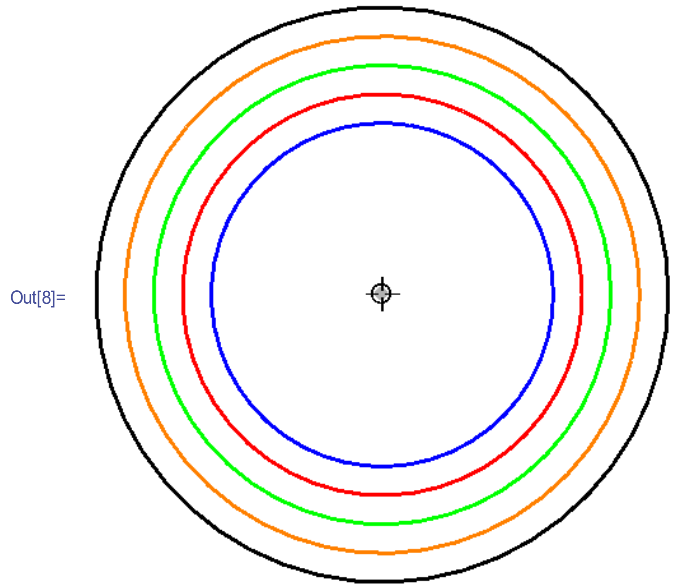


Fig. Sphere (blue) with radius r. Sphere (black) with radius R. Spherical shells (regions between spheres with black and brown, brown and green, green and red, red and blue), and so on are schematically shown.

Then we have

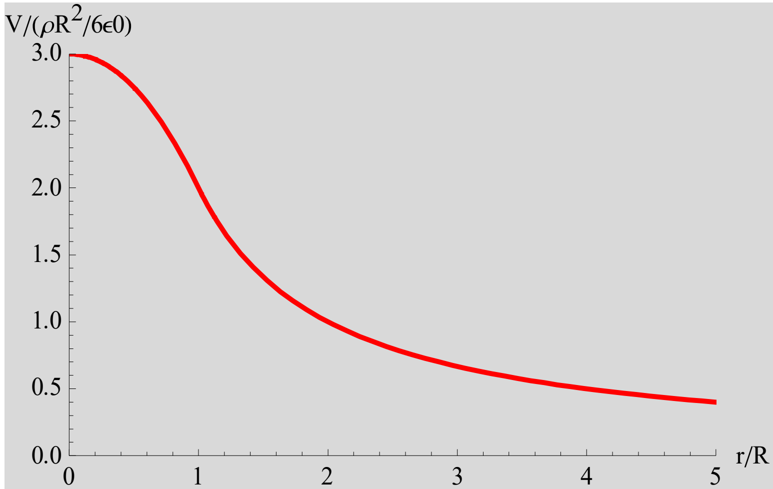
$$dV_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r'}\rho(4\pi r'^2\,dr'),$$

or

$$V_2 = \frac{1}{4\pi\varepsilon_0}\int_r^R \frac{\rho(4\pi r'^2)dr'}{r'} = \frac{\rho}{\varepsilon_0}\int_r^R r'dr' = \frac{\rho}{2\varepsilon_0}(R^2 - r^2)\,.$$

Thus the resulting electric potential is

$$\begin{aligned} V = V_1 + V_2 &= \frac{\rho r^2}{3\varepsilon_0} + \frac{\rho}{2\varepsilon_0}(R^2 - r^2) \\ &= \frac{\rho r^2}{3\varepsilon_0} - \frac{\rho r^2}{2\varepsilon_0} + \frac{\rho R^2}{2\varepsilon_0} = -\frac{\rho r^2}{6\varepsilon_0} + \frac{\rho R^2}{2\varepsilon_0} \\ &= \frac{\rho R^2}{6\varepsilon_0}(3 - \frac{r^2}{R^2}) \end{aligned}$$



10.2 Method-2 (using the Gauss' law)

The electric field E is given by

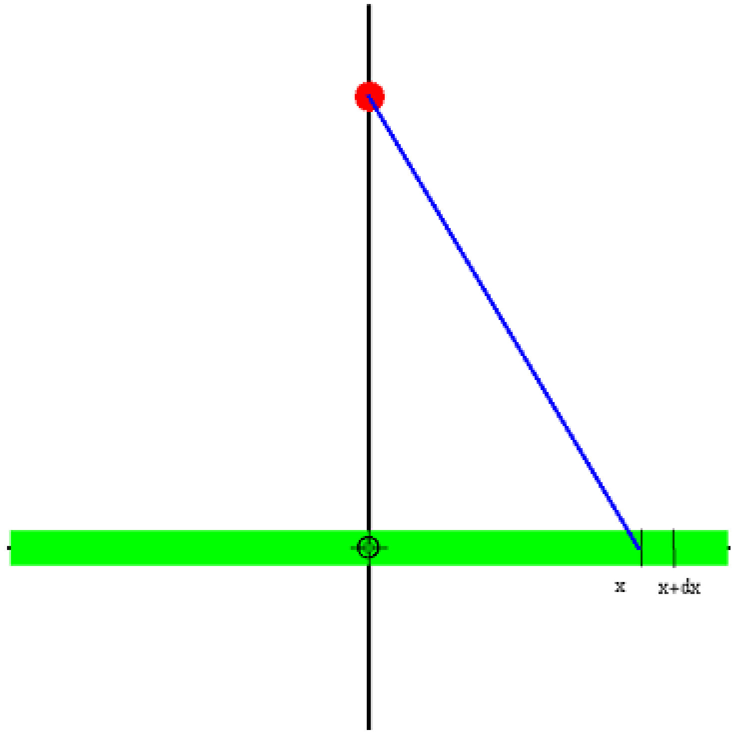
$$E_{out} = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \text{for } r>R.$$

$$E_{in} = \frac{Qr}{4\pi\varepsilon_0 R^3} \qquad \text{for } 0<r<R$$

Then the electric potential inside the sphere is

$$\begin{aligned}
V &= -\int_{\infty}^{R} E_{out} dr - \int_{R}^{r} E_{in} dr \\
&= -\int_{\infty}^{R} \frac{Q}{4\pi\varepsilon_0 r^2} dr - \int_{R}^{r} \frac{1}{4\pi\varepsilon_0 r^2} \frac{Qr^3}{R^3} dr \\
&= -\int_{\infty}^{R} \frac{Q}{4\pi\varepsilon_0 r^2} dr - \int_{R}^{r} \frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3} dr \\
&= \frac{Q}{4\pi\varepsilon_0} \left(\frac{3}{2R} - \frac{r^2}{2R^3}\right) \\
&= \frac{\rho R^2}{6\varepsilon_0} \left(3 - \frac{r^2}{R^2}\right)
\end{aligned}$$

11. Electric potential due to a charged rod (case-I)



A total charge Q is distributed uniformly along a straight rod of length L. Find the potential at point P at a distance h from the midpoint of the rod. The potential at P due to a small segment of the rod, with length dx and charge dQ, located at the position is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{\sqrt{x^2+h^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{\sqrt{x^2+h^2}}$$

with the line charge density $\lambda = Q/L$

$$V = \int dV = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dx}{\sqrt{x^2+h^2}} = \frac{Q}{2\pi L\varepsilon_0}\int_0^{L/2}\frac{dx}{\sqrt{x^2+h^2}}$$

$$= \frac{Q}{2\pi L\varepsilon_0}\ln[\frac{L+\sqrt{4h^2+L^2}}{2h}]$$

((**Mathematica**))

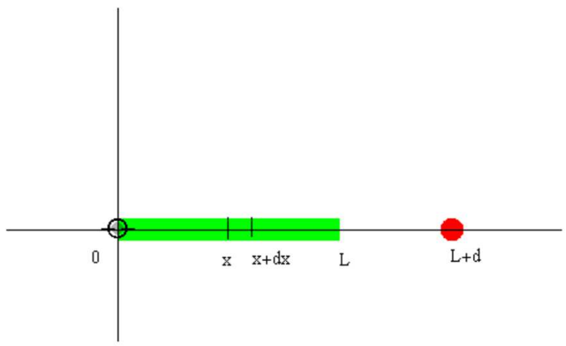
$$\int_0^{L/2} \frac{1}{\sqrt{x^2+h^2}}\, dx \; //$$

```
Simplify[#, {L > 0, h > 0}] &
```

$$\mathrm{Log}\left[\frac{L+\sqrt{4\,h^2+L^2}}{2\,h}\right]$$

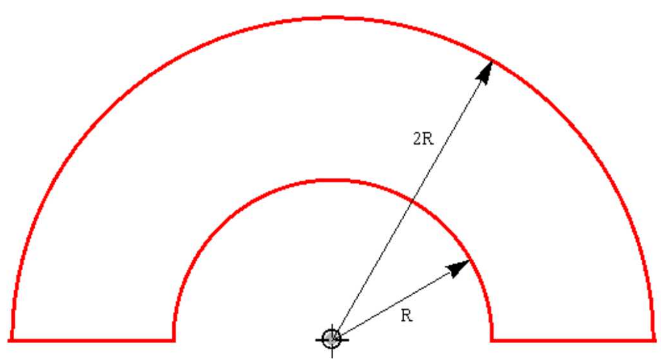
12. Electrical potential due to a charged rod (Case-II)

A rod of length L has a uniform linear density λ. Determine the potential at a point P on the axis of the rod a distance from one end.



$$V=\int dV=\frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dx}{L+d-x}=-\frac{\lambda}{4\pi\varepsilon_0}\ln[L+d-x]\,|_0^L=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{d+L}{d}\right)$$

13. Electric potential due to a circled wire



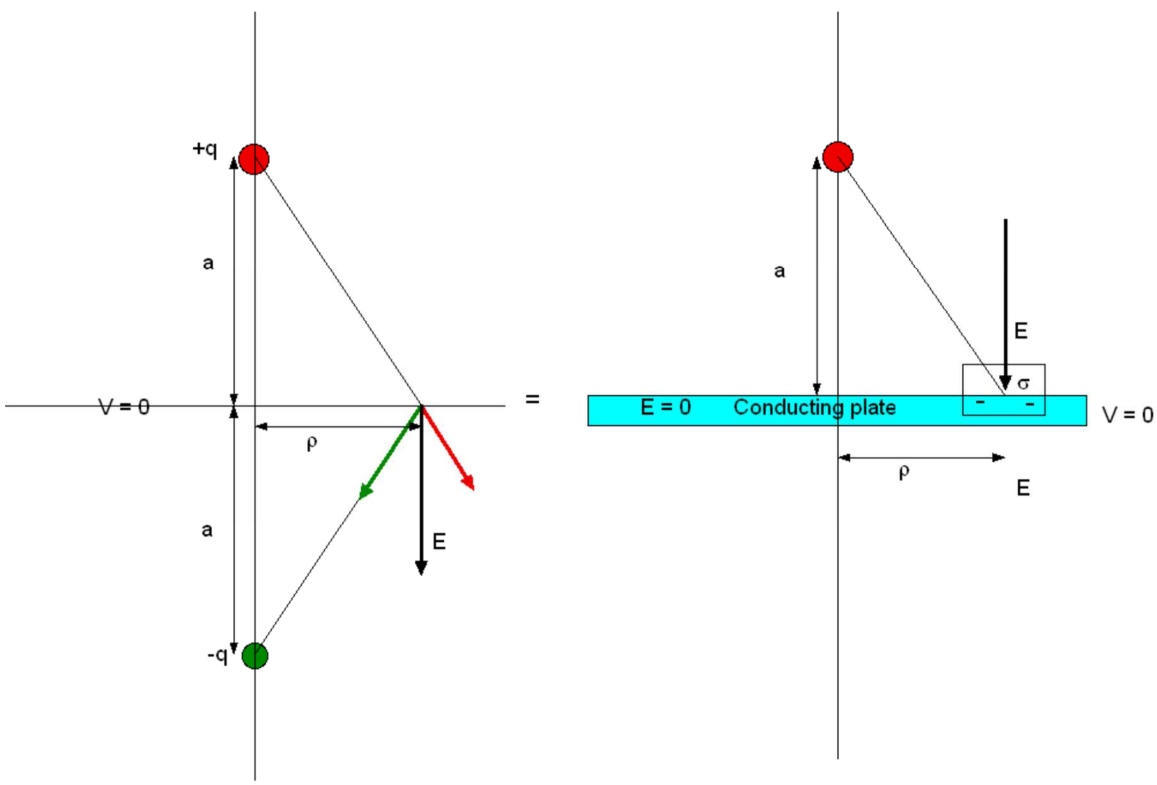
A rod with uniform linear charge density λ is bent into the shape above. Find the potential at the center for this configuration. We use the above results. Then we have

$$
\begin{aligned}
V &= 2\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{R+R}{R}\right)+\frac{1}{4\pi\varepsilon_0}\frac{\lambda\pi(2R)}{2R}+\frac{1}{4\pi\varepsilon_0}\frac{\lambda\pi(R)}{R} \\
&= \frac{\lambda}{2\pi\varepsilon_0}\ln 2+\frac{\lambda}{4\varepsilon_0}+\frac{\lambda}{4\varepsilon_0} \\
&= \frac{\lambda}{2\pi\varepsilon_0}\ln 2+\frac{\lambda}{2\varepsilon_0} \\
&= \frac{\lambda}{4\pi\varepsilon_0}2(\pi+\ln 2)
\end{aligned}
$$

14. Method of images

(a) A point charge near a conductor plane

We assume that there are two point charges as shown in this figure. The distance between the two point charges is $2a$. We notice that the plane, since it is halfway between the two charges, has zero potential. This implies that the right-half plane picture is the same as that obtained from a point charge in front of a conducting sheet.



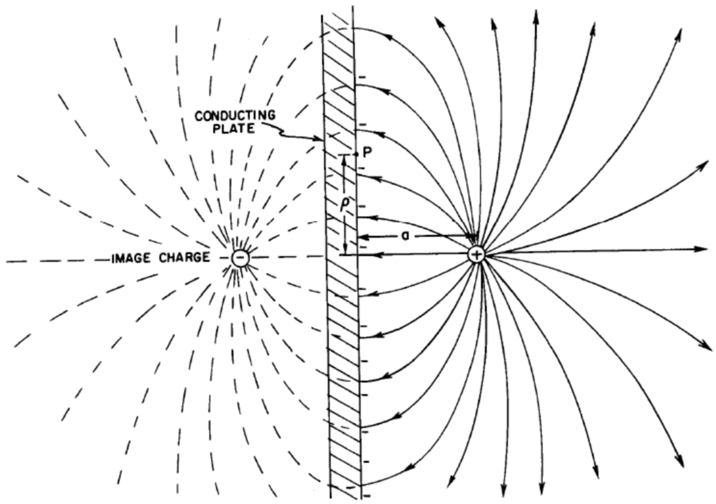
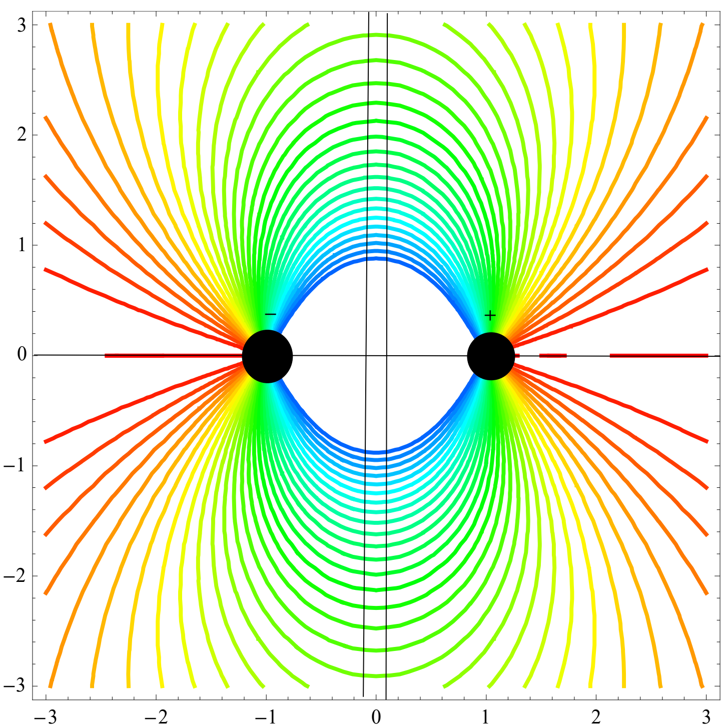


Fig. 6–10. The field of a charge near a plane conducting surface, found by the method of images.

(Feynman Lectures on Physics)



We consider a point on the conducting sheet at the distance ρ from the point directly beneath the positive charge. The electric field at this point is normally to the surface and is directed into it. The electric field resulting from both the positive point charge and the negative point charge is normal to the sheet and is given by

$$E(\rho) = -\frac{2aq}{4\pi\varepsilon_0(\rho^2 + a^2)^{3/2}}.$$

From the Gauss' law, the surface charge density is given by

$$\sigma(\rho) = \varepsilon_0 E(\rho) = -\frac{2aq}{4\pi(\rho^2 + a^2)^{3/2}}.$$

We note that the total induced charge is - q as it should be.

$$\int_0^\infty \sigma(\rho) 2\pi\rho d\rho = -q$$

We show the ContourPlot of the normalized surface charge density which is given by

$$-\frac{\sigma(\rho)}{\left(\dfrac{q}{2\pi a^2}\right)} = \frac{1}{(\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + 1)^{3/2}}$$

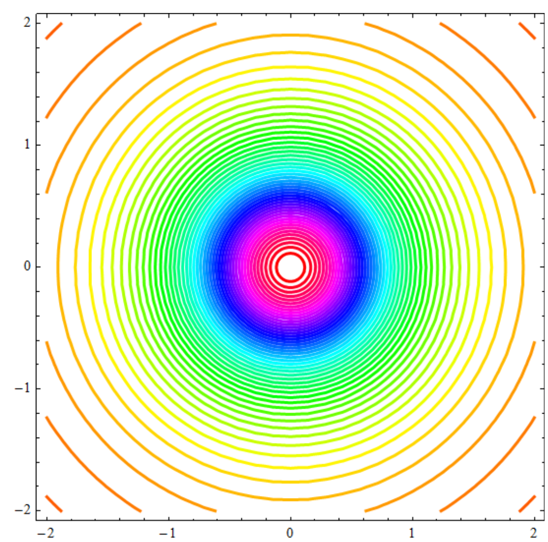


Fig. ContourPlot of the normalized surface charge density in the (x/a, y/a) plane.

((**Mathematica**))

$$\int_0^\infty \frac{-2\, a\, q\, 2\, \pi\, \rho}{4\, \pi\, \left(\rho^2 + a^2\right)^{3/2}} \, d\rho \;\texttt{// Simplify[\#, a > 0] \&}$$

$-q$

(b) A point charge near a conducting sphere

We find the fields around a conducting sphere which has a point charge near it. Now we must look for a simple physical situation which gives a sphere for an equipotential surface. As shown in this figure, the field of two unequal point charges has an equipotential that is a sphere. If we choose the location of an image charge- and pick the right amount of charge- may be we can make the equipotential surface fit our sphere.

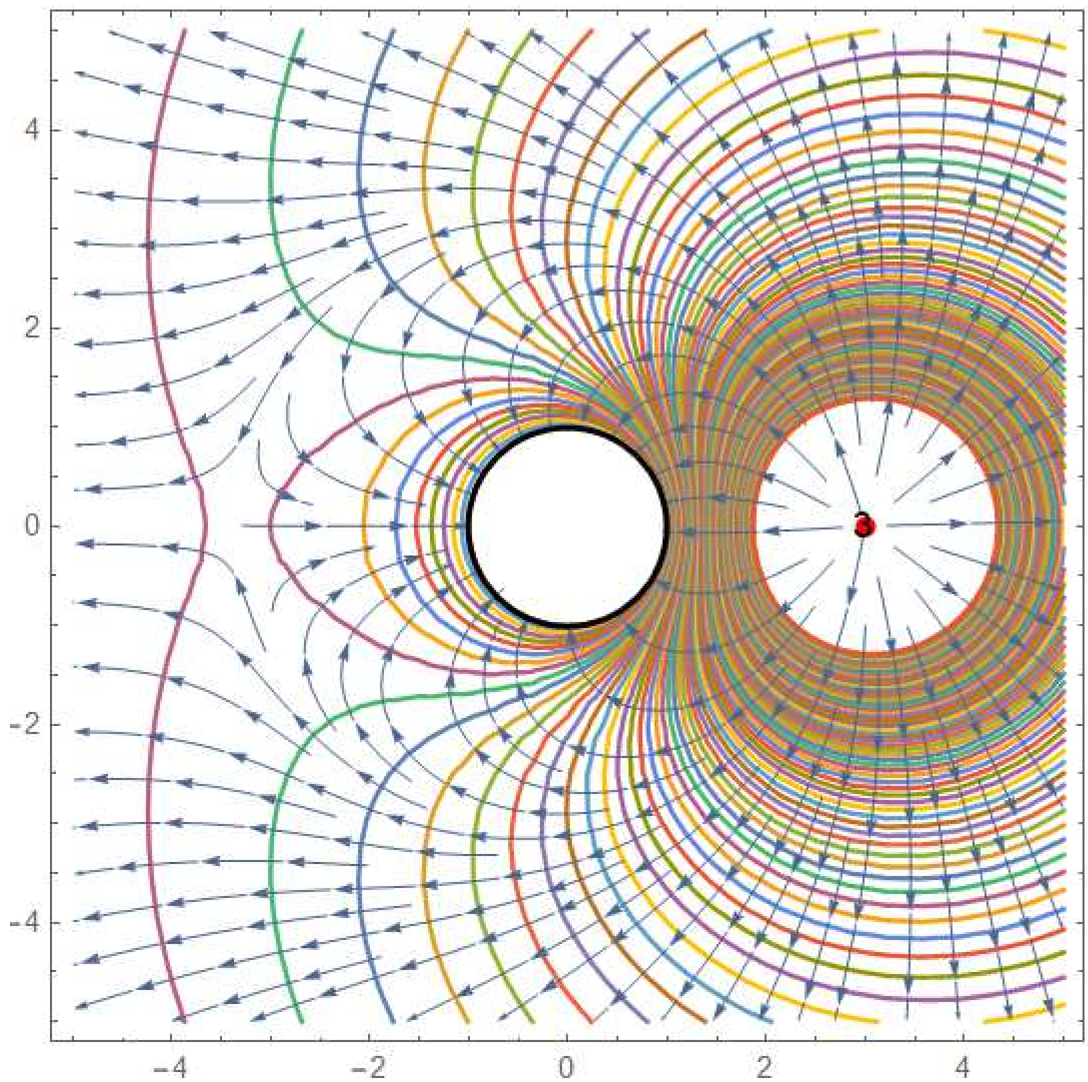


Fig. Electric field lines for a = 1, b = 3, q = 3, and $q' = -q(a/b) = -1$. The zero-equipotential surface exists as a sphere of radius 1 centered at x = 0 and y = 0. The charge of (q = 3) and the charge (q'=-1) are located at x = 1/3 and x = 3, respectively.

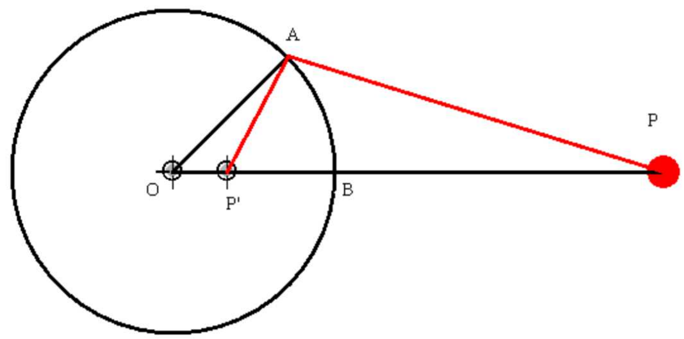


Fig. The point charge q induces charges on a grounded conducting sphere whose fields are those of an image charge q' placed at the point shown. AP = r_1, P'A = r_2, OA = a, OP = b, OP' = a^2/b, q' = $(-a/b)q$. We choose b = 3, a = 1, q = 3, q'=-1. OP' = 1/3, OA = 1, OP = 3 in the above figure (using Mathematica).

Assume that one wants the equipotential surface to be a sphere of radius a with its center at the distance b from the charge. Put an image of the strength $q' = -(a/b)q$ on the line from the charge to the center of the sphere, and at a distance a^2/b from the center. The sphere will be at zero potential.

The mathematical proof is given as follows. The potential V at P from q and q' is given by

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right).$$

The potential will be zero at all points for which

$$\frac{q}{r_1} + \frac{q'}{r_2} = 0 \qquad \text{or} \qquad \frac{r_2}{r_1} = -\frac{q'}{q}.$$

If q' is placed at the distance OP' = a^2/b from the center, the ratio (r_2/r_1) has the constant value a/b.

In fact, suppose that ΔAOP is similar to ΔPOA. Then we have

$$\frac{AP'}{AP} = \frac{OA}{OP} = \frac{OP'}{OA} \qquad \text{or} \qquad \frac{r_2}{r_1} = \frac{a}{b} = \frac{OP'}{a}$$

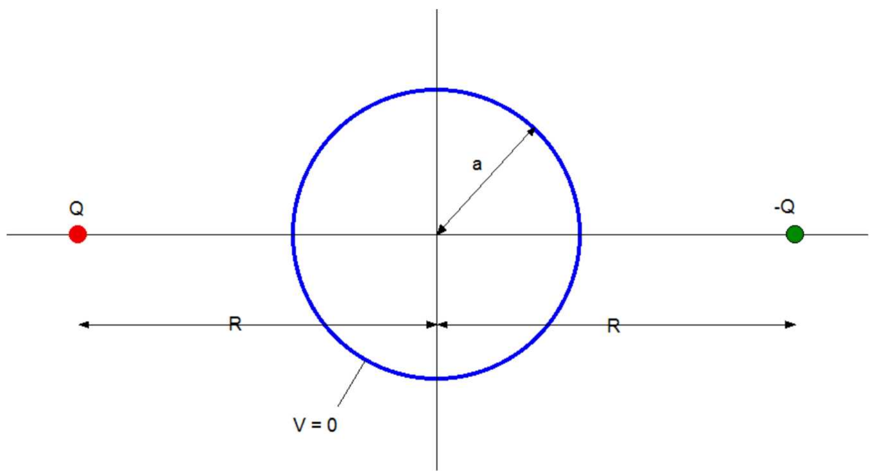
or

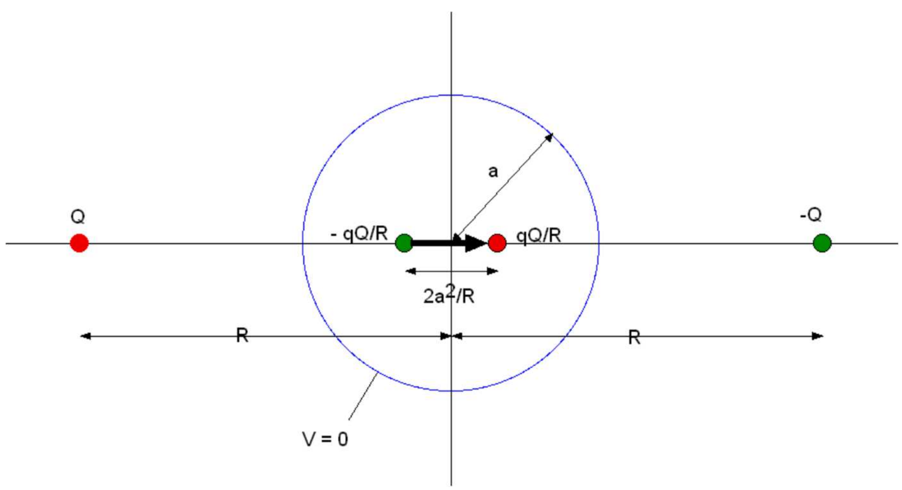
$$OP = \frac{a^2}{b} \qquad \text{and} \qquad \frac{q'}{q} = -\frac{a}{b}$$

What is the attractive force between the point charge q and the conducting sphere? This force is equal to the attractive Coulomb force between q and q'.

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{P'P^2} = -\frac{1}{4\pi\varepsilon_0}\frac{abq^2}{(b^2 - a^2)^2}$$

(c) A conducting sphere in the presence of a uniform electric field





How does the electric field change when the conducting sphere (radius a) is put in the uniform electric field $\boldsymbol{E}_0$ along the x axis. Suppose that this electric field is generated by two point charges of Q at x = -R and $-Q$ at x = R. The electric field at the sphere is $2Q/(4\pi\varepsilon_0 R^2)$. In the limit of $R \rightarrow 0$, the field is parallel to the x axis.

$$E_0 = \frac{2Q}{4\pi\varepsilon_0 R^2}$$

Now we consider the image charges of Q at x = -R and –Q at x = R, concerning the conducting sphere. The image charges ($\pm Q\frac{a}{R}$) are located at $x = \pm a^2 / R$. Then these image charges form an electric dipole with the electric dipole moment,

$$p = \frac{2Qa}{R}\frac{a^2}{R} = \frac{2Qa^3}{R^2} = \frac{4\pi\varepsilon_0 R^2 E_0 a^3}{R^2} = 4\pi\varepsilon_0 E_0 a^3$$

The electric potential of the electric dipole moment is given by

$$V_{dipole} = \frac{p}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{4\pi\varepsilon_0 E_0 a^3}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{E_0 a^3}{r^2}\cos\theta$$

From the superposition principle, the resulting potential is a sum of V_{dipole} and original uniform field (E_0) along the x axis,

$$V_{tot} = -E_0 x + \frac{E_0 a^3}{r^2}\cos\theta + const = E_0(-r + \frac{a^3}{r^2})\cos\theta + const$$

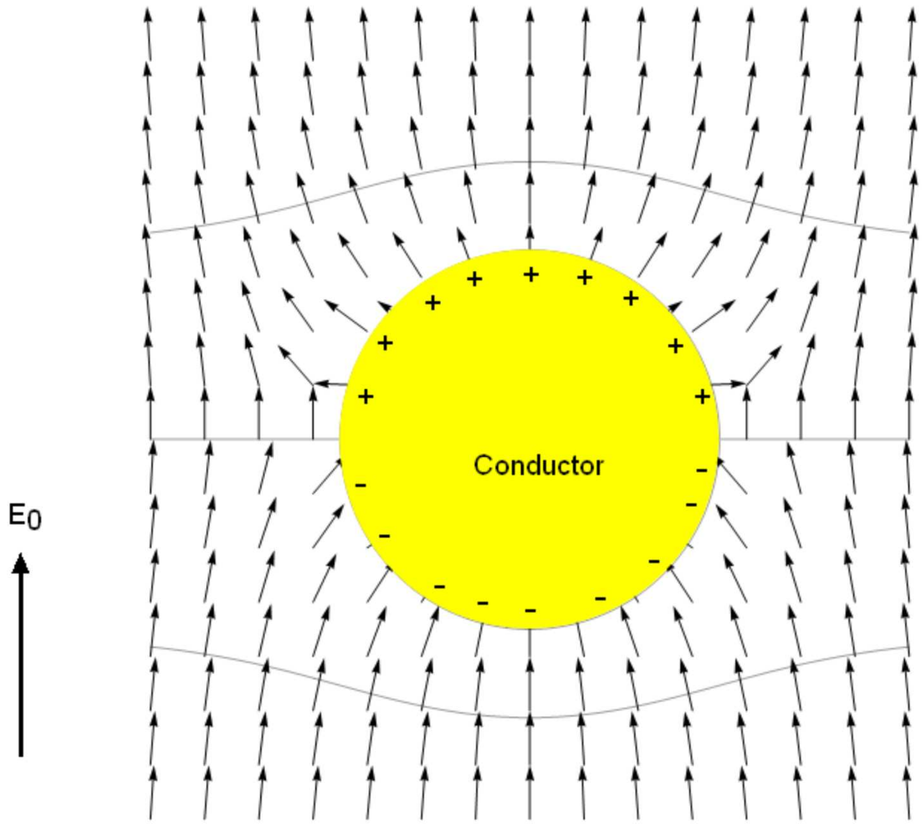
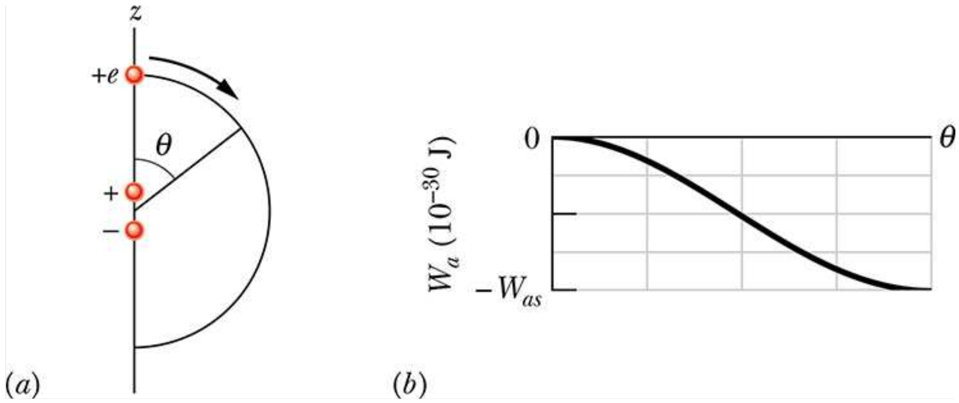


Fig. Electric field distribution near a conducting sphere in the presence of a uniform electric field. $a = 3$ and $E_0 = 1$. The electric field should be zero everywhere inside the sphere.

15. Typical examples

15.1 Problem 24-22 (SP-24)

In Fig.(a), a particle of charge $+e$ is initially at coordinate $z = 20$ nm on the dipole axis through an electric dipole, on the positive side of the dipole. (The origin of z is at the dipole center.) The particle is then moved along a circular path around the dipole center until it is at coordinate z = -20 nm. Figure (b) gives the work W_a done by the force moving the particles versus the angle θ that locates the particle. The scale of the vertical axis is set by $W_{as} = 2.0 \times 10^{-30}$ J. What is the magnitude of the dipole moment?



((**My solution**))
R = 20 nm

$$V_p = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = \frac{pR\cos\theta}{4\pi\varepsilon_0 R^3} = \frac{p\cos\theta}{4\pi\varepsilon_0 R^2}$$

$$U = eV_p = \frac{ep\cos\theta}{4\pi\varepsilon_0 R^2}$$

U can be rewritten as

$$U = W_{as}\cos\theta$$

where

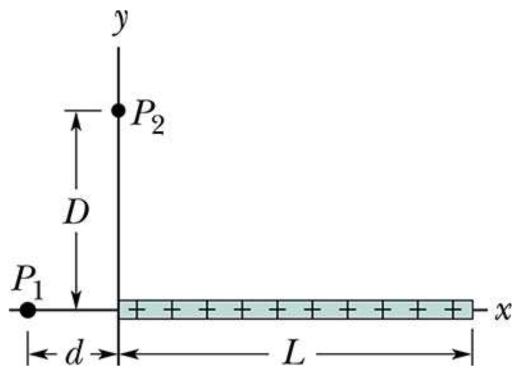
$$W_{as} = \frac{ep}{4\pi\varepsilon_0 R^2} = 4.0\times10^{-30}\,J$$

Then we have

$$p = \frac{W_{as}4\pi\varepsilon_0 R^2}{e} = 1.11114 \text{ x } 10^{-36}\text{ Cm}$$

15.2 Problem 24-30 (SP-24)

Figure shows a thin plastic rod of length L = 12.0 cm and uniform positive charge Q = 56.1 fC lying on an x axis. With V = 0 at infinity, find the electric potential at point P_1 on the axis, at distance d = 2.50 cm from one end of the rod.



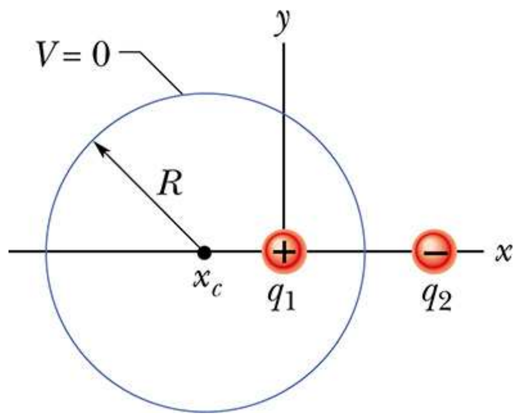
((**Solution**))
Q = 56.1 fC
d = 2.50 cm
L = 12.0 cm

$$\lambda = \frac{Q}{L}$$

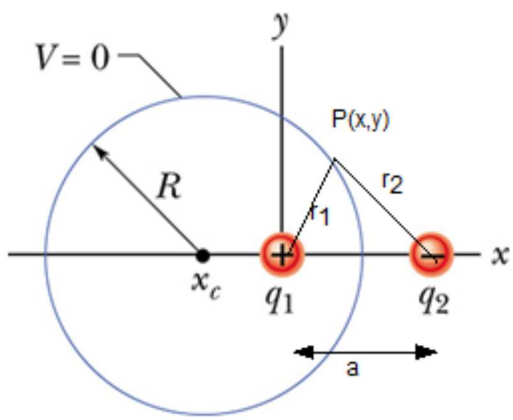
$$\begin{aligned}
V &= \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dx}{d+x} \\
&= \frac{1}{4\pi\varepsilon_0}\lambda\int_0^L \frac{dx}{d+x} = \frac{\lambda}{4\pi\varepsilon_0}[\ln(d+x)]_0^L \\
&= \frac{\lambda}{4\pi\varepsilon_0}\ln(\frac{d+L}{d}) \\
&= \frac{Q}{4\pi\varepsilon_0 L}\ln(\frac{d+L}{d})] = 7.39 mV
\end{aligned}$$

15.3 Problem 24-114

A point charge $q_1 = +6.0e$ is fixed at the origin of a rectangular coordinate system, and a point charge $q_2 = -10e$ is fixed at $x = 8.6$ nm, $y = 0$. The locus of all points in the xy plane for which $V = 0$ (other than infinity) is a circle centered on the x axis, as shown in Fig. Find (a) the location x_c of the center of the circle and (b) the radius R of the circle. (c) Is the xy cross section of the 5 V equipotential surface also a circle?



((**Solution**))
$q_1 = 6$ e
$q_2 = -10$ e
$a = 8.6$ nm
$e = 1.602176487 \times 10^{-19}$ C Charge of electron



(a) and (b)

The electric potential V is equal to zero,

$$V = \frac{q_1}{4\pi\varepsilon_0 r_1} + \frac{q_2}{4\pi\varepsilon_0 r_2} = 0$$

or

$$\frac{r_1}{r_2} = -\frac{q_1}{q_2}$$

Noting that

$$r_1 = \sqrt{x^2 + y^2}$$
$$r_2 = \sqrt{(x-a)^2 + y^2}$$
$$\frac{q_1}{q_2} = -\frac{3}{5}$$

we have

$$25(x^2 + y^2) = 9[(x-a)^2 + y^2]$$

or

$$x^2 + y^2 = -\frac{9}{8}ax + \frac{9}{16}a^2$$

or

$$[(x^2 + \frac{9}{8}ax + \left(\frac{9a}{16}\right)^2] + y^2 \ = +\frac{9}{16}a^2 + \left(\frac{9a}{16}\right)^2]$$
$$(x + \frac{9a}{16})^2 + y^2 = \left(\frac{15}{16}a\right)^2$$

This is the circle of radius (15*a*/16) at the center (-9*a*/16, 0).

(c)

$$V = \frac{q_1}{4\pi\varepsilon_0 r_1} + \frac{q_2}{4\pi\varepsilon_0 r_2} = \frac{q_1}{4\pi\varepsilon_0\sqrt{x^2 + y^2}} - \frac{|q_2|}{4\pi\varepsilon_0\sqrt{(x-a)^2 + y^2}}$$
$$= \frac{q_1}{4\pi\varepsilon_0 a\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2}} - \frac{|q_2|}{4\pi\varepsilon_0\sqrt{(\frac{x}{a} - 1)^2 + \left(\frac{y}{a}\right)^2}}$$
$$= \frac{A_1}{\sqrt{\xi^2 + \eta^2}} - \frac{A_1}{\sqrt{(\xi - 1)^2 + \eta^2}}$$

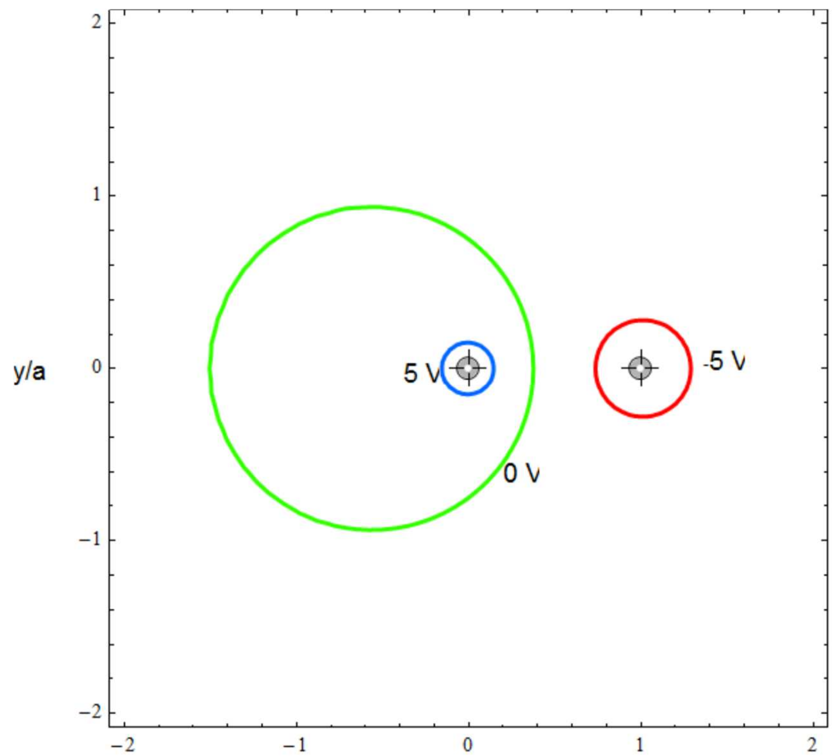
where

$$\xi = \frac{x}{a}$$

$$\eta = \frac{y}{a}$$

$$A_1 = \frac{q_1}{4\pi\varepsilon_0 a}$$

$$A_2 = \frac{|q_2|}{4\pi\varepsilon_0 a}$$

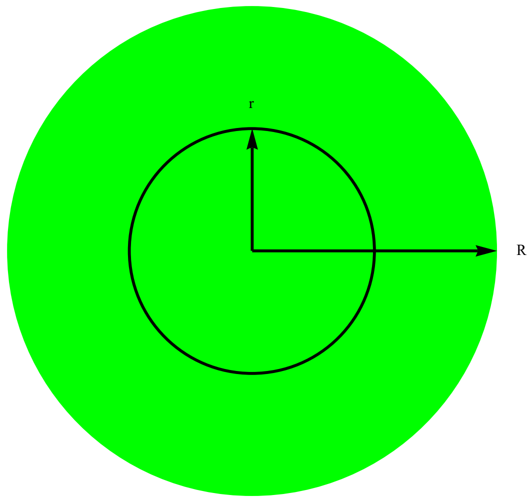


16. Hint of HW-24

16.1 **Problem 24-11** ***

A nonconducting sphere has radius R = 2.31 cm and uniformly distributed charge q = + 3.50 fC. Take the electric potential at the sphere's center to be $V_0 = 0$. What is V at radial distance (a) r = 1.45 cm and (b) $r = R$.

((**Solution**))



$$q = \frac{4\pi}{3}\rho R^3 = 3.50\ fC$$

$R = 2.31$ cm

Gauss' theorem

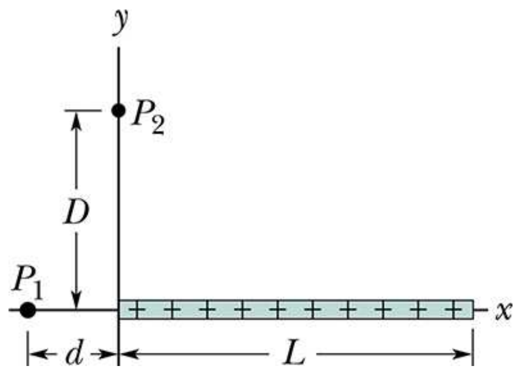
$$E(4\pi r^2) = \frac{\rho}{4\pi\varepsilon_0}\frac{4\pi}{3}r^3$$

$$E = \frac{\rho r}{3\varepsilon_0}$$

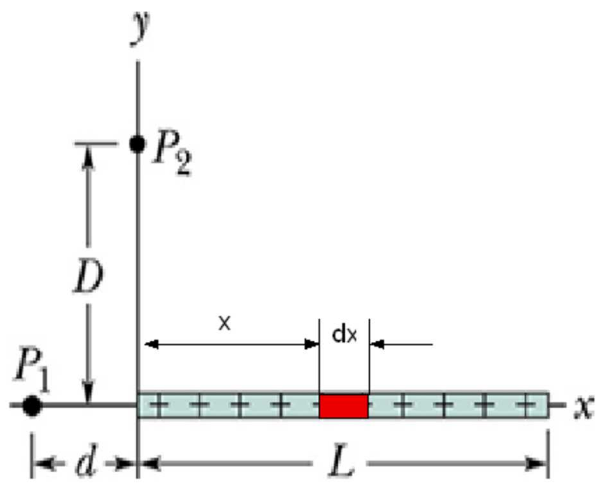
$$V = -\int_0^r \frac{\rho r}{3\varepsilon_0}dr$$

16.2 Problem 24-33***

The thin plastic rod shown in Fig has length L = 12.0 cm and a nonuniform linear charge density $\lambda = cx$', where c = 28.9 pC/m^2. With V = 0 at infinity, find the electric potential at point P_1 on the axis, at distance d = 3.00 cm from one end.



((**Solution**))
$L = 12.0$ cm
$c = 28.9$ PC/m^2
$d = 3$ cm

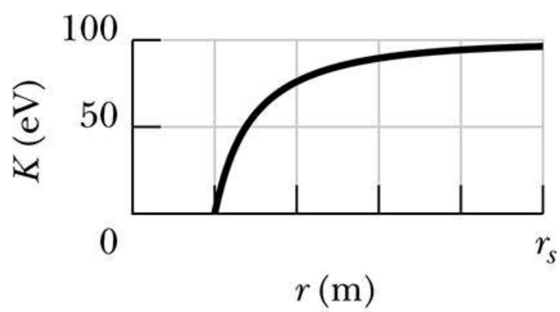


$$V = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dx}{d+x} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{d+L}{d}\right)$$

16.3 Problem 24-56

Figure a shows an electron moving along an electric dipole axis toward the negative side of the dipole. The dipole is fixed in place. The electron was initially very far from the dipole, with kinetic energy 100 eV. Figure b gives the kinetic energy K of the electron versus its distance r from the dipole center. The scale of the horizontal axis is set by $r_s = 0.10$ m. What is the magnitude of the dipole moment?

(a)



(b)

((**Solution**))
r_s = 0.1 m
Intial kinetic energy K_i = 100 eV at $x = \infty$

$$V = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = \frac{(-p\hat{x})\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = \frac{-px}{4\pi\varepsilon_0 r^3}$$

When $r = x$

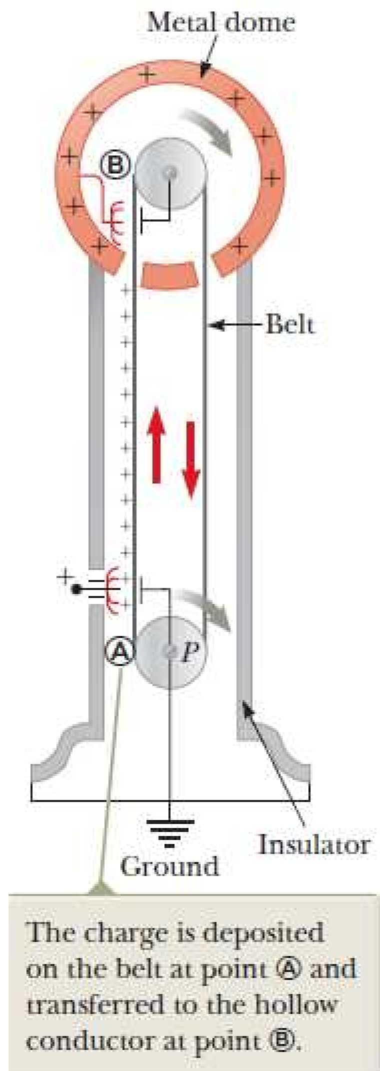
$$V = \frac{-px}{4\pi\varepsilon_0 x^3} = \frac{-p}{4\pi\varepsilon_0 x^2}$$

The potential energy is given by

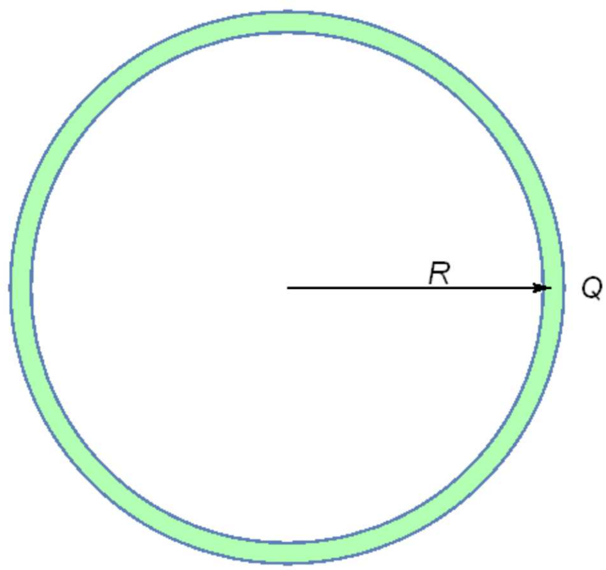
$$U = (-e)V = \frac{ep}{4\pi\varepsilon_0 x^2}$$

APPENDIX-I   Van der Graaff generator

R.A. Serway and J.W. Jewett, Jr. Physics for Scientists and Engineers, 8th edition (Brooks/Cole, 2010).



In 1929, Robert J. Van de Graaff (1901–1967) used this principle to design and build an electrostatic generator, and a schematic representation of it is given in Fig. This type of generator was once used extensively in nuclear physics research. Charge is delivered continuously to a high-potential electrode by means of a moving belt of insulating material. The high-voltage electrode is a hollow metal dome mounted on an insulating column. The belt is charged at point A by means of a corona discharge between comb-like metallic needles and a grounded grid. The needles are maintained at a positive electric potential of typically 10^4 V. The positive charge on the moving belt is transferred to the dome by a second comb of needles at point B. Because the electric field inside the dome is negligible, the positive charge on the belt is easily transferred to the conductor regardless of its potential. In practice, it is possible to increase the electric potential of the dome until electrical discharge occurs through the air. Because the “breakdown” electric field in air is about 3×10^6 V/m, a sphere 1.00 m in radius can be raised to a maximum potential of 3×10^6 V. The potential can be increased further by increasing the dome’s radius and placing the entire system in a container filled with high-pressure gas.



The electric potential V of the Van der Graaff generator is given by

$$V = \frac{Q}{4\pi\varepsilon_0 R},$$

where Q is the surface charge of the conducting sphere (radius R) for the Van der Graaf generator. The electric field E is expressed by

$$E = \frac{Q}{4\pi\varepsilon_0 R^2},$$

or

$$V = ER.$$

The capacitance C is given by

$$C = 4\pi\varepsilon_0 R.$$

For the Earth, C = 708.981 μF. For the Sun, C = 0.0774393 F. For the Van de Graaff, C = 700 pF (for R = 0.27 m).

Suppose that $E = 3 \times 10^6$ V/m.

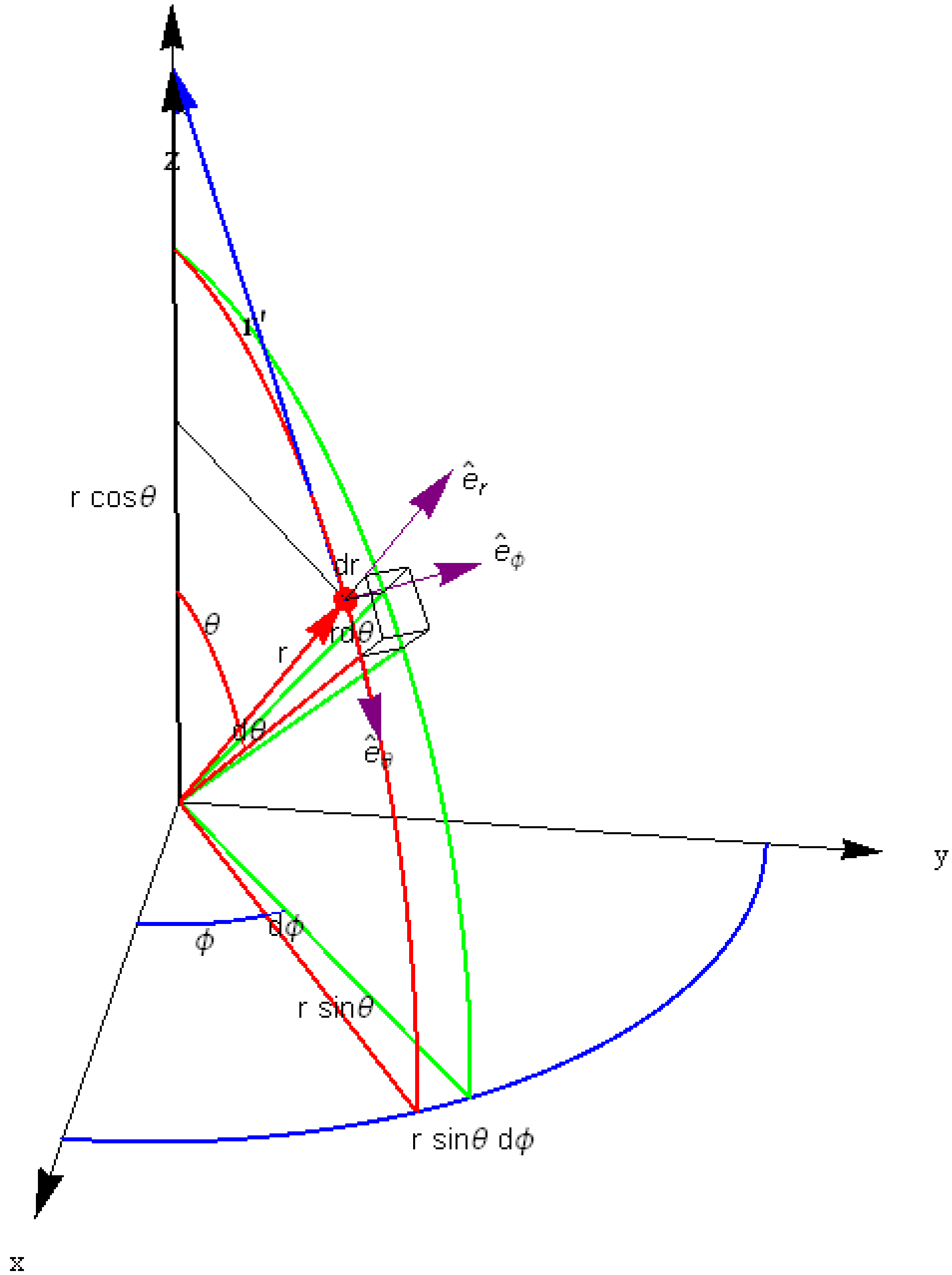
$$V = ER = (3\times10^6)R$$

Using this relation, we have

R	V
3mm	10 kV
3 cm	100 kV
30 cm	1 MV
3 m	10 MV

APPENDIX-II

The electric potential from a uniformly charged sphere
(Griffiths Example 2.8)



We find the electric potential of a charged sphere of radius *R*, which carries a uniform density ρ . The electric potential at the point P is given by

$$V(z) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r'} d^3\boldsymbol{r}$$

where

$$d^3\mathbf{r} = r^2 \sin\theta dr d\theta d\phi$$

$$r' = \sqrt{r^2 + z^2 - 2rz\cos\theta}$$

$$\rho = \frac{Q}{\frac{4\pi}{3}R^3}$$

Noting that $\int_0^{2\pi} d\phi = 2\pi$, we have

$$V(z) = \frac{\rho}{4\pi\varepsilon_0} 2\pi \int_0^R dr \int_0^\pi \frac{r^2 dr \sin\theta d\theta}{\sqrt{r^2 + z^2 - 2rz\cos\theta}}$$

Using the integral formula

$$\int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r^2 + z^2 - 2rz\cos\theta}} = \frac{1}{rz}[r + z - |r - z|]$$

we get

$$V(z) = \frac{\rho}{4\pi\varepsilon_0} 2\pi \int_0^R r^2 dr \frac{1}{rz}[r + z - |r - z|]$$

(a) $z > R$ (outside the sphere)

$$\begin{aligned} V(z) &= \frac{\rho}{4\pi\varepsilon_0} 2\pi \int_0^R r^2 dr \frac{1}{rz}[r + z - (z - r)] \\ &= \frac{\rho}{4\pi\varepsilon_0} \frac{4\pi}{z} \int_0^R r^2 dr \\ &= \frac{\rho}{4\pi\varepsilon_0} \frac{4\pi}{z} \frac{R^3}{3} \\ &= \frac{Q}{4\pi\varepsilon_0 z} \end{aligned}$$

(b) $z < R$ (inside the sphere)

$$
\begin{aligned}
V(z) &= \frac{\rho}{4\pi\varepsilon_0} 2\pi[\int_0^z r^2 dr \frac{1}{rz}[r+z-|r-z|] + \int_z^R r^2 dr \frac{1}{rz}[r+z-|r-z|]] \\
&= \frac{\rho}{4\pi\varepsilon_0} 4\pi[\int_0^z r^2 dr \frac{1}{rz} r + \int_z^R r^2 dr \frac{1}{rz} z] \\
&= \frac{\rho}{\varepsilon_0}(\frac{1}{z}\int_0^z r^2 dr + \int_z^R r dr) \\
&= \frac{\rho}{\varepsilon_0}(\frac{R^2}{2} - \frac{z^2}{6}) \\
&= \frac{Q}{8\pi\varepsilon_0 R}(3 - \frac{r^2}{R^2})
\end{aligned}
$$

APPENDIX-III: Lightning rod

A lightning rod (US, AUS) or lightning conductor (UK) is a metal rod mounted on a structure and intended to protect the structure from a lightning strike. If lightning hits the structure, it will preferentially strike the rod and be conducted to ground through a wire, instead of passing through the structure, where it could start a fire or cause electrocution. Lightning rods are also called finials, air terminals or strike termination devices.

In a lightning protection system, a lightning rod is a single component of the system. The lightning rod requires a connection to earth to perform its protective function. Lightning rods come in many different forms, including hollow, solid, pointed, rounded, flat strips or even bristle brush-like. The main attribute common to all lightning rods is that they are all made of conductive materials, such as copper and aluminum. Copper and its alloys are the most common materials used in lightning protection.
https://en.wikipedia.org/wiki/Lightning_rod

Now, suppose two metal spheres with radii r_1 and r_2 are connected by a thin conducting wire, as shown in Figure 1.

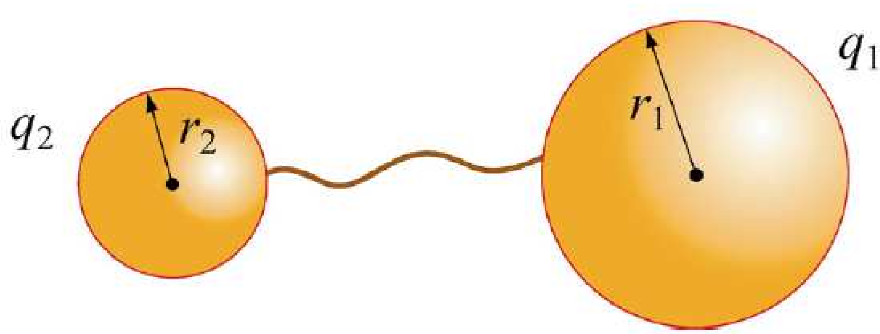


Fig. Two conducting spheres connected by a wire.

Charge will continue to flow until equilibrium is established such that both spheres are at the same potential $V_1 = V_2 = V$. Suppose the charges on the spheres at equilibrium are q_1 and q_2. Neglecting the effect of the wire that connects the two spheres, the equipotential condition implies

$$V_1 = \frac{q_1}{4\pi\varepsilon_0 r_1}, , \qquad V_2 = \frac{q_2}{4\pi\varepsilon_0 r_2}, , \qquad V_1 = V_2 = V$$

or

$$\frac{q_1}{r_1} = \frac{q_2}{r_2}$$

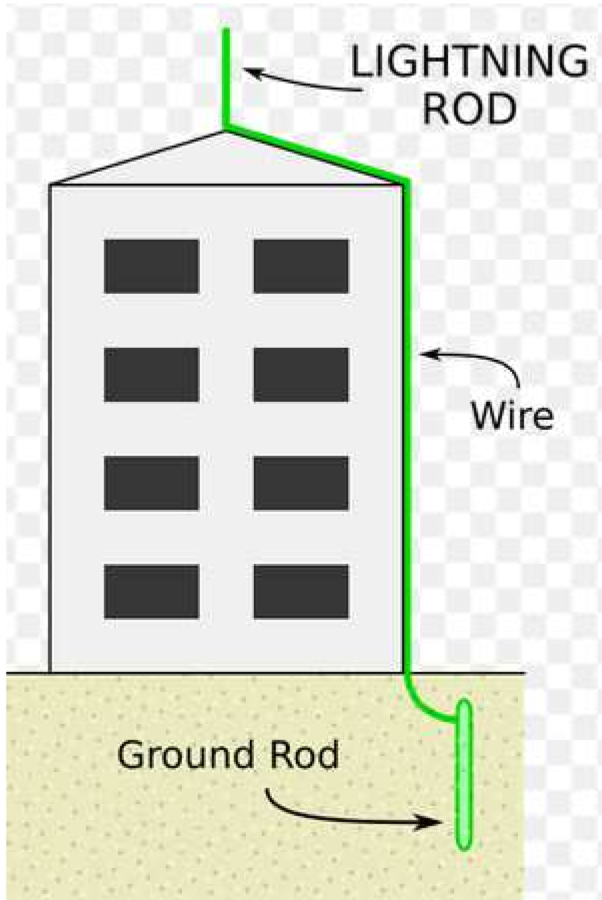
assuming that the two spheres are very far apart so that the charge distributions on the surfaces of the conductors are uniform. The surface charge densities on spheres 1 and 2 are related to the charges q_1 and q_2 as

$$\sigma_1 = \frac{q_1}{4\pi r_1^2}, \quad \sigma_2 = \frac{q_2}{4\pi r_2^2}.$$

The two equations can be combined to yield

$$\frac{r_1^2 \sigma_1}{r_1} = \frac{r_2^2 \sigma_2}{r_2} \qquad \text{or} \qquad r_1\sigma_1 = r_2\sigma_2 = \sigma r$$

The surface charge density σ is inversely proportional to the radius *r*. It is concluded that the regions with the smallest radii of curvature have the greatest σ. Thus, the electric field strength on the surface of a conductor is greatest at the sharpest point. The design of a lightning rod is based on this principle.



((**Feynman**))

R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics vol. II (Basic Books, 2010).

We would like now to discuss qualitatively some of the characteristics of the fields around conductors. If we charge a conductor that is not a sphere, but one that has on it a point or a very sharp end, as, for example, the object sketched in Fig., the field around the point is much higher than the field in the other regions. The reason is, qualitatively, that charges try to spread out as much as possible on the surface of a conductor, and the tip of a sharp point is as far away as it is possible to be from most of the surface. Some of the charges on the plate get pushed all the way to the tip. A relatively *small* amount of charge on the tip can still provide a large surface *density*; a high charge density means a high field just outside.

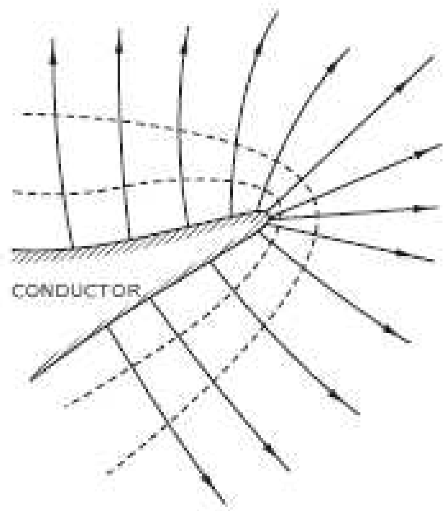


Fig.1 The electric field near a sharp point on a conductor is very high.

One way to see that the field is highest at those places on a conductor where the radius of curvature is smallest is to consider the combination of a big sphere and a little sphere connected by a wire, as shown in Fig.2 It is a somewhat idealized version of the conductor of Fig.1. The wire will have little influence on the fields outside; it is there to keep the spheres at the same potential. Now, which ball has the biggest field at its surface?

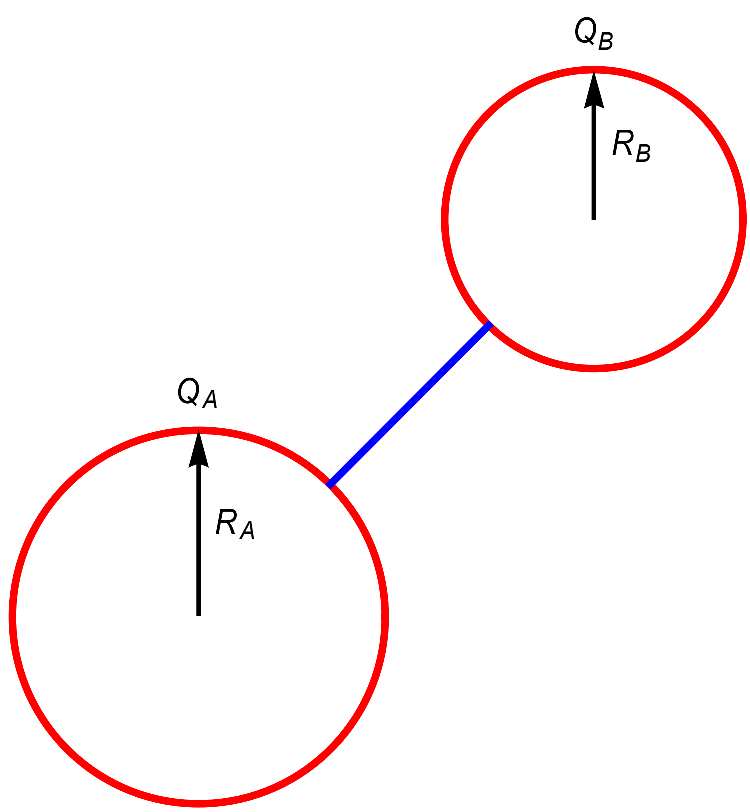


Fig.2 The field of a pointed object can be approximated by that of two spheres at the same potential.

The electric potential is the same when two spheres are connected with a wire,

$$\frac{Q_A}{4\pi\varepsilon_0 R_A} = \frac{Q_B}{4\pi\varepsilon_0 R_B}$$

or

$$\frac{Q_A R_A}{4\pi R_A^{\ 2}} = \frac{Q_B R_B}{4\pi R_B^{\ 2}}$$

or

$$R_A \sigma_A = R_B \sigma_B$$

where

$$A_A = 4\pi R_A^{\ 2}, \qquad\qquad A_B = 4\pi R_B^{\ 2}.$$

Noting that the electric field is

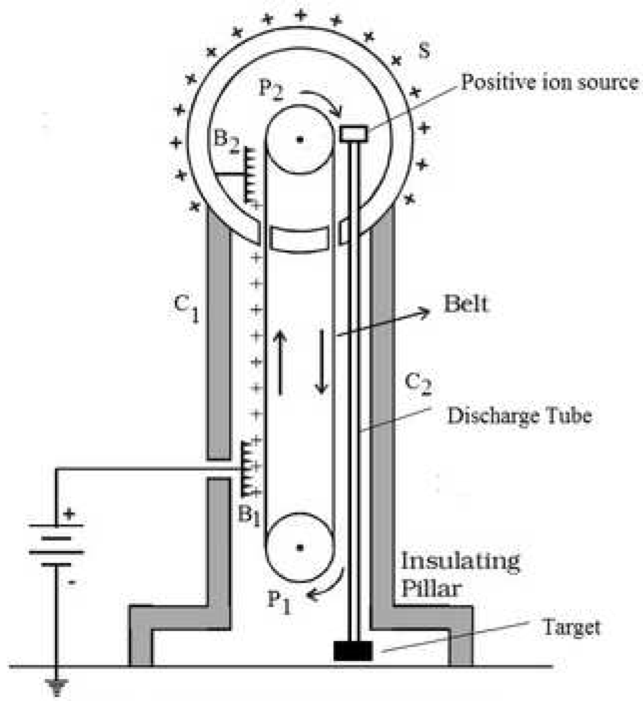
$$E_A = \frac{\sigma_A}{\varepsilon_0}, \qquad E_B = \frac{\sigma_B}{\varepsilon_0}$$

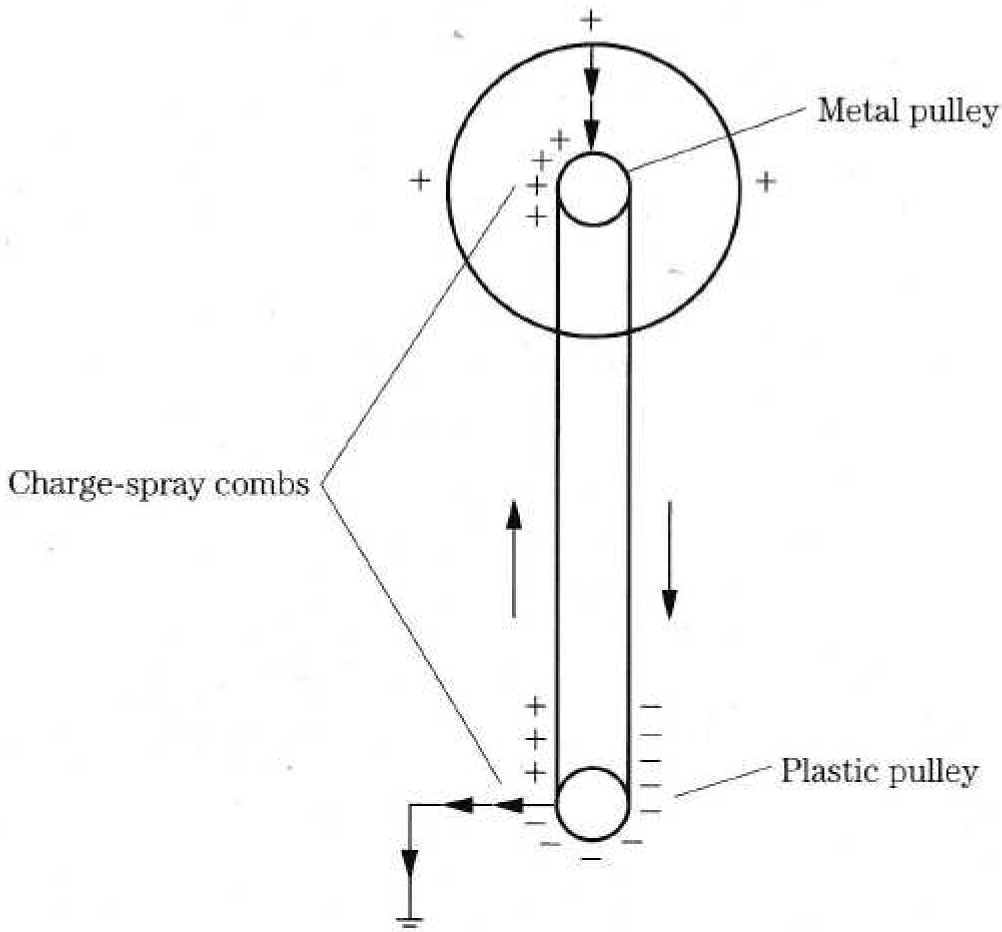
we have

$$R_A E_A = R_B E_B$$

Thus the electric field becomes large when the radius of curvature becomes small.

APPENDIX-IV Van de Graaff generator and standard value of electric field for air dielectric break down


S
Positive ion source
P2
B2
Belt
C1
C2
Discharge Tube
B1
Insulating Pillar
P1
Target



Although air is normally an excellent insulator, when stressed by a sufficiently high voltage (an electric field of about 3×10^6 V/m or 3 kV/mm), air can begin to break down, becoming partially conductive. Across relatively small gaps, breakdown voltage in air is a function of gap length times pressure.

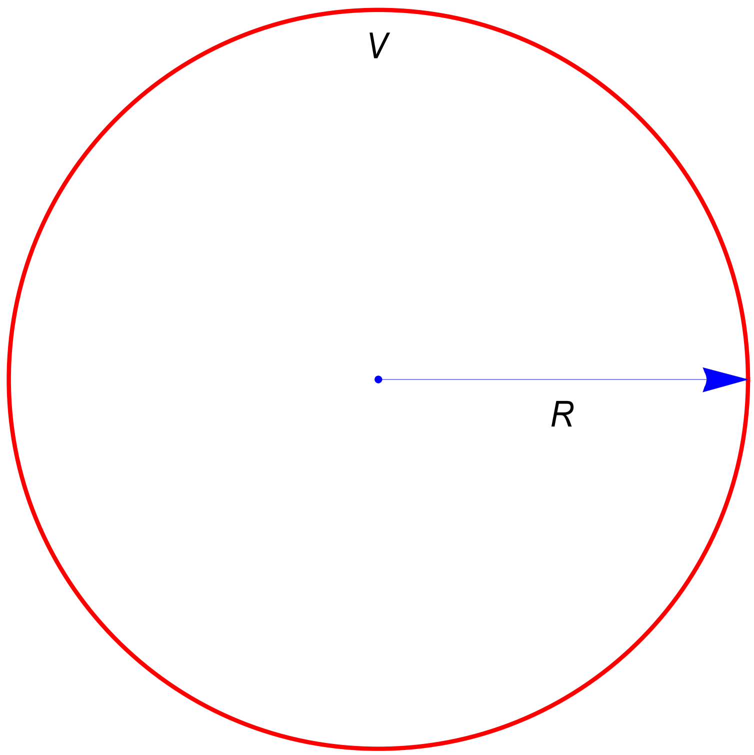


Fig. Schematic diagram of the van de Graaff. R = 0.3 m. C = 33.4 pF. The standard value of air dielectric breakdown is 3kV/mm = 3000 kV/m=3MV/m).

The voltage and the electric field of the Van de Graaff;

$$V = \frac{Q}{4\pi\varepsilon_0 R}.$$

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}.$$

$$V = ER$$

The capacitance:

$$C = 4\pi\varepsilon_0 R$$

When $R = 0.3$ m, $C = 33.4$ pF.

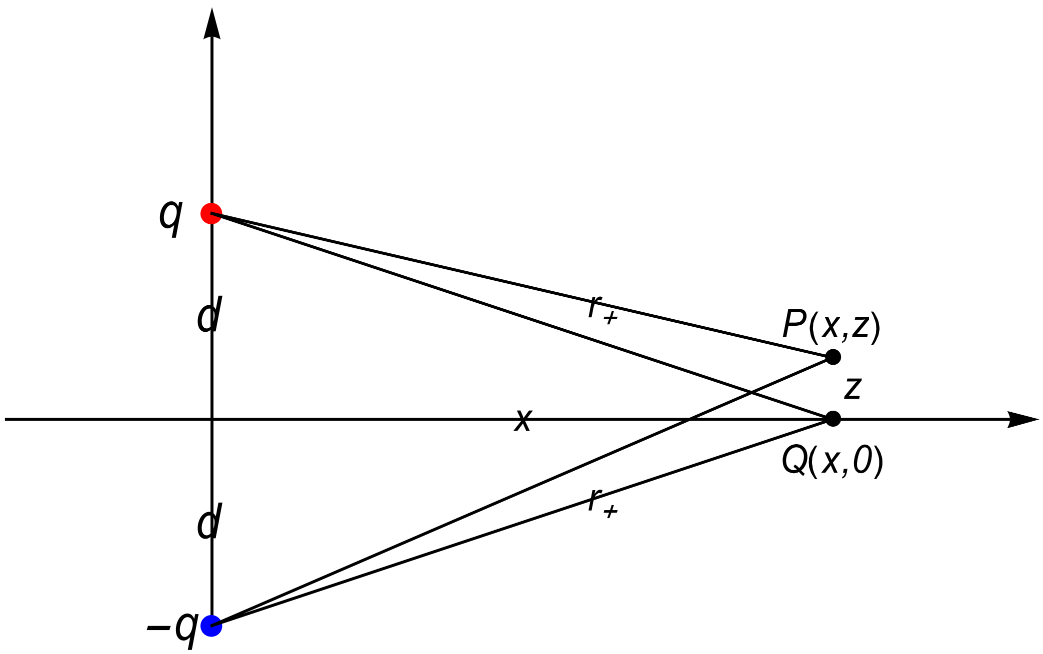
For $E = 3\times10^6$ V/m = 3000 kV/m (which is the breakdown electric field of air),

$V = 3\text{x}10^5$ V = 300 kV, $Q = 10$ μC.

REFERENCES

R.A. Ford, Homemade Lightning, Creative Experiments in Electricity, 3rd edition (McGraw-Hill, 2002).

APPENDIX



Here we find the electric potential at the point $P(x,z)$ due to the electric dipole moment with $p = 2qd$. We also find the expression of the electric field at the point $Q(x,0)$.

Electric potential from the electric dipole moment:
For $x^2 >> (d-z)^2$ and $x^2 >> (d+z)^2$, the distances between the electric dipolements and the point $P(x,z)$;

$$r_+ = \sqrt{(d-z)^2 + x^2} \approx x\sqrt{1 + \frac{(d-z)^2}{x^2}}$$

$$r_- = \sqrt{(d+z)^2 + x^2} \approx x\sqrt{1 + \frac{(d+z)^2}{x^2}}$$

or

$$r_- \approx x + \frac{(d+z)^2}{2x},$$

$$r_+ \approx x + \frac{(d-z)^2}{2x}$$

where

$$\begin{aligned} r_+ - r_- &\approx \frac{1}{2x}[(d-z)^2 - (d+z)^2] \\ &= \frac{2zd}{x} \end{aligned}$$

The electric potential:

$$\begin{aligned} V(x,z) &= \frac{q}{4\pi\varepsilon_0}(\frac{1}{r_+} - \frac{1}{r_-}) \\ &\approx \frac{q}{4\pi\varepsilon_0 x^2}(r_+ - r_-) \\ &= \frac{pz}{4\pi\varepsilon_0 x^3} \end{aligned}$$

where $p = 2qd$. Thus the electric field is obtained as

$$E_x = -\frac{\partial V}{\partial x} = \frac{3z}{4\pi\varepsilon_0 x^4} \approx 0$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\varepsilon_0 x^3}$$