Course Description:
Functions, Limits and continuity, Differentiation, Applications of derivatives, Integration, Inverse functions. Applications of the Integral

Recommended Textbook(s):
Calculus, Early Transcendental By James Stewart, 6th Edition, 2008, Brooks/Cole

Prerequisites:
None

Course Topics:

1. **Functions and models**: four ways to represent a function , mathematical models: a catalogue of essential functions , new functions from old functions , exponential functions, inverse functions and logarithms
2. **Limits**: the tangent and velocity problems. The limit of a function, calculating limits using the limit laws. Continuity, limits at infinity, horizontal asymptote. Infinite limits, vertical asymptotes. derivatives and rates of change
3. **Differentiation rules:** Differentiation of Polynomials. The Product and Quotient Rules. Derivatives of Trigonometric Functions. The Chain Rule, Implicit Differentiation. Related Rates, Indeterminate forms and l'hospital's rule.
7. **Applications of differentiation**: maximum and minimum values. The mean value theorem. How derivatives affect the shape of a graph. Summary of curve sketching. Optimization problems. Antiderivatives.
10.**Integrals**: the definite integral. The fundamental theorem of calculus. The indefinite integral and net change theorem. The substitution rule.
11. **Applications of integrals**: areas between curves. Volumes. Volumes by cylindrical shells. Average value of a function.
12.**Exponential and logarithmic functions**. Derivative and integrals involving logarithmic functions. Inverse functions. Derivative and integrals involving exponential functions. Derivative and integrals involving inverse trig functions. Hyperbolic functions.

Program and Course Outcomes:

1. Evaluate Limits of functions using various techniques including L'Hopital's Rule
2. Discuss the continuity functions
3. Identify the properties of inverse functions and their derivatives
4. Find the derivative of algebraic, trigonometric, exponential, and logarithmic functions
5. Sketch the graph of a function using the information for the first and second derivatives
6. Solve problems involving applications of derivatives including, related rates and optimization
7. Identify the definition and properties associated with definite integrals
8. Solve problems using the Fundamental Theorem of Calculus
9. Evaluate integrals using the method of substitution
10. Solve problems involving applications of integrals including finding volume of solids of revolution and area between curves

Selected References

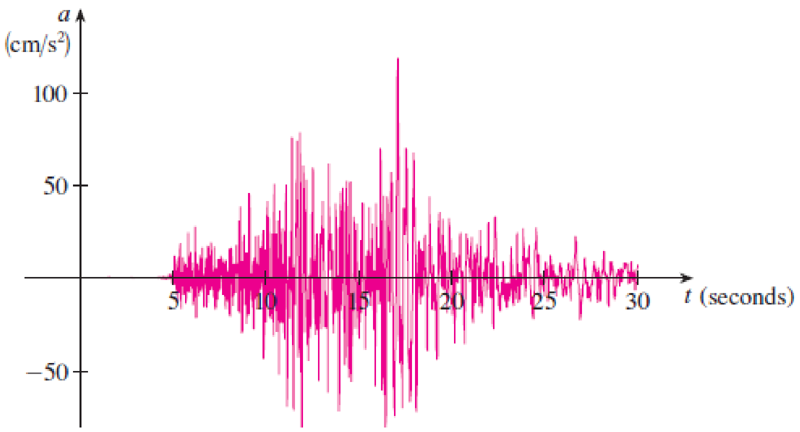
1 – Advanced Engineering Mathematics, Kreyszig
2 – Advanced Engineering Mathematics, Wyle
3 – Further Engineering Mathematics, Stroud.
4 – Engineering Mathematics, Kandasamy.
5 – Advanced Engineering Mathematics, Gustafson
6 – Elementary Differential Equations, Boyce.
7 – Numerical Analysis, Burden.
9 - Calculus by Thomas & Finney.

1 Functions

Examples of functions

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula A= πr^2

B. The vertical acceleration of the ground as measured by a seismograph during an earthquake is a function of the elapsed time. Figure below shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of the graph provides a corresponding value of a.



So, function is y = f(x), expressing y as a dependent variable on f and x is an independent variable.

For example f(x) = 2x-1

If x = 1 then 2*1-1 = 1
X= -1 then 2 *-1-1 = -3

And so on

If an absolute value like f(x) = IxI then $x = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

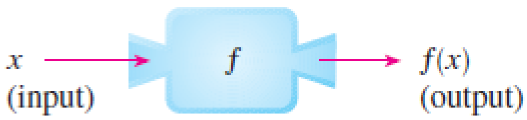
Note:

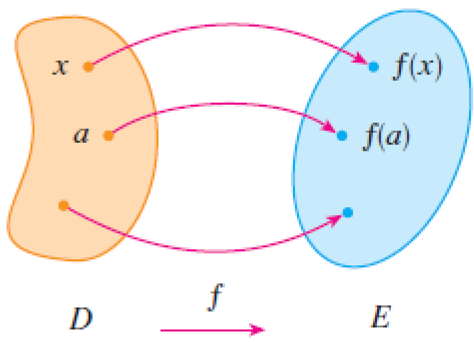
$|-a| = |a|$
$|ab| = |a|\,|b|$
$|a/b| = |a|/|b|$ but $b \neq 0$
$|a+b| = |a|+|b|$

2. Domain and Range

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the domain of the function. The number f(x) is the value of f at x and is read “f of x” The range of f is the set of all possible values of f(x) as varies throughout the domain.

Then domains and ranges of many functions are intervals of real numbers.





Example5: Find the domain and range of the following functions:

(a) $f(x) = 2x - 1$

(b) $f(x) = x^2$

(c) $f(x) = \tan x$

(d) $y = \sqrt{x}$

(e) $y = \dfrac{x - 12}{x^2 - 5x + 6}$

Solution

(a) $f(x) = 2x - 1$

Domain: x = R $-\infty \leq x \leq \infty$

Range f(x) = R R: denotes as all real number

(b) $f(x) = x^2$

Domain: x = R $-\infty \leq x \leq \infty$

Range f(x) = R

(c) $f(x) = \tan x$

Domain: x = R excluding $\pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2}, ...$

(d) $f(x) = \sqrt{x}$

Domain $0 \leq x$ and Range $0 \leq y$

(e) $y = \dfrac{x - 12}{x^2 - 5x + 6}$

The denominator not equal to zero

$x^2 - 5x + 6 = 0$

(x-3)(x-2) then domain all but $x \neq 3$ and $x \neq 2$

2. Sketch of functions

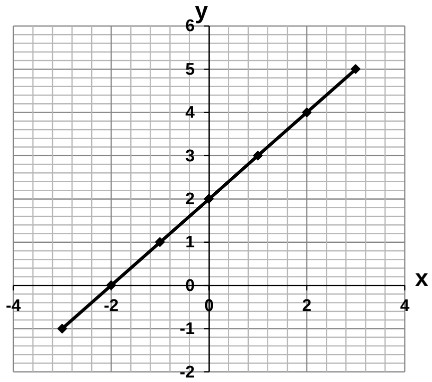
The points in the plane whose (x,y) are the input and output pairs of a function make up the graph of the function.

Definition:
Even function: if f(x) = f(-x)
Odd function: if f(x) = -f(-x)

Example6: sketch the function y= x+2

Solution:

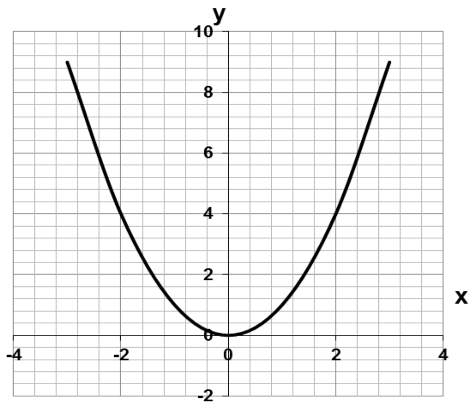
x	0	1	2	…..	-1	-2
y	2	3	4	….	1	0



Example7: sketch the power function $y= x^2$

Solution:

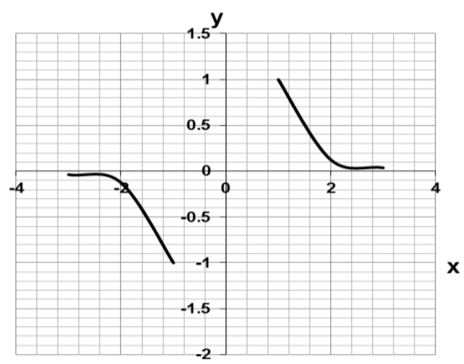
x	0	1	2	…..	-1	-2
y	0	1	4	….	1	4



Notes: if n is odd then symmetric about origin and pass through (1,1) and (-1,-1)

if n is even then symmetric about y axis and pass through (1,1) and (-1,1)
if the power is negative then $y=1/x^n$

like $y= 1/x^3$, $1/x$ …..



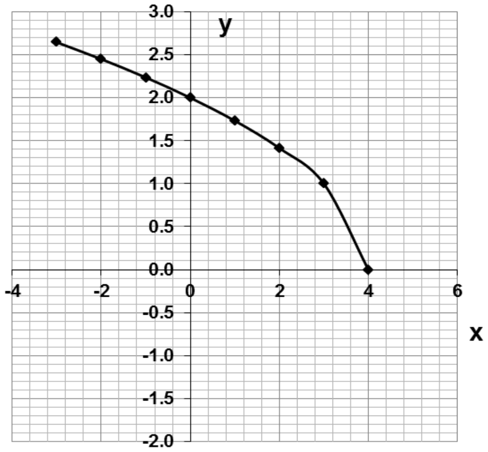
Graph of $y= 1/x^3$

Example8: Find the domain and range then sketch the function $y = \sqrt{4-x}$

Solution

Domain : $4-x \geq 0$ then $x \leq 4$

Range $y \geq 0$



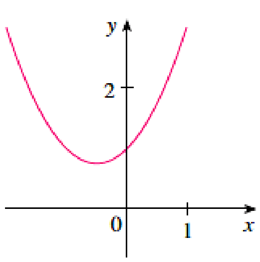
2. Polynomials

The general form is

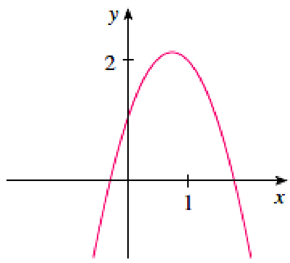
$K_n x^n + K_{n-1} x^{n-1} + K_{n-2} x^{n-2} + \ldots\ldots + K_1 x + K_0$

Ex: $x^3 + 5x^2 + 3$
$x^5 - x^3 + x^{0.5}$ ………….. etc.

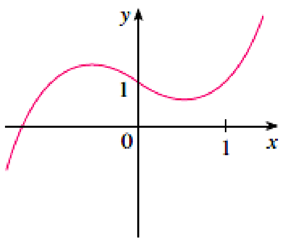
The graph is



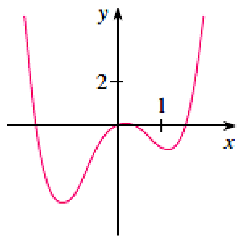
(a) $y = x^2 + x + 1$



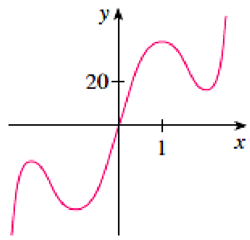
(b) $y = -2x^2 + 3x + 1$



(a) $y = x^3 - x + 1$



(b) $y = x^4 - 3x^2 + x$



(c) $y = 3x^5 - 25x^3 + 60x$

Example8: sketch the function $f_{(x)} = (x-2)(x+1)$

Solution

$f_{(x)} = y = x^2 + x - 2x - 2$

$y = x^2 - x - 2$ $\qquad$ $y = ax^2 + bx + c$

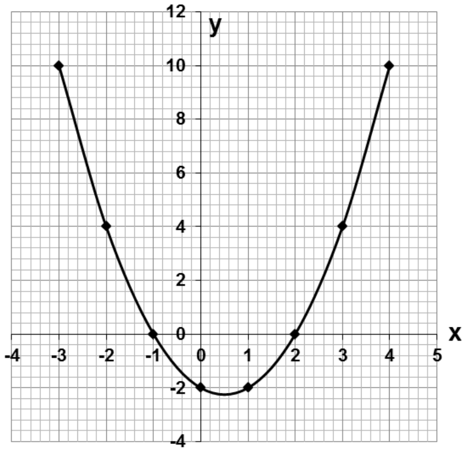
The vertex = x= -b/2a , $\qquad$ x = -(-1)/2*1 = ½
y = $(1/2)^2$ -1/2 -2 = -9/4

vertex (1/2, -9/4)

points of intercept

at x= 0 y = -2
at y= 0 0 = (x-2)(x-1)

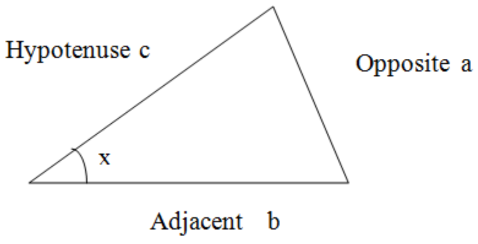
x= 2 (2,0)
x = -1 (-1,0)



2. Trigonometric functions

Sine sinx = a/c
Cosine cosx = b/c
Tangent tanx = a/b = sinx/cosx

Cotangent cotx = b/a = cosx/sinx
Secant secx = c/a = 1/cosx
Cosecant csecx = c/a = 1/sinx



Identities

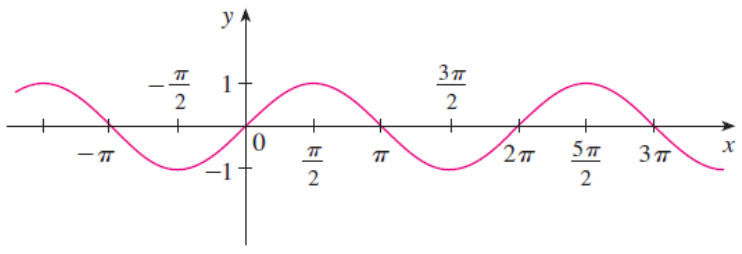
Trigonometric Identities – part 1 www.GIMathS.Com

Reciprocal Identities		Half Angle Identities	Double Angle Identities	Pythagoras Identities
$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$	$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin^2\theta + \cos^2\theta = 1$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$	$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$	$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$	$1 + \cot^2\theta = \csc^2\theta$

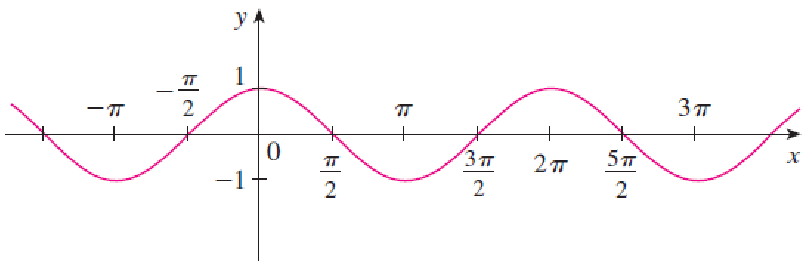
Sum to Product Identities	Product to Sum Identities
$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Even/Odd Identities
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$
$\sec(-\theta) = \sec\theta$
$\cot(-\theta) = -\cot\theta$

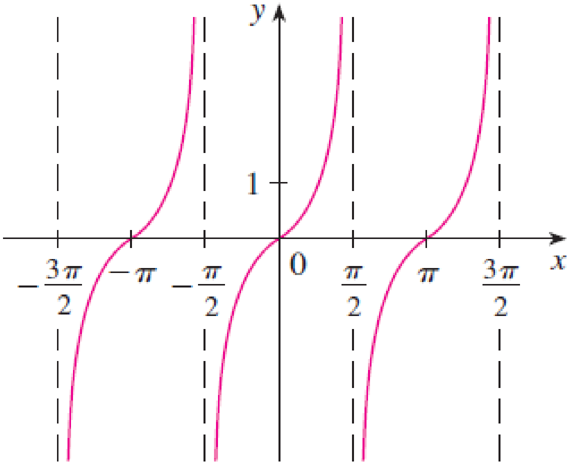
Graphs:



(a) $f(x) = \sin x$



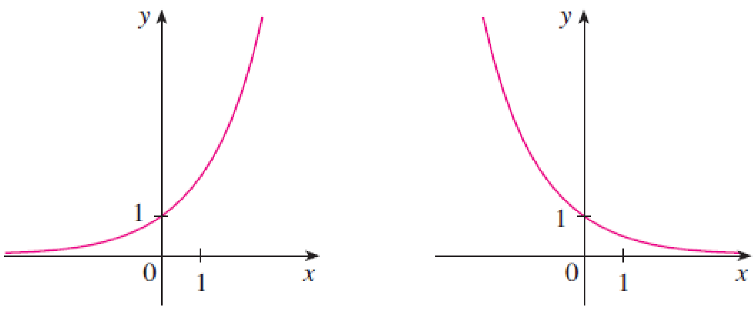
(b) $g(x) = \cos x$



$y = \tan x$

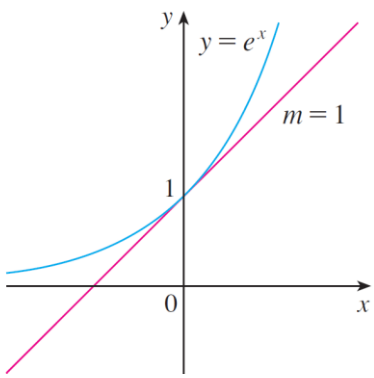
3. Exponential functions

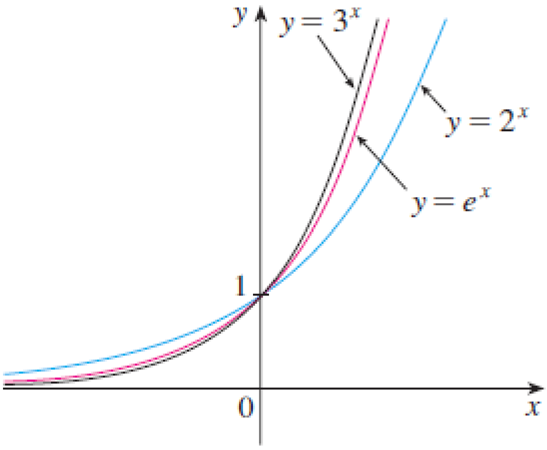
The exponential functions are the functions of the form f(x) = a^x , where the base a is a positive constant.



(a) $y = 2^x$ (b) $y = (0.5)^x$

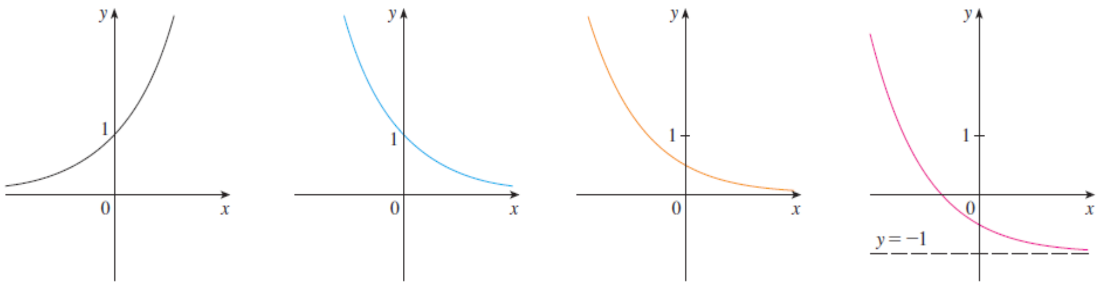
If we choose the base a so that the slope of the tangent line tangent line to the y = ax at (0,1) is exactly. In fact, there is such a number and it is denoted by the letter e. e =2.71828





Ilustrative example: Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

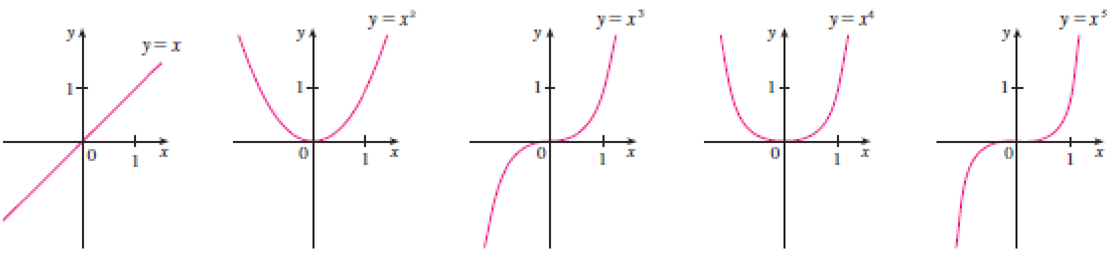
Solution We start with the graph of $y = e^x$ from Figure below, and reflect about the y-axis to get the graph of $y = e^{-x}$ in . Figure (b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure (c). . Finally, we shift the graph downward one unit to get the desired graph in Figure (d). The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.

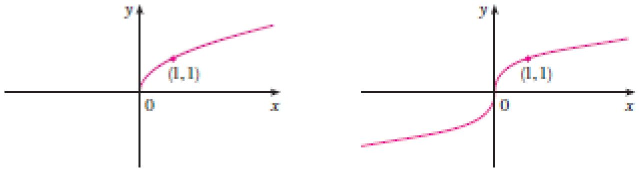


(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

4. Power functions

A function of the form f(x) = x^a , where is a constant, is called a power function. We consider several cases.

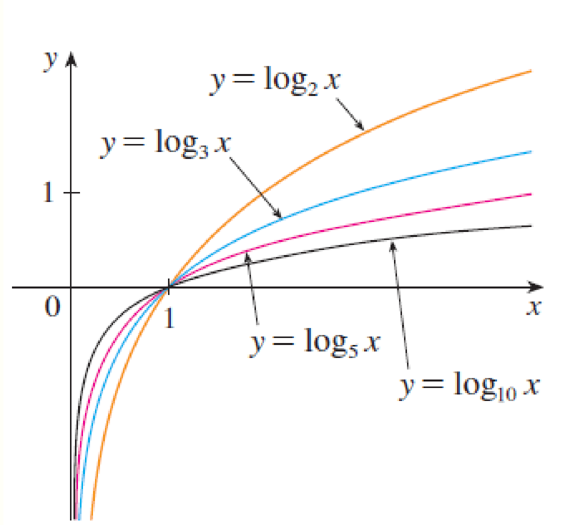




(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

5. Logarithmic functions

The logarithmic function f(x) = $\log_a$, where a is a positive constant, are the inverse function of the exponential functions. In each case the domain is(0, ∞) and the range is (-∞, ∞) and the function increases slowly when x>1.



Example9: Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$ (b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ (d) $u(t) = 1 - t + 5t^4$

Solution

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

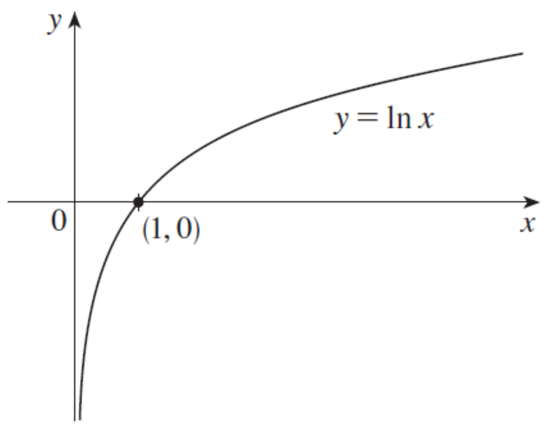
(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

NATURAL LOGARITHMS

The logarithm with base is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

$$\ln e = 1$$

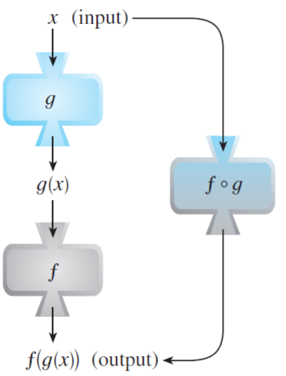


x	y= lnx
0	∞
1	0
2	0.693
3	1.098
4	1.386
5	1.609
-1	∞
0.9	-0.105
0.5	-0.693
0.2	-1.609
0.1	-2.302

6. Algebra of functions

Let f is a fynction of x then we get f(x) and g is a function of x also we get g(x)

Df is the domain of f(x)
Dg is the domain of g(x)



Then:

f+g = f(x) + g(x) and Df ∩ Dg
f – g = f(x) - g(x)
f.g = f(x) . g(x)

and the domain is as same before

if f/g then Df ∩ Dg but g(x) ≠ 0
if g/f then Dg ∩ Df but f(x) ≠ 0

and Df∘g = {x: x ∈ Dg, g(x) ∈ Df}
where

f∘g(x) = f(g(x)) also called the composition of f and g

Example10: Find f∘g and g∘f if $f_{(x)} = \sqrt{1-x}$ and $g_{(x)} = \sqrt{5+x}$

Solution

(f∘g)x= f(g(x)) = f($\sqrt{5+x}$) = $\sqrt{1-\sqrt{5+x}}$
(1-x) ≥ 0 then x ≤ 1 Df: x≤1
5+x ≥ 0 then x ≥ -5 Dg: x≥-5
D f∘g = {x: x ≥ -5, $\sqrt{5+x}$ ≤1} = {x: -5 ≤ x ≤-4}

Example 11: Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9))$$
$$= [\cos(x + 9)]^2 = F(x)$$

□

Example 12: If $f_{(x)} = \sqrt{x}$ and $g_{(x)} = \sqrt{1-x}$

Find:

f+g, f-g, g-f, f∘g, f/g, g/f then graph f∘g and also f+g.

Solution

$f_{(x)} = \sqrt{x}$ domain $x \geq 0$

$g_{(x)} = \sqrt{1-x}$ domain $x \leq 1$

f+g = (f+g)x = $\sqrt{x} + \sqrt{1-x}$ domain $0 \leq x \leq 1$ or [0,1]

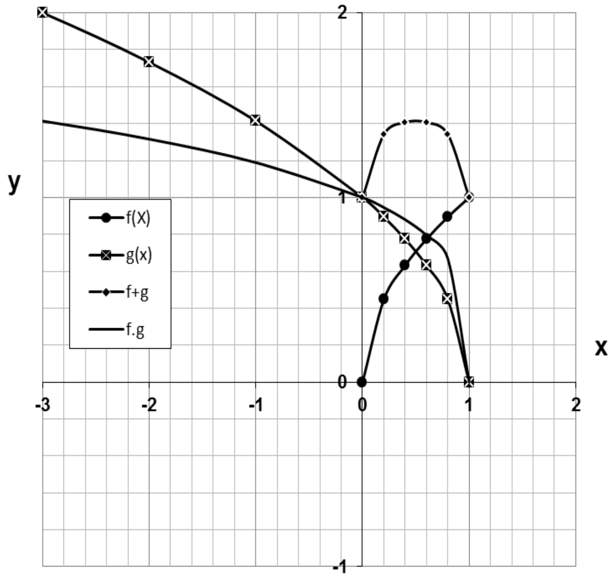
f-g = $\sqrt{x} - \sqrt{1-x}$ domain $0 \leq x \leq 1$

g-f = $\sqrt{1-x} - \sqrt{x}$ domain $0 \leq x \leq 1$

f∘g = f(x) g(x) = f(g(x)) =f($\sqrt{1-x}$) = $\sqrt{\sqrt{1-x}}$ = $\sqrt[4]{1-x}$ domain $(-\infty, 1]$ (why?)

f/g = f(x)/g(x) = $\sqrt{\dfrac{x}{1-x}}$ domain $(-\infty, 1]$

g/f = g(x)/f(x) = $\sqrt{\dfrac{1-x}{x}}$ domain $(0, 1]$


y
x
f(X)
g(x)
f+g
f.g
2
1
0
-1
-3
-2
-1
0
1
2

Inverse functions

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function y = ln x as the inverse of the exponential function y = e^x, and we also give examples of several inverse trigonometric functions.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Example 63:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Note:

Only a one-to-one function can have an inverse

Q: What is the one to one function ?

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Note

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

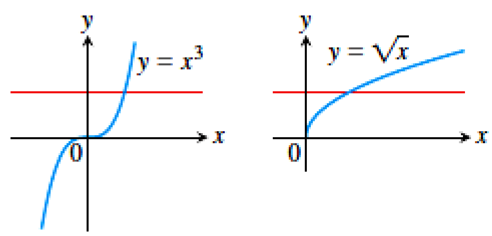
Example:

two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.

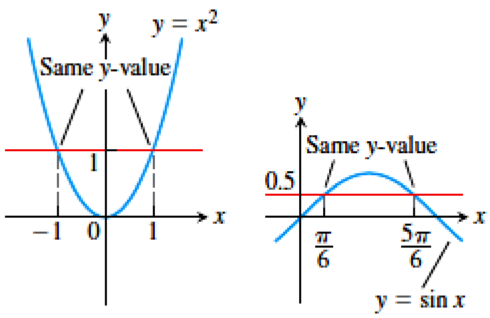
(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 64:

Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

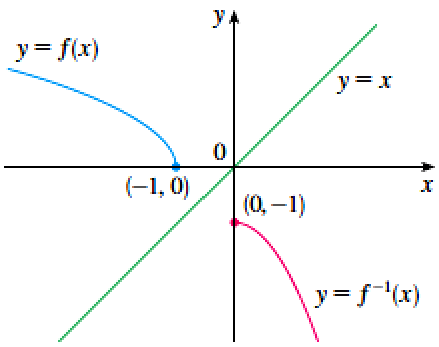
$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

Example 65:

Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

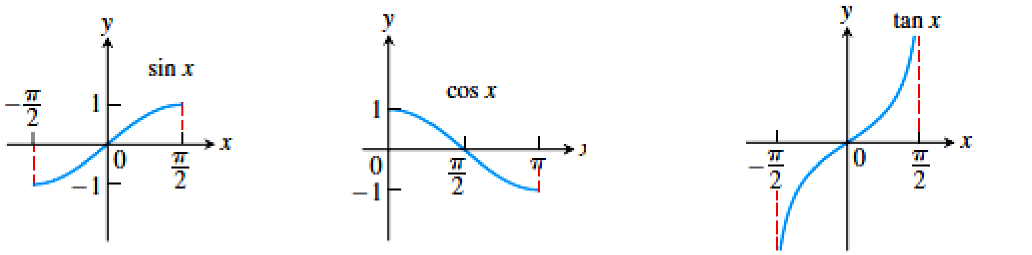
SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure



Inverse Trigonometric Functions

The six basic trigonometric functions of a general radian angle x were reviewed in Chapter 2. These functions are not one-to-one (their values repeat periodically).

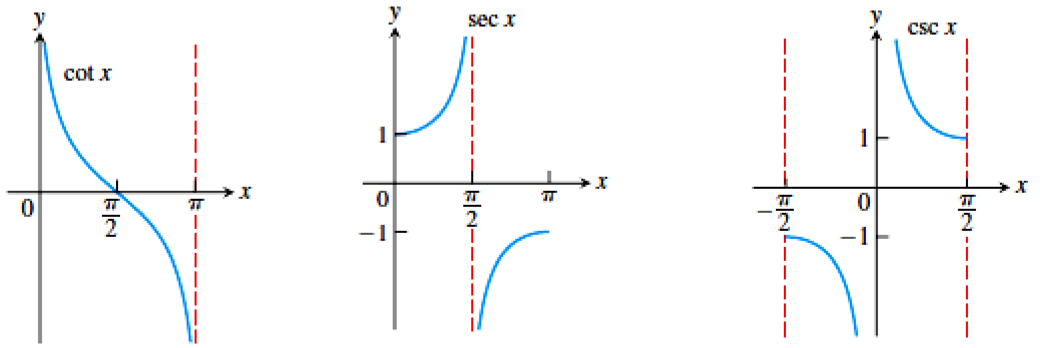
Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

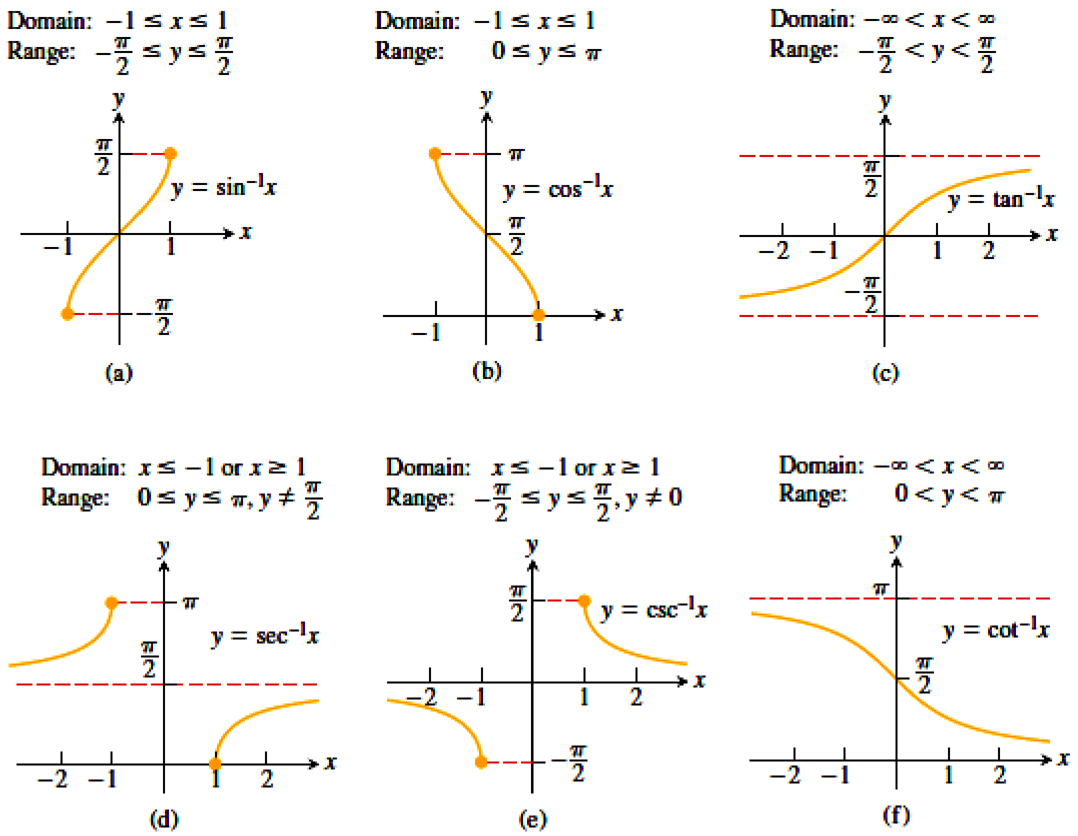
$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

Graph of inverse trig functions



Notes:

To convert from degree to radian

$$\pi \text{ radians} = 180°$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

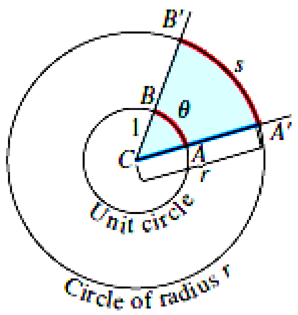


TABLE 1.1 Angles measured in degrees and radians

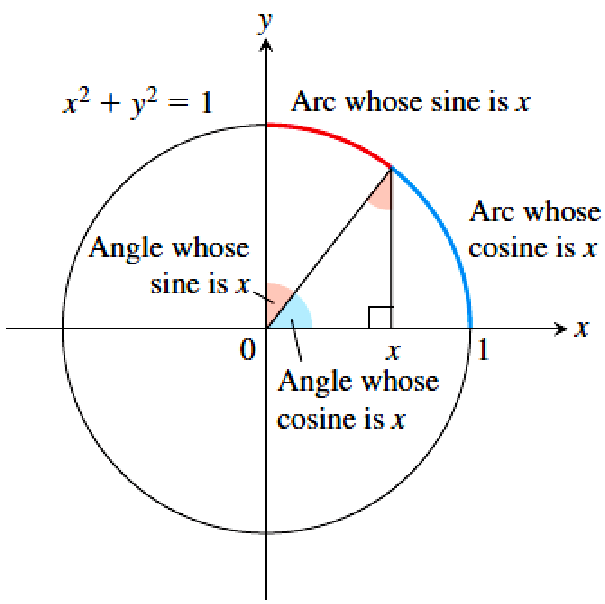
Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

TABLE 1.2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

The "Arc" in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



Example 66:

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.68a.

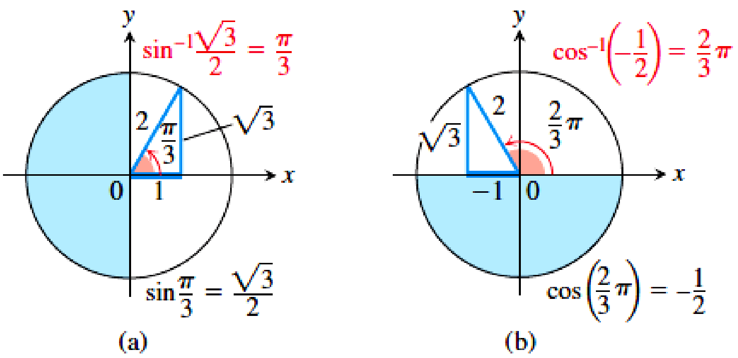
(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine

We can create the following table of common values for the arcsine and arccosine functions

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$



Example 67:

Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$. This enables us to read from the triangle that

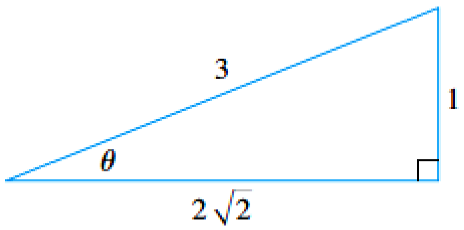
$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$

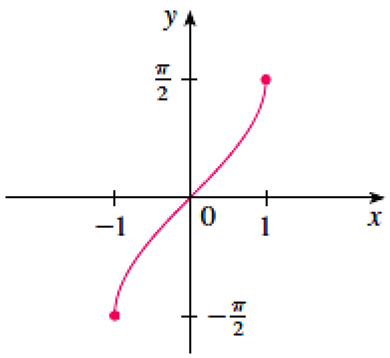
The cancellation equations for inverse functions become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function by reflection about the line $y = x$.





$y = \sin^{-1}x = \arcsin x$

Example 68:

Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

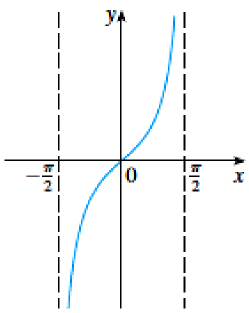
Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

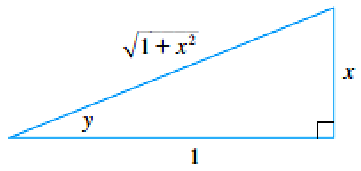
$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ ■

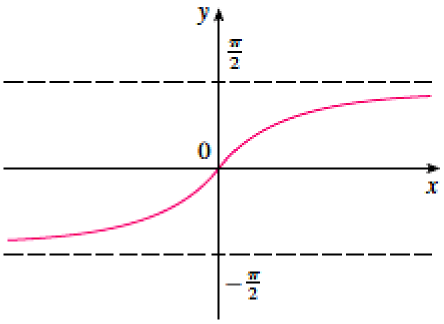
The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure .

$y = \cos^{-1}x = \arccos x$



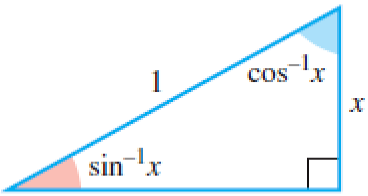
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



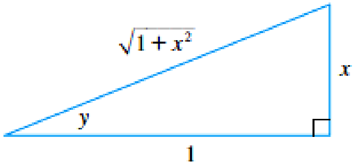


Identities of inverse trig functions

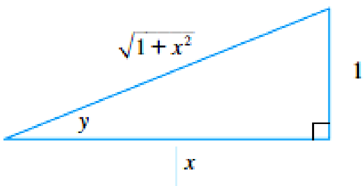
$$\sin^{-1}x + \cos^{-1}x = \pi/2.$$



$\sin^{-1}x$



$\cos^{-1}x$



and so on ..