5 Integrals

The Indefinite Integral

An integral can be considered to be an **antiderivative**. Thus, if we know that the derivative of $F(x)$ is $f(x)\,[=F'(x)]$, an integral of $f(x)$ is $F(x)$. For example, the derivative of $\frac{1}{3}x^3$ is x^2, and an integral of x^2 is $\frac{1}{3}x^3$. Note that we have used the article an. Since the derivative of a constant is zero, $F(x)$ is arbitrary to the extent of an arbitrary constant. The integral we have defined is known as an **indefinite integral** which is usually denoted by the symbol $\int$. Thus, we write

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

Example 79: Evaluate the following indifinite integral $\int x^4 + 3x - 9\,dx$

Solution

The indefinite integral is

$$\int x^4 + 3x - 9\,dx = \frac{1}{x}x^5 + \frac{3}{2}x^2 - 9x + c$$

PROPERTIES OF INDEFINITE INTEGRALS

1. *A constant factor can be taken outside the integral sign:*

$$\int af(x)\,dx = a\int f(x)\,dx \qquad (a = \text{const}).$$

2. *Integral of the sum or difference of functions (additivity):*

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Computing Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Example 80: Evaluate each of the following integrals

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\int x^8 + x^{-8}\,dx$

(c) $\displaystyle\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\displaystyle\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution:

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

$$\int 5t^3 - 10t^{-6} + 4\,dt = 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c$$
$$= \frac{5}{4}t^4 + 2t^{-5} + 4t + c$$

(b) $\int x^8 + x^{-8}\,dx$

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

$$\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx = \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\,dx$$
$$= 3\frac{1}{7/4}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}} + c$$
$$= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c$$

(d) $\int dy$

$$\int dy = \int 1\,dy = y + c$$

(e) $\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw$

$$\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw=\int 4w-w^3+4w^{\frac{1}{3}}-w^{\frac{7}{3}}dw$$

$$=2w^2-\frac{1}{4}w^4+3w^{\frac{4}{3}}-\frac{3}{10}w^{\frac{10}{3}}+c$$

(f) $\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx$

$$\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx=\int \frac{4x^{10}}{x^3}-\frac{2x^4}{x^3}+\frac{15x^2}{x^3}dx$$

$$=\int 4x^7-2x+\frac{15}{x}dx$$

$$=\frac{1}{2}x^8-x^2+15\ln|x|+c$$

Furtehr examples:

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

You should check this answer by differentiating it. ■

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

SOLUTION This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta\,\cot\theta\,d\theta = -\csc\theta + C \qquad \blacksquare$$

Substitution Rule for Indefinite Integrals

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1+x^2}\,dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.*

> $\boxed{4}$ **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then
>
> $$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 1 Find $\int x^3\cos(x^4+2)\,dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using

$x^3\,dx = \frac{1}{4}\,du$ and the Substitution Rule, we have

$$\int x^3\cos(x^4+2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$

$$= \tfrac{1}{4}\sin u + C$$

$$= \tfrac{1}{4}\sin(x^4+2) + C$$

Notice that at the final stage we had to return to the original variable x.

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$

$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x+1}$. Then

$$du = \frac{dx}{\sqrt{2x+1}} \qquad \text{so} \qquad dx = \sqrt{2x+1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\int \sqrt{2x+1}\,dx = \int u \cdot u\,du = \int u^2\,du$$

$$= \frac{u^3}{3} + C = \tfrac{1}{3}(2x+1)^{3/2} + C$$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$

$$= -\tfrac{1}{8}(2\sqrt{u}) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C$$

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^u\,du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3 \cos(x^4+2)\,dx = \int \cos(x^4+2)\cdot x^3\,dx = \tfrac{1}{4}\int \cos(x^4+2)\cdot(4x^3)\,dx$$

$$= \tfrac{1}{4}\int \cos(x^4+2)\cdot \frac{d}{dx}(x^4+2)\,dx = \tfrac{1}{4}\sin(x^4+2) + C$$

Similarly, the solution to Example 4 could be written like this:

$$\int e^{5x}\,dx = \tfrac{1}{5}\int 5e^{5x}\,dx = \tfrac{1}{5}\int \frac{d}{dx}(e^{5x})\,dx = \tfrac{1}{5}e^{5x} + C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\,x^5\,dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\int \sqrt{1+x^2}\,x^5\,dx = \int \sqrt{1+x^2}\,x^4\cdot x\,dx$$

$$= \int \sqrt{u}\,(u-1)^2\cdot \tfrac{1}{2}du = \tfrac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du$$

$$= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C$$

$$= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C \qquad \blacksquare$$

EXAMPLE 6 Calculate $\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 6 can also be written as

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$$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$