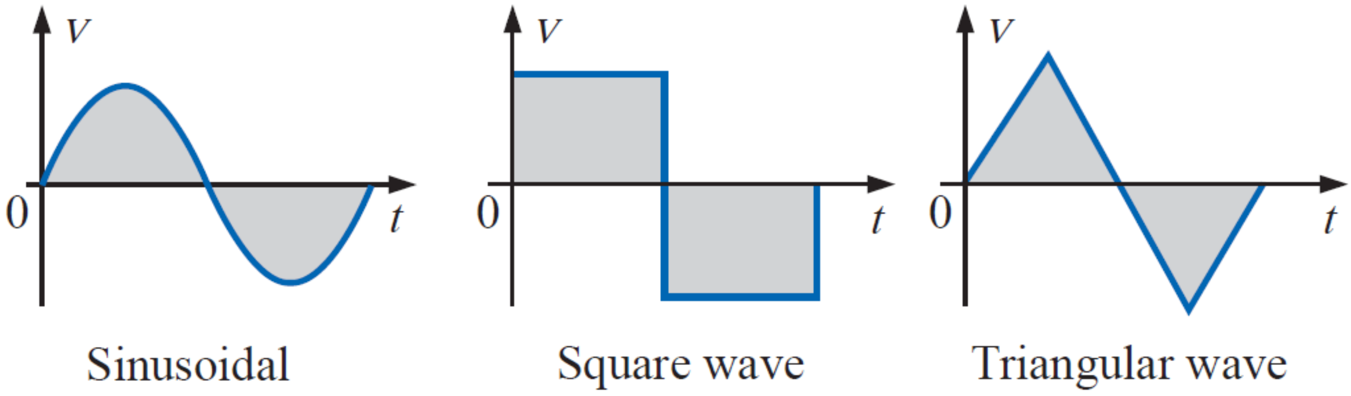
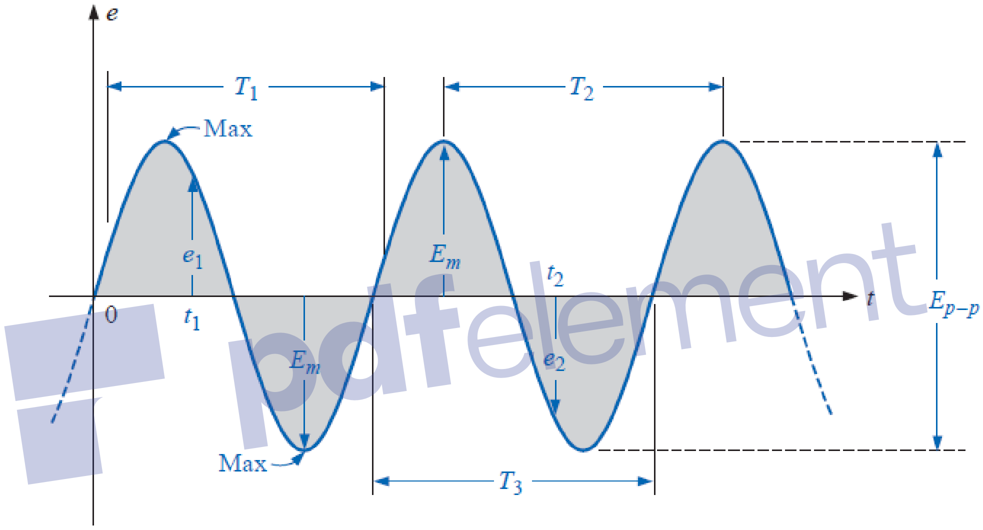
Sinusoidal Alternating Waveforms

Introduction

The term *alternating* indicates only that the waveform alternates between two prescribed levels in a set time sequence



The vertical scaling of the sinusoidal waveform is in volts or amperes and the horizontal scaling is *always* in units of time and can be represented as.



Instantaneous value, Peak amplitude, Peak value, Peak-to-peak value and Period.

Frequency (f): The number of cycles that occur in 1 s.

1 hertz (Hz) = 1 cycle per second (c/s)

$$f = \frac{1}{T}$$

Example:

Find the period of a periodic waveform with a frequency of

a. 60 Hz.

b. 1000 Hz.

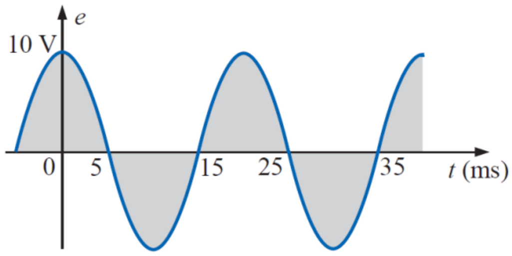
Solution

a- $T = \frac{1}{f} = \frac{1}{60} = 16.67\ ms$

b- $T = \frac{1}{f} = \frac{1}{1000} = 1ms$

Example:

Determine the frequency of the waveform of following Fig



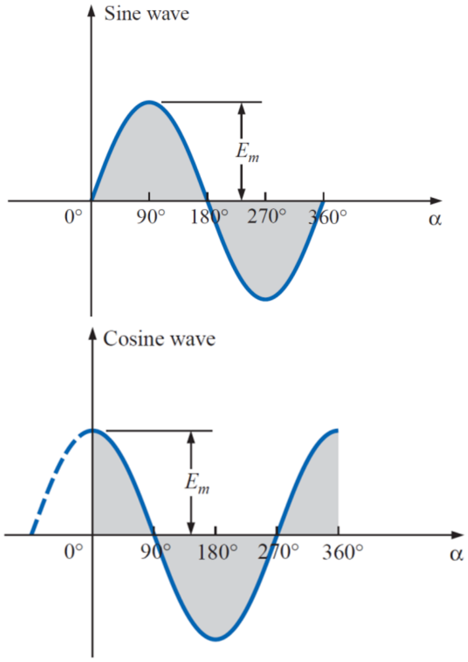
Solution:

From the figure, T = (25 ms - 5 ms) = 20 ms, and

$$f = \frac{1}{T} = \frac{1}{20 \times 10^{-3}} = 50\ Hz$$

THE SINE WAVE

The unit of measurement for the horizontal axis of Figure below is the *degree.* A second unit of measurement frequently used is the **radian** (rad).



$$2\pi\ (rad) = 360\ (degree)$$

$$(rad) = \frac{degree}{360} 2\pi$$

$$(degree) = \frac{rad}{2\pi} 360$$

$$\pi = 3.14159$$

$$30^0 = \frac{\pi}{6}\ rad$$

$$45^0 = \frac{\pi}{4}\ rad$$

$$60^0 = \frac{\pi}{3}\ rad$$

$$90^0 = \frac{\pi}{2}\ rad$$

The velocity with which the radius vector rotates about the centre, called the **angular velocity,** can be determined from the following equation:

$$\boxed{\text{Angular velocity} = \frac{\text{distance (degrees or radians)}}{\text{time (seconds)}}}$$

$$\boxed{\omega = \frac{\alpha}{t}}$$

$$\boxed{\alpha = \omega t}$$

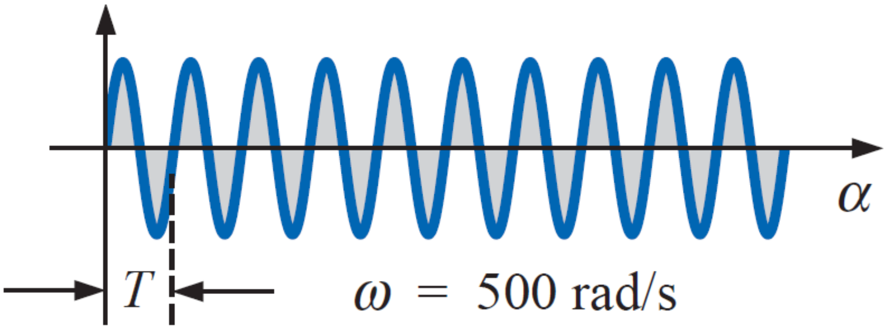
For sinusoidal waveform, the angular velocity can be expressed as

$$\boxed{\omega = \frac{2\pi}{T}} \qquad \text{(rad/s)}$$

Example:
Determine the angular velocity of a sine wave having a frequency of 60 Hz.

Solution
$\omega = 2\pi f\ = 2 \times 3.14 \times 60 = 377\ rad/s$

Example:
Determine the frequency and period of the sine wave of following Figure.



Solution

$\omega = \frac{2\pi}{T}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{500} = 12.57\ ms$

$f = \frac{1}{T} = 79.58\ Hz$

Example:
Given $\omega = 200$ rad/s, determine how long it will take the sinusoidal waveform to pass through an angle of 90°.

$t = \frac{\alpha}{\omega} = \frac{\pi/2}{500} = 7.85\ ms$

Example:
Find the angle through which a sinusoidal waveform of 60 Hz will pass in a period of 5 ms.

$\alpha = \omega t = 2\pi f t = 2 \times 3.14 \times 60 \times 5 \times 10^{-3} = 1.885\ rad$

$\alpha(\ ^0) = \frac{180}{\pi} 1.885 = 108^0$

GENERAL FORMAT FOR THE SINUSOIDAL VOLTAGE OR CURRENT

The basic mathematical format for the sinusoidal waveform is

$i = I_m \sin\alpha = I_m \sin\omega t$

$v = V_m \sin\alpha = V_m \sin\omega t$

Example
Given $E = 5\sin\alpha$, determine E at $\alpha = 40^0$ and $\alpha = 0.8\pi$.

Solution:

$\alpha = 40^0$

$E = 5\sin\alpha$= 5(0.6428) = **3.214 V**

For $\alpha = 0.8\pi$,

α (°) =$\frac{180}{\pi}$ (0.8π) = 144°

and $E = 5\sin 144°$ = 5(0.5878) = **2.939 V**

The angle at which a particular voltage level is attained can be determined by rearranging the equation

$v = V_m sin\alpha$

$sin\alpha = \frac{v}{V_m}$

$\alpha = sin^{-1}\frac{v}{V_m}$

Similarly, for a particular current level,

$$\boxed{\alpha = \sin^{-1}\frac{i}{I_m}}$$

Example

a. Determine the angle at which the magnitude of the sinusoidal function $v = 10 \sin 377t$ is 4 V.
b. Determine the time at which the magnitude is attained.

Solutions:

$$\alpha_1 = \sin^{-1}\frac{v}{E_m} = \sin^{-1} 0.4 = \mathbf{23.578°}$$

The magnitude of 4 V (positive) will be attained at two points between 0° and 180°. The second intersection is determined by

$$\alpha_2 = \mathbf{180° - 23.578° = 156.422°}$$

For the first intersection,

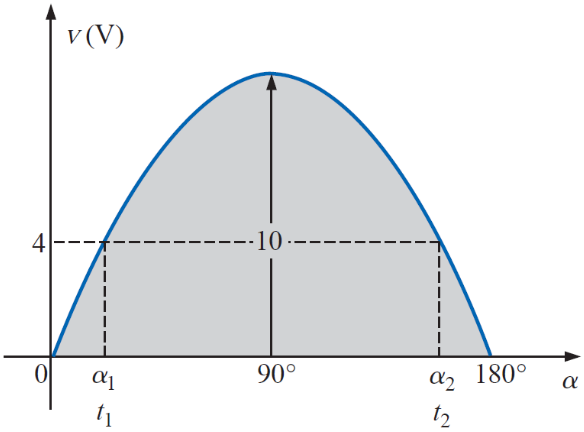
$$\alpha\ (rad) = \frac{\pi}{180} \times \mathbf{23.578} = 0.411$$

$$t_1 = \frac{\alpha}{\omega} = \frac{0.411}{377} = \mathbf{1.09\ ms}$$

For the second intersection,

$$\alpha\ (rad) = \frac{\pi}{180} \times \mathbf{156.422} = 2.73$$

$$t_2 = \frac{\alpha}{\omega} = \frac{2.73}{377} = \mathbf{7.24\ ms}$$

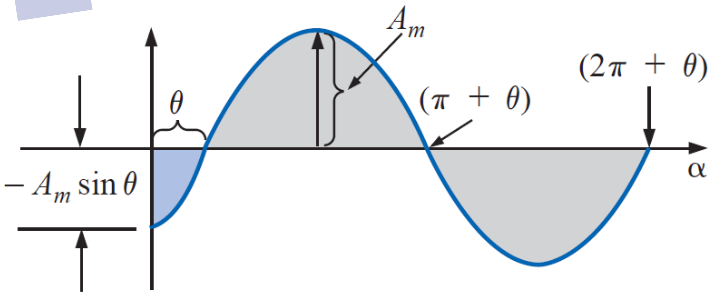


PHASE RELATIONS

If the waveform is shifted to the right or left of 0°, the expression becomes

$$\boxed{A_m \sin(\omega t \pm \theta)}$$

where θ is the angle in degrees or radians that the waveform has been shifted.

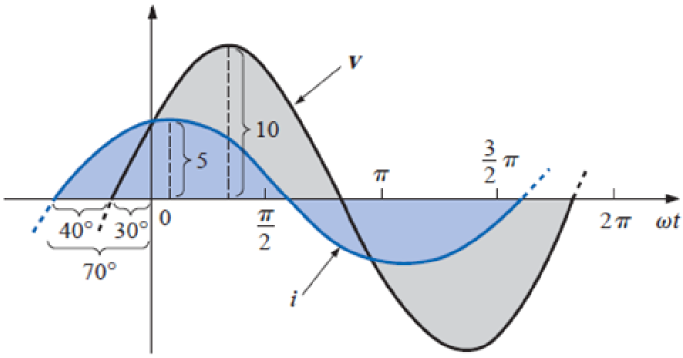


Example:

What is the phase relationship between the sinusoidal waveforms of each of the following sets?

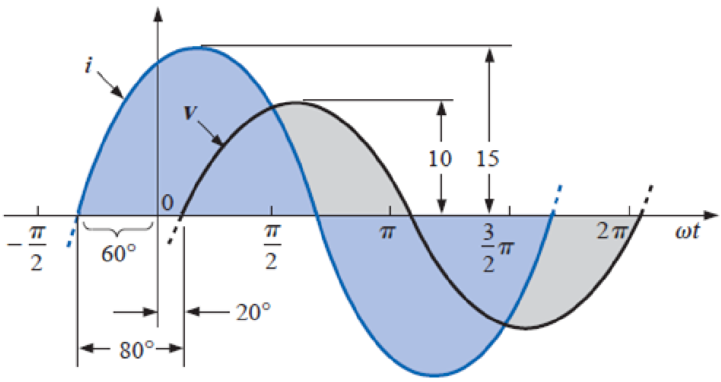
a. $v = 10\sin(wt + 30)$
$i = 5\sin(wt + 70)$

***i* leads *v* by 40°, or *v* lags *i* by 40°.**



b. $i = 15\sin(wt + 60)$
$v = 10\sin(wt - 20)$

i leads v by 80°, or v lags i by 80°.



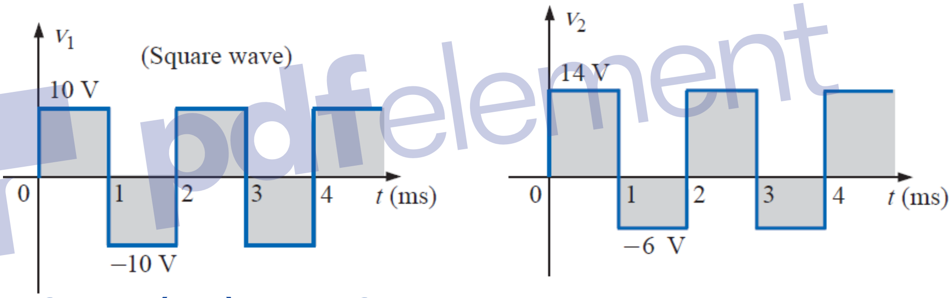
AVERAGE VALUE

For half wave rectifier

$$V_{av} = \frac{1}{\pi}\int_0^{\pi} v(t)\,dt$$

$$V_{av} = \frac{1}{\pi}\int_0^{\pi} V_m \sin\omega t\,d\omega t = \frac{2V_m}{\pi} = 0.636V_m$$

Example:
Determine the average value of the waveforms



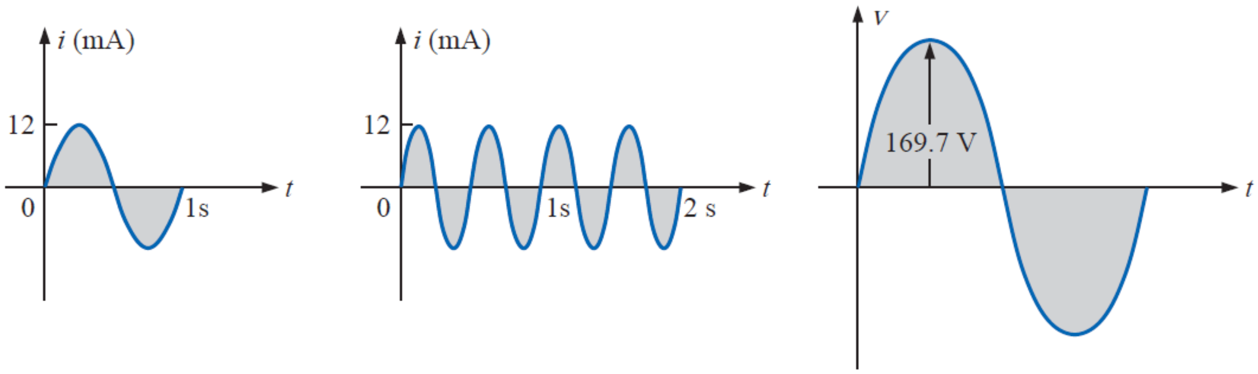
EFFECTIVE Root Mean Square (rms) VALUES

$$V_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(t)\,dt}$$

$$V_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} {V_m}^2 \sin^2\omega t\,d\omega t} = \frac{V_m}{\sqrt{2}} = 0.707V_m$$

Example:

Find the rms values of the sinusoidal waveform in each part



RESPONSE OF BASIC *R*, *L*, AND *C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

Resistor

Let $v(t) = V_m \sin \omega t$

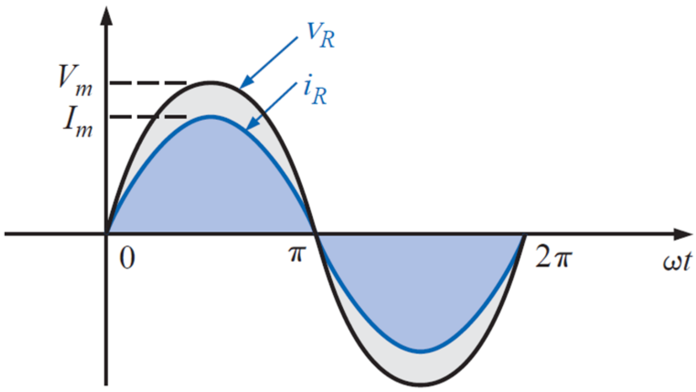
$$i(t) = \frac{v(t)}{R} = \frac{V_m \sin \omega t}{R} = I_m \sin \omega t$$

$$I_m = \frac{V_m}{R}$$

In addition, for a given $i(t) = I_m \sin \omega t$,

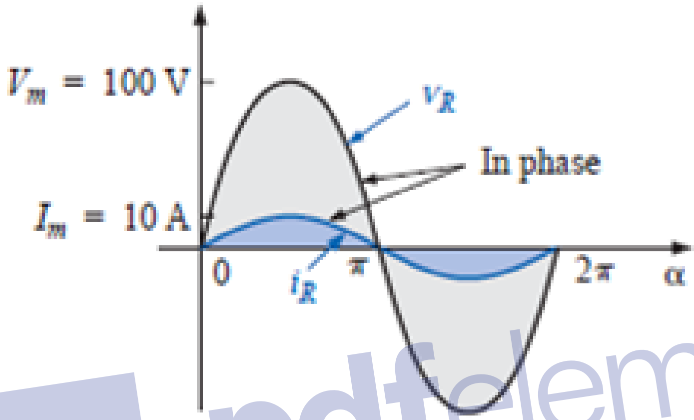
$v(t) = RI_m \sin \omega t = V_m \sin \omega t$

$V_m = RI_m$



Example:

The voltage across 10 Ω resistor is 100 $sin377t$, sketch the curves for the voltage and current.



Example:

The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10 Ω. Sketch the curves for *v* and *i*.

a. $v = 100 \sin 377t$

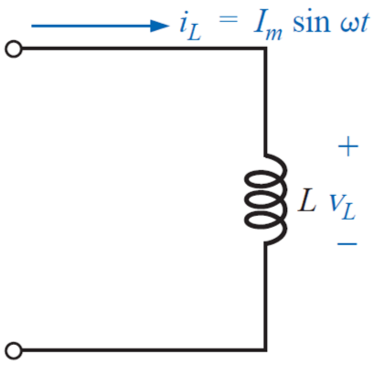
b. $v = 25 \sin(377t + 60°)$

Inductor

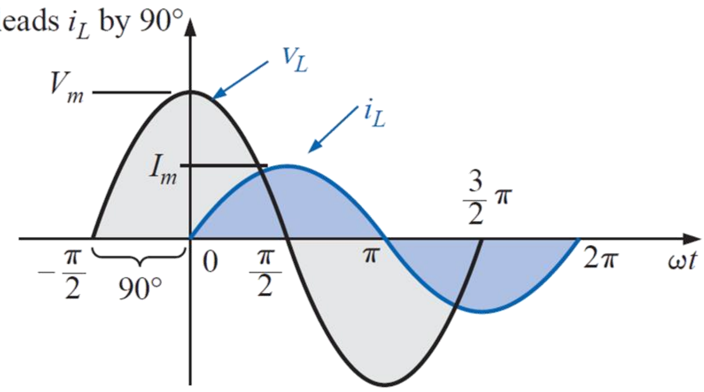
$$v_L = L\frac{di_L}{dt} = L\omega I_m \cos \omega t$$

$$v_L = V_m \cos \omega t = V_m \sin(\omega t + 90)$$

$$V_m = L\omega I_m$$



for an inductor, v_L leads i_L by 90°, or i_L lags v_L by 90°.



Reactance X_L

$$X_L = \frac{V_m}{I_m} = \frac{\omega L I_m}{I_m} = \omega L$$

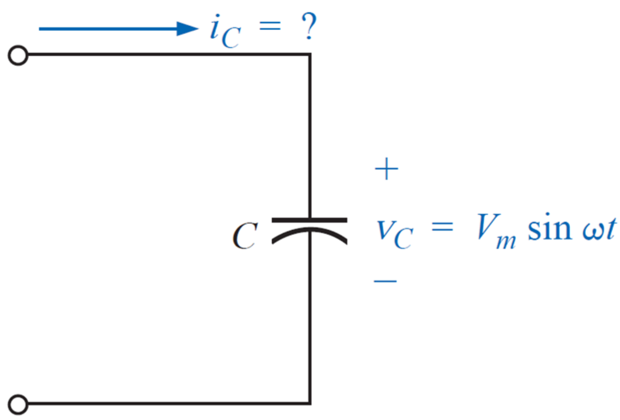
Example:
The voltage across a 0.5-H coil is provided below. What is the sinusoidal expression for the current?
$v = 100 \sin 20t$

$$X_L = \omega L = 20 \times 0.5 = 10\ \Omega$$

$$i = \frac{V_m}{X_L} \sin(20t - 90) = 10 \sin(20t - 90) A$$

Example:
The current through a 0.1-H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the v and i curves.
a. $i = 10 \sin 377t$
b. $i = 7 \sin(377t - 70°)$

Capacitor



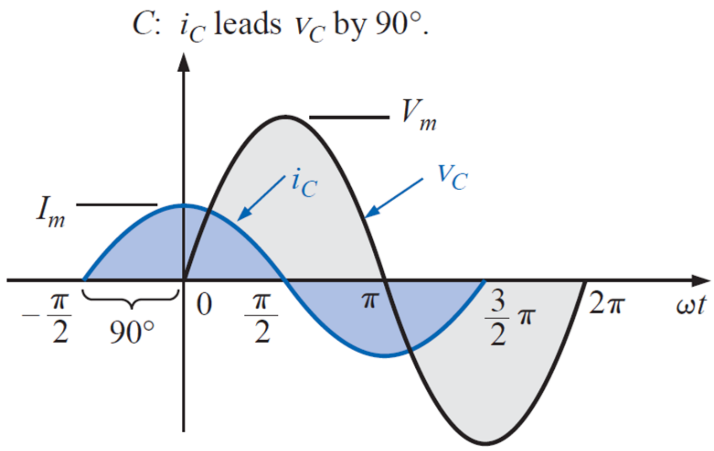
$$i_C = C\frac{dv_C}{dt} = C\omega V_m \cos \omega t$$
$$i_C = I_m \cos \omega t = I_m \sin(\omega t + 90)$$

$$I_m = \omega C V_m$$

Reactance X_C

$$X_C = \frac{V_m}{I_m} = \frac{V_m}{\omega C V_m} = \frac{1}{\omega C}$$

for a capacitor, i_C leads v_C by 90°, or v_C lags i_C by 90°.



Example:

The voltage across a 1-µF capacitor is provided below. What is the sinusoidal expression for the current? Sketch the *v* and *i* curves.

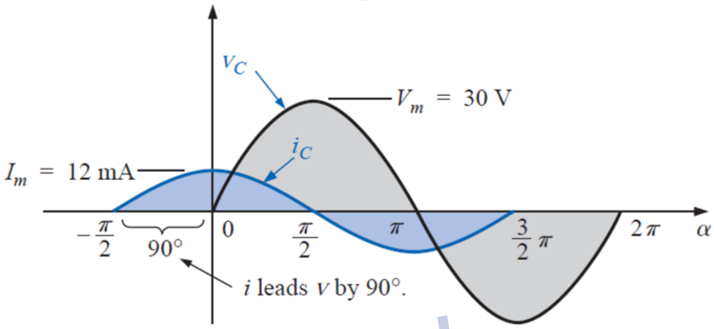
$\boldsymbol{v = 30 sin400t}$

Solution

$$X_C = \frac{1}{\omega C} = \frac{1}{400 \times 1 \times 10^{-6}} = 2500\Omega$$

$$I_m = \frac{V_m}{\omega C} = \frac{30}{2500} = 12mA$$

$$v = 12 sin(400t + 90)$$



Example:

The current through a 100-mF capacitor is *i* = 40 sin(500*t* - 60°). Find the sinusoidal expression for the voltage across the capacitor.

Example:

For the following pairs of voltages and currents, determine whether the element involved is a capacitor, an inductor, or a resistor, and determine the value of *C*, *L*, or *R* if sufficient data are provided :

a. *v* = 100 sin(wt - 40°)
   i = 20 sin(wt - 40°)
b. *v* = 1000 sin(377*t* - 10°)
   i = 5 sin(377*t* - 80°)
c. *v* = 500 sin(157*t* - 30°)
   i = 1 sin(157*t* - 120°)
d. *v* = 50 cos(wt - 20°)
   i = 5 sin(wt - 110°)

AVERAGE POWER and power factor

Let we have

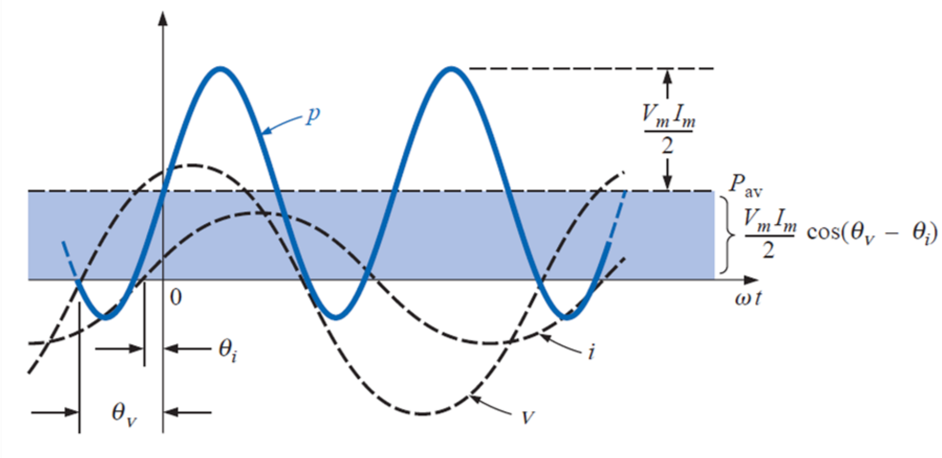
$v = V_m \sin(\omega t + \theta_v)$

$i = I_m \sin(\omega t + \theta_i)$

then the power is defined by

$p = vi = V_m \sin(\omega t + \theta_v)\, I_m \sin(\omega t + \theta_i)$

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) - \frac{V_m I_m}{2}\cos(2\omega t + \theta_v + \theta_i)$$



$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m I_m}{2}\cos\theta$$

$$\text{power factor} = F_p = \cos\theta = \frac{p}{\frac{V_m I_m}{2}} = \frac{p}{\frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}} = \frac{p}{V_{eff} I_{eff}}$$

Resistor

$$p = \frac{V_m I_m}{2}\cos(0) = \frac{V_m I_m}{2} = \frac{V_m I_m}{\sqrt{2}\sqrt{2}} = V_{eff} I_{eff} = \frac{V_{eff} V_{eff}}{R} = \frac{V_{eff}^{\ 2}}{R} = I_{eff}^{\ 2} R$$

Inductor

$$p = \frac{V_m I_m}{2}\cos(90) = 0$$

Capacitor

$$p = \frac{V_m I_m}{2}\cos(90) = 0$$

Example:

Find the average power dissipated in a network whose input current and voltage are the following:

i = 5 sin(wt + 40°)

v = 10 sin(wt + 40°)

Example:

Determine the average power delivered to networks having the following input voltage and current:

a. v = 100 sin(wt + 40°)
   i = 20 sin(wt + 70°)

b. v =150 sin(wt - 70°)
   i = 3 sin(wt - 50°)

Example:

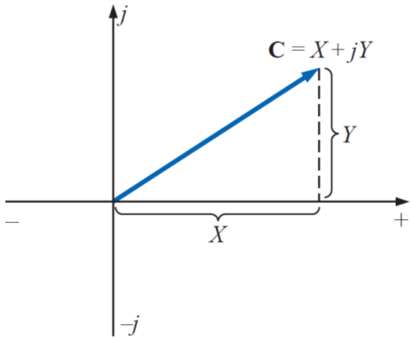
Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. v = 50 sin(wt - 20°)
   i = 2 sin(wt + 40°)

b. v =120 sin(wt + 80°)
   i = 5 sin(wt + 30°)

c. $I_{eff} = 5A, V_{eff} = 20\ V\ and\ p = 100\ w$

COMPLEX NUMBERS



RECTANGULAR FORM

The format for the **rectangular form** is

$C = X + jY$

$X = Z\ cos\theta$

$y = Z\ sin\theta$

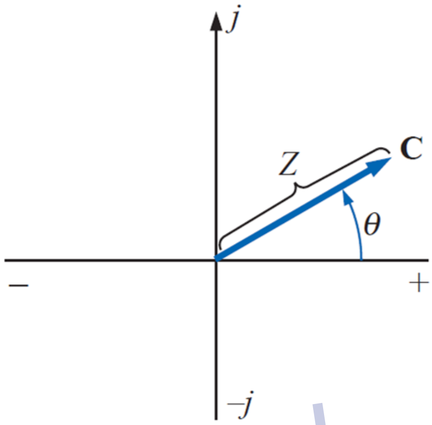
Example:

Sketch the following complex numbers in the complex plane:

a. **C** = 3 + j 4

b. **C** = 0 - j 6

c. **C** = -10 - j20



POLAR FORM

$C = Z\angle\theta$

$Z = \sqrt{X^2 + Y^2} \qquad \theta = tan^{-1}\frac{Y}{X}$

Example:

Sketch the following complex numbers in the complex plane:

a. **C** = 5∠30°

b. **C** =7∠ − 120°

c. **C** =−4.2∠60°

Example:

Convert the following from rectangular to polar form:

a. C = 3 + j 4

b. C = -6 + j 3

Example:

Convert the following from polar to rectangular form:

a. **C** = 10∠40°

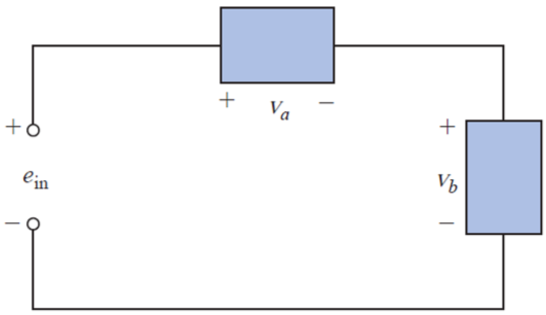
b. **C** = 10∠230°

Example:

Find the input voltage of the circuit shown below when the frequency is 60 Hz

$v_a = 50\ \sin(377t + 30)$

$v_b = 30\ \sin(377t + 60)$



Solution:

$$v_a = \frac{50}{\sqrt{2}}\angle 30 = 35.35\angle 30\ V = 30.61\text{ V} + j\ 17.68\text{ V}$$
$$v_b = \frac{30}{\sqrt{2}}\angle 60 = 21.21\angle 60\ V = 10.61\text{ V} + j\ 18.37\text{ V}$$

Applying Kirchhoff's voltage law, we have

Rectangular form

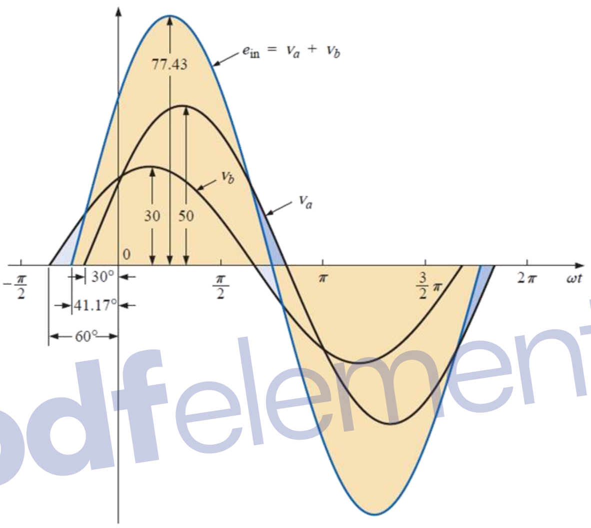
$$E_{in} = v_a + v_b = 30.61\text{ V} + j\ 17.68 + 10.61\text{ V} + j\ 18.37 = 41.22\text{ V} + j\ 36.05\text{ V}$$

Polar form

$$E_{in} = 54.76\text{ V}\angle 41.17°\text{ V}$$

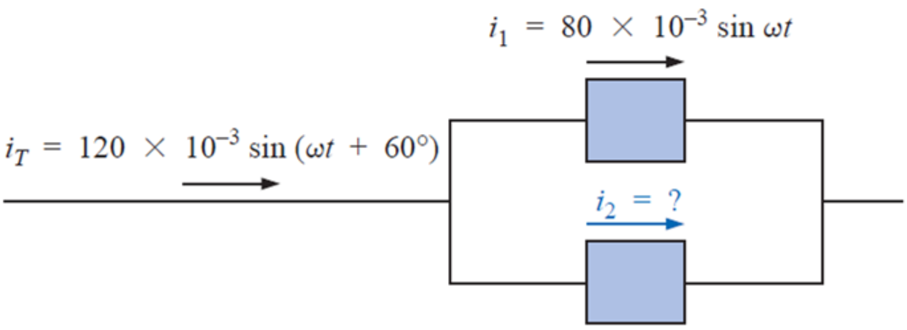
Time domain

$$E_{in} = \sqrt{2}(54.76)\sin(377t + 41.17) = 77.43\ \sin(377t + 41.17)$$



Example:

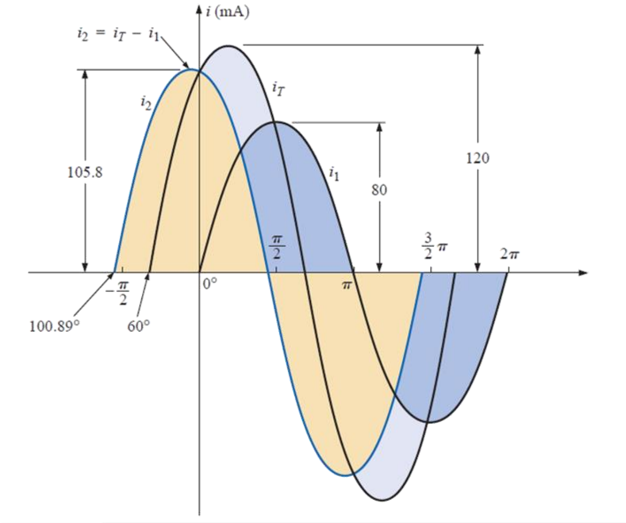
Determine the current i_2 for the network



Solution:

Applying Kirchhoff's current law, we obtain

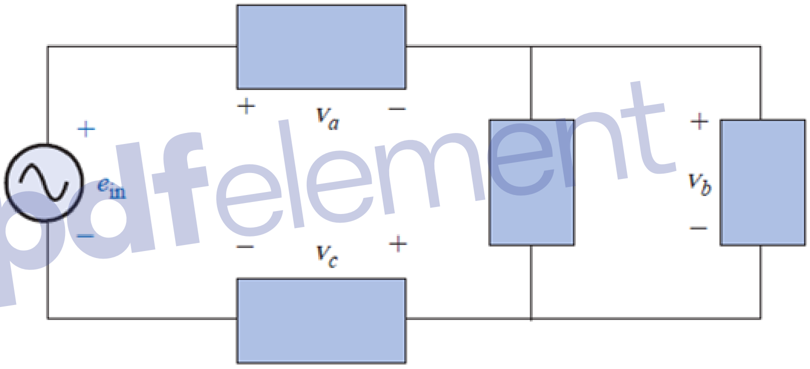
$$i_T = i_1 + i_2$$
$$i_2 = i_T - i_1$$
$$i_T = 120 \times 10^{-3}\sin(\omega t + 60) = \frac{120 \times 10^{-3}}{\sqrt{2}}\angle 60 = 84.84\angle 60\ mA = 42.42 + j\ 73.47\text{ mA}$$
$$i_1 = 80 \times 10^{-3}\sin\omega t = \frac{80 \times 10^{-3}}{\sqrt{2}}\angle 0 = 56.56\angle 0\ mA = 56.56 + j\ 0\text{ mA}$$
$$i_2 = i_T - i_1 = -14.14 + j\ 73.47\text{ mA} = 74.82\text{ mA}\angle 100.89° = 105.8 \times 10^{-3}\sin(\omega t + 100.89)\ A$$



H.W

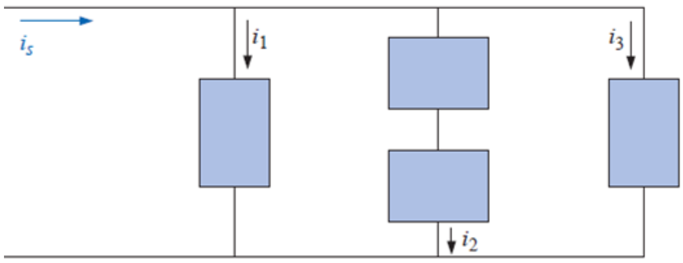
Find the sinusoidal expression for the applied voltage *e* for the system

$v_a = 60\sin(\omega t + 30)$
$v_b = 30\sin(\omega t - 30)$
$v_c = 40\sin(\omega t + 120)$



Find the sinusoidal expression for the current i_s for the system

$i_1 = 120 \times 10^{-3}\sin(377t + 180)$
$i_2 = 120 \times 10^{-3}\sin(377t)$
$i_3 = 2i_1$



MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

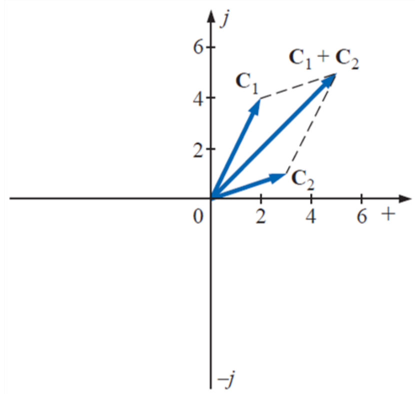
$j = \sqrt{-1}$
$j^2 = -1$
$\frac{1}{j} = -j$

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of $C = 2 + j3$ is $2 - j3$
The conjugate of **C** = $2\angle 30°$ is $2\angle -30°$

Addition

To add two or more complex numbers, simply add the real and imaginary parts separately. For example, if
$C_1 = X_1 + jY_1$ and $C_2 = X_2 + jY_2$
Then $C_1 + C_2 = X_1 + X_2 + j(Y_1 + Y_2)$

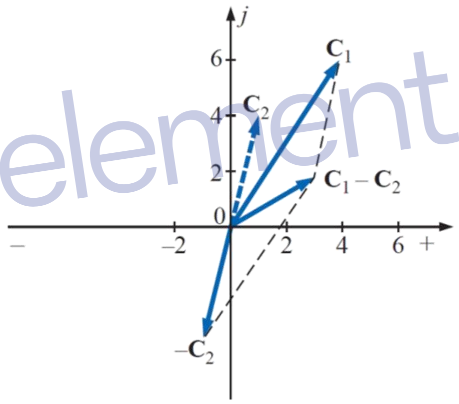


Example:

Add $C_1 = 2 + j4$ and $C_2 = 3 + j1$
Add $C_1 = 3 + j6$ and $C_2 = -6 + j3$

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if
$C_1 = X_1 + jY_1$ and $C_2 = X_2 + jY_2$
Then $C_1 - C_2 = X_1 - X_2 + j(Y_1 - Y_2)$



Example:

Subtract $C_2 = 1 + j4$ and $C_1 = 4 + j6$
Subtract $C_2 = -2 + j5$ and $C_1 = 3 + j3$

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if
$C_1 = X_1 + jY_1$ and $C_2 = X_2 + jY_2$
Then $C_1 \times C_2 = X_1X_2 - Y_1Y_2 + j(X_1Y_2 + X_2Y_1)$
Example:
Find **C**1 . **C**2 if
C1 = 2 + *j* 3 and **C**2 = 5 + *j* 10

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for
$C_1 = Z_1\angle\theta_1$ and $C_2 = Z_2\angle\theta_2$
Then $C_1 . C_2 = Z_1Z_2\angle\theta_1 + \theta_2$
Example:
Find **C**1 . **C**2 if $C_1 = 5\angle 20$ and $C_2 = 10\angle 30$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$C_1 = X_1 + jY_1$ and $C_2 = X_2 + jY_2$

Then

$$\frac{C_1}{C_2} = \frac{X_1+jY_1}{X_2+jY_2} = \frac{X_1+jY_1}{X_2+jY_2} \times \frac{X_2-jY_2}{X_2-jY_2} = \frac{X_1X_2+Y_1Y_2}{X_2^2+Y_2^2} + j\,\frac{X_2Y-X_1Y_2}{X_2^2+Y_2^2}$$

Example:

Find **C**1 / **C**2 if

C1 = 1 + *j* 4 and **C**2 = 4 + *j* 5