

University of Anbar
College of Engineering
Mechanical Engineering Dept.



Advanced Heat Transfer/ I Conduction and Radiation

Handout Lectures for MSc. / Power Chapter One/ Introductory Concepts

Course Tutor

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- J. P. Holman, “*Heat Transfer*”, McGraw-Hill Book Company, 6th Edition, 2006.
- T. L. Bergman, A. Lavine, F. Incropera, D. Dewitt, “*Fundamentals of Heat and Mass Transfer*”, John Wiley & Sons, Inc., 7th Edition, 2007.
- Vedat S. Arpaci, “*Conduction Heat Transfer*”, Addison-Wesley, 1st Edition, 1966.
- P. J. Schneider, “*Conduction Teat Transfer*”, Addison-Wesley, 1955.
- D. Q. Kern, A. D. Kraus, “*Extended surface heat transfer*”, McGraw-Hill Book Company, 1972.
- G. E. Myers, “*Analytical Methods in Conduction Heat Transfer*”, McGraw-Hill Book Company, 1971.
- J. H. Lienhard IV, J. H. Lienhard V, “*A Heat Transfer Textbook*”, 4th Edition, Cambridge, MA : J.H. Lienhard V, 2000.

Chapter One

Introductory Concepts

1.1 Modes of Heat Transfer

Heat transfer (or **heat**) is thermal energy in transit due to a spatial temperature difference.

Whenever a temperature difference exists in a medium or between media, heat transfer must occur.

As shown in Figure 1.1, we refer to different types of heat transfer processes as modes. When a temperature gradient exists in a stationary medium, which may be a solid or a fluid, we use the term "**conduction**" to refer to the heat transfer that will occur across the medium. In contrast, the term "**convection**" refers to heat transfer that will occur between a surface and a moving fluid when they are at different temperatures. The third mode of heat transfer is termed "**thermal radiation**". All surfaces of finite temperature emit energy in the form of electromagnetic waves. Hence, in the absence of an intervening medium, there is net heat transfer by radiation between two surfaces at different temperatures.

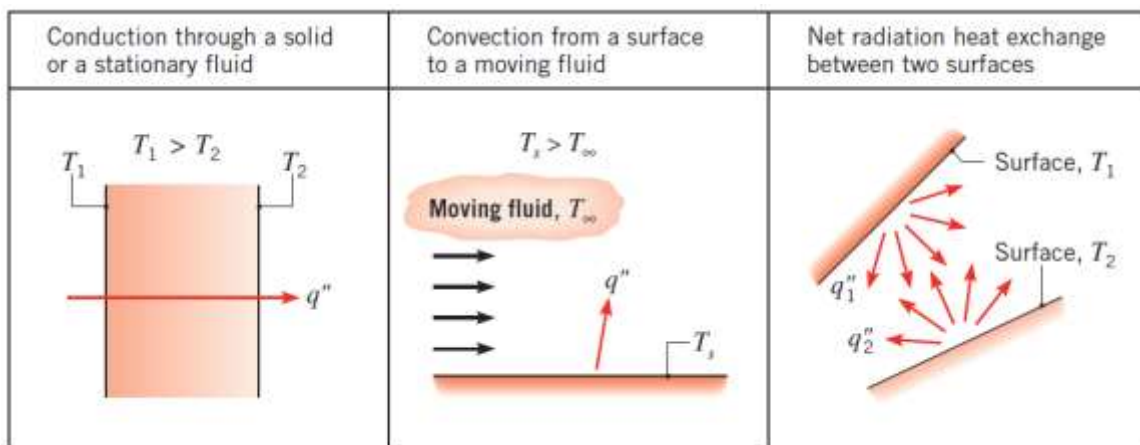


Figure 1.1: Conduction, convection, and radiation heat transfer modes.

As engineers, it is important that we understand the physical mechanisms which underlie the heat transfer modes and that we be able to use the rate equations that quantify the amount of energy being transferred per unit time.

1.1.1 Conduction Heat Transfer

At mention of the word *conduction*, we should immediately conjure up concepts of *atomic* and *molecular activity* because processes at these levels sustain this mode of heat transfer. Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

The physical mechanism of conduction is most easily explained by considering a gas and using ideas familiar from your thermodynamics background. Consider a gas in which a temperature gradient exists, and assume that there is *no bulk, or macroscopic, motion*. The gas may occupy the space between two surfaces that are maintained at different temperatures, as shown in Figure 1.2. We associate the temperature at any point with the energy of gas molecules in proximity to the point. This energy is related to the random translational motion, as well as to the internal rotational and vibrational motions, of the molecules.

Higher temperatures are associated with higher molecular energies. When neighboring molecules collide, as they are constantly doing, a transfer of energy from the more energetic to the less energetic molecules must occur. In the presence of a temperature gradient, energy transfer by conduction must then occur in the direction of decreasing temperature. This would be true even in the absence of collisions, as is evident from Figure 1.2. The hypothetical plane at is constantly being crossed by molecules from above and below due to their *random* motion. However, molecules from above are associated with a higher temperature than those from below, in which case there must be a *net* transfer of energy in the positive x -direction. Collisions between molecules enhance this energy transfer.

We may speak of the net transfer of energy by random molecular motion as a *diffusion* of energy.

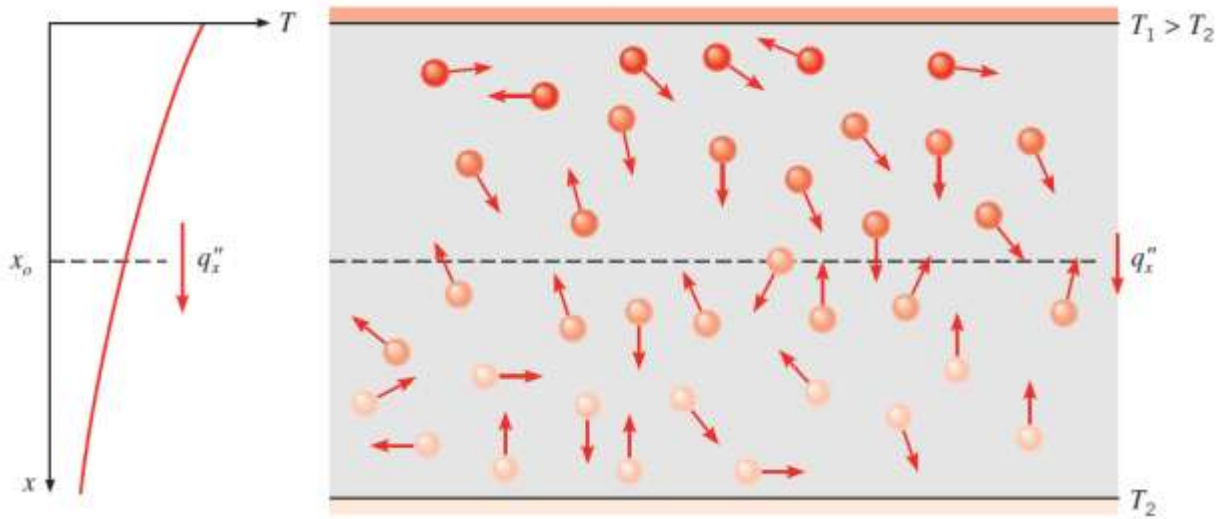


Figure 1.2: Association of conduction heat transfer with diffusion of energy due to molecular activity.

The situation is much the same in liquids, although the molecules are more closely spaced and the molecular interactions are stronger and more frequent. Similarly, in a solid, conduction may be attributed to atomic activity in the form of lattice vibrations. The modern view is to ascribe the energy transfer to *lattice waves* induced by atomic motion. In an electrical nonconductor, the energy transfer is exclusively via these lattice waves; in a conductor, it is also due to the translational motion of the free electrons.

Examples of conduction heat transfer are legion. The exposed end of a metal spoon suddenly immersed in a cup of hot coffee is eventually warmed due to the conduction of energy through the spoon. On a winter day, there is significant energy loss from a heated room to the outside air. This loss is principally due to conduction heat transfer through the wall that separates the room air from the outside air.

Heat transfer processes can be quantified in terms of appropriate *rate equations*. These equations may be used to compute the amount of energy being transferred

per unit time. For heat conduction, the rate equation is known as *Fourier's law*. For the one-dimensional plane wall shown in Figure 1.3, having a temperature distribution $T(x)$, the rate equation is expressed as,

$$q_x'' = -k \frac{dT}{dx} \quad (1-1)$$

The heat flux (W/m^2) is the **heat transfer rate** in the x -direction per unit area perpendicular to the direction of transfer, and it is proportional to the **temperature gradient**, dT/dx , in this direction. The parameter k is a transport property known as the **thermal conductivity** (W/mK) and is a characteristic of the wall material. The **minus sign** is a consequence of the fact that heat is transferred in the direction of decreasing temperature. Under the steady-state conditions shown in Figure 1.3, where the temperature distribution is linear, and the temperature gradient may be expressed as,

$$q_x'' = -k \frac{dT}{dx} = -k \frac{T_2 - T_1}{L} = k \frac{T_1 - T_2}{L} \quad (1-2)$$

The **heat rate** by conduction, q_x (W), through a plane wall of area A is then the product of the flux and the area, $q_x = q_x'' \times A$

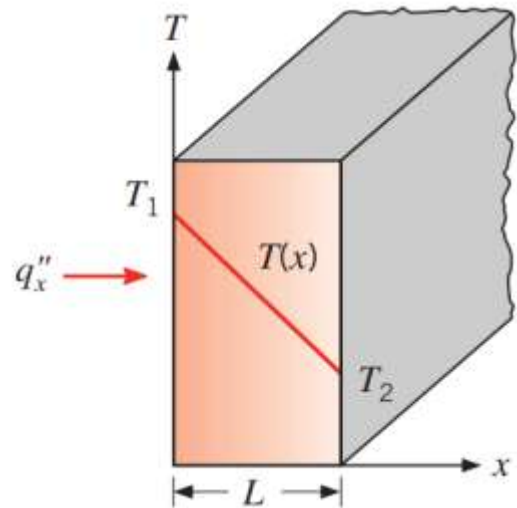


Fig. 1-3: One-dimensional heat transfer by conduction

1.1.2 Convection Heat Transfer

The convection heat transfer mode is comprised of *two mechanisms*. In addition to *energy transfer* due to *random molecular motion* (diffusion), energy is also transferred by the *bulk, or macroscopic, motion of the fluid*. This fluid motion is associated with the fact that, at any instant, large numbers of molecules are moving

collectively or as aggregates. Such motion, in the presence of a temperature gradient, contributes to heat transfer. Because the molecules in the aggregate retain their random motion, the total heat transfer is then due to a superposition of energy transport by the random motion of the molecules and by the bulk motion of the fluid. The term *convection* is customarily used when referring to this cumulative transport and the term *advection* refers to transport due to bulk fluid motion

Convection heat transfer may be classified according to the nature of the flow. We speak of *forced convection* when the flow is caused by *external means*, such as by a fan, a pump, or atmospheric winds. As an example, consider the use of a fan to provide forced convection air cooling of hot electrical components on a stack of printed circuit boards (Figure 1.4a). In contrast, for *free (or natural) convection*, the flow is induced by *buoyancy forces*, which are due to density differences caused by temperature variations in the fluid. An example is the free convection heat transfer that occurs from hot components on a vertical array of circuit boards in air (Figure 1.4b). Air that makes contact with the components experiences an increase in temperature and hence a reduction in density. Since it is now lighter than the surrounding air, buoyancy forces induce a vertical motion for which warm air ascending from the boards is replaced by an inflow of cooler ambient air.

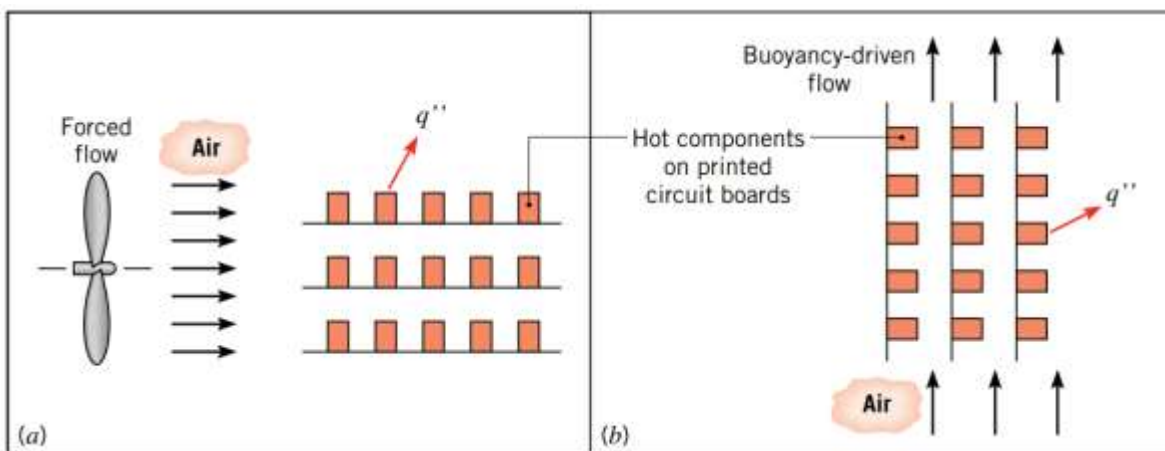


Fig. 1-4: Convection heat transfer processes. (a) Forced convection. (b) Natural convection.

Regardless of the nature of the convection heat transfer process, the appropriate rate equation is of the form,

$$q'' = h (T_s - T_\infty) \quad (1-3)$$

where q'' , the convective heat flux (W/m^2), is proportional to the difference between the surface and fluid temperatures, T_s and T_∞ , respectively. This expression is known as *Newton's law of cooling*, and the parameter h ($\text{W}/\text{m}^2\text{K}$) is termed the *convection heat transfer coefficient*. This coefficient depends on conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an assortment of fluid thermodynamic and transport properties. In the solution of such problems we presume h to be known, using typical values given in Table 1.1.

Table 1.1: Typical values of the convection heat transfer coefficient.

Process	h ($\text{W}/\text{m}^2 \cdot \text{K}$)
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	100–20,000
Convection with phase change	
Boiling or condensation	2500–100,000

1.1.3 Radiation Heat Transfer

Thermal radiation is *energy emitted* by matter that is at a nonzero temperature. Although we will focus on radiation from solid surfaces, emission may also occur from liquids and gases. Regardless of the form of matter, the emission may be attributed to changes in the electron configurations of the constituent atoms or

molecules. The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons). While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum. Consider radiation transfer processes for the surface of Figure 1.5a. Radiation that is emitted by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which energy is released per unit area (W/m^2) is termed the surface *emissive power*, E . There is an upper limit to the *emissive power*, which is prescribed by the *Stefan Boltzmann law*:

$$E_b = \sigma T_s^4 \tag{1-4}$$

where T_s is the absolute temperature (K) of the surface and σ is the *Stefan Boltzmann constant* ($\sigma = 5.67 \times 10^{-8} \text{ W}/\text{m}^2\text{K}^4$). Such a surface is called an *ideal radiator* or *blackbody*. The heat flux emitted by a *real surface* is less than that of a blackbody at the same temperature and is given by,

$$E = \varepsilon \sigma T_s^4 \tag{1-5}$$

Where ε is a radiative property of the surface termed the emissivity. With values in the range $0 \leq \varepsilon \leq 1$, this property provides a measure of how efficiently a surface emits energy relative to a blackbody.

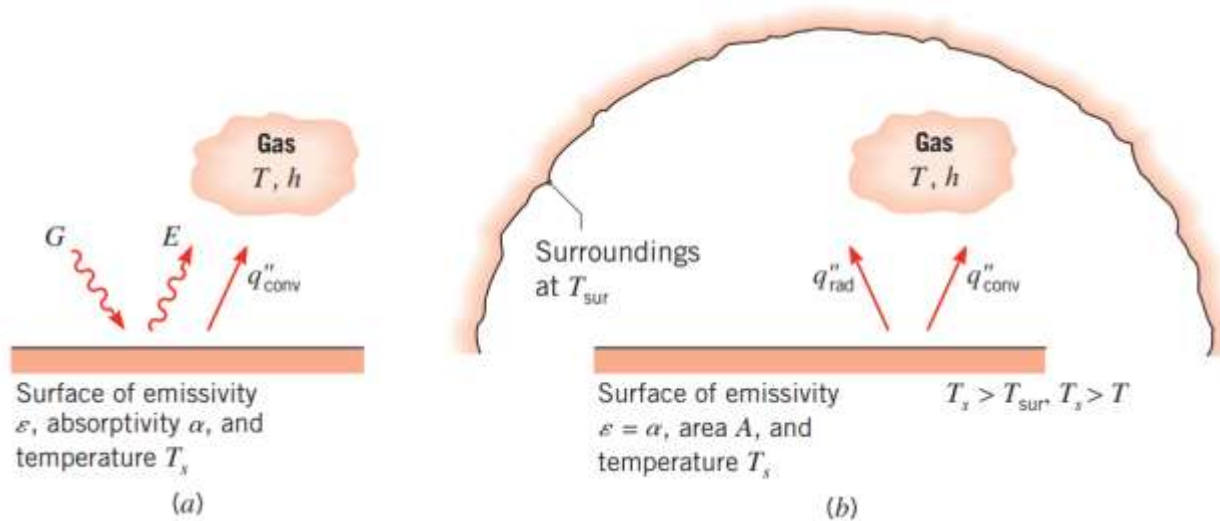


Fig. 1-5: Radiation exchange: (a) at a surface and (b) between a surface and large surroundings.

A special case that occurs frequently involves radiation exchange between a small surface at T_s and a much larger, isothermal surface that completely surrounds the smaller one (Figure 1.5b). The surroundings could, for example, be the walls of a room or a furnace whose temperature T_{sur} differs from that of an enclosed surface ($T_{sur} \neq T_s$). For such a condition, the irradiation may be approximated by emission from a blackbody at T_{sur} , in which case $G = \sigma T_{sur}^4$. If the surface is assumed to be one for which $\alpha = \varepsilon$ (a *gray* surface), the net rate of radiation heat transfer from the surface, expressed per unit area of the surface, is

$$q_{rad}'' = \frac{q}{A} = \varepsilon E_b(T_s) - \alpha G = \varepsilon \sigma (T_s^4 - T_{sur}^4) \quad (1-6)$$

This expression provides the difference between thermal energy that is released due to radiation emission and that gained due to radiation absorption.

A surface which absorbs all radiation incident upon it ($\alpha=1$) or at a specified temperature emits the maximum possible radiation is called (black surface). The emissivity of a surface, ε , is defined as;

$$\frac{q}{q_b} = \varepsilon$$

where (q and q_b) are the radiant heat fluxes from this surface and from a black surface respectively at the same temperature. Under thermal equilibrium ($\alpha=\varepsilon$) for all surfaces (*Kirchhoff's law*).

When two bodies exchange heat by radiation, the net heat exchange is given by Stefan-Boltzmann's law of radiation which was found experimentally by Stefan and later proved thermodynamically by Boltzmann. Thus;

$$q = F_G \varepsilon \sigma (T_s^4 - T_{sur}^4)$$

Where F_G is geometric view factor, configuration factor or shape factor.

1.2 Fourier's law of conduction

Microscopic theories such as the *kinetic theory of gases* and the *free-electron theory of metals* have been developed to the point where they can be used to predict conduction through media. However, the *macroscopic* or *continuum* theory of conduction, which is the subject matter of this course, disregards the molecular structure of continua. Thus conduction is taken to be phenomenological and its effects are determined by experiment as described in details in *Section 1.1.1*.

The molecular structure of material (continua) may be classified according to variations in thermal conductivity. A material (continuum) is said to be *homogeneous* if its conductivity does not vary from point to point within the continuum, and *heterogeneous* if there is such variation. Furthermore, continua in which the conductivity is the same in all directions are said to be *isotropic*, whereas those in which there exists directional variation of conductivity are said to be *anisotropic*. Some materials consisting of a fibrous structure exhibit anisotropic character, for example, wood and asbestos. Materials having a porous structure, such as wool or cork, are examples of heterogeneous continua. In this course, except where explicitly stated otherwise, we shall be studying only the problems of isotropic continua. Because of the symmetry in the conduction of heat in isotropic continua, the flux of heat at a point must be normal to the isothermal surface through this point.

According to the *first law of thermodynamics*, under steady conditions there must be a constant rate of heat q through any cross section of the geometry (such, walls, cylinders and spheres). From the *second law of thermodynamics* we know that the direction of this heat is from the higher temperature to the lower. Therefore, equations 1.1 and 1.2 give Fourier's law for *homogeneous isotropic continua*. Equations (1.1 and 1.2) may also be used for a fluid (liquid or gas) placed between two plates a distance L apart, provided that suitable precautions are taken to

eliminate convection and radiation. Therefore, equations (1.1 and 1.2) describe the conduction of heat in fluids as well as in solids.

Let the temperatures of two isothermal surfaces corresponding to the locations x and $x+\Delta x$ be T and $T+\Delta T$, respectively (Figure 1.6). Since this plate may be assumed to be locally homogeneous, equation (1.1) can be used for a layer of the plate having the thickness Δx as $\Delta x \rightarrow 0$. Thus it becomes possible to state the differential form of Fourier's law of conduction, giving the heat flux at x in the direction of increasing x , as follows:

$$q_x = -k \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta T}{\Delta x} \right) = -k \frac{\partial T}{\partial x} \quad (1-7)$$

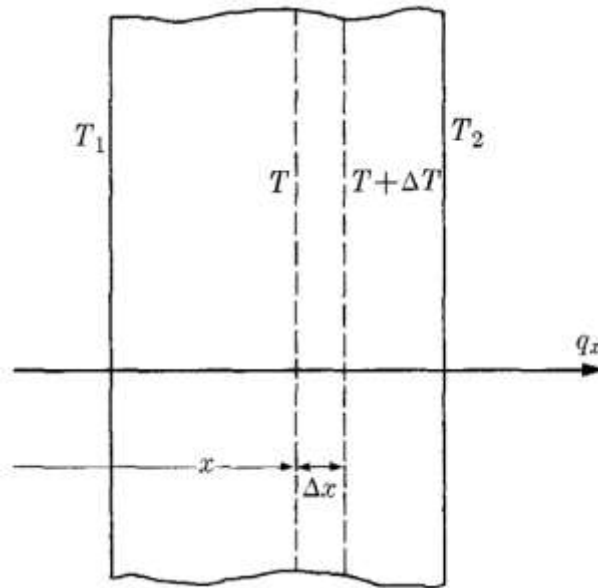


Fig. 1-6: Isothermal surfaces.

Fourier's law for heterogeneous isotropic continua. In equation (1.7), by introducing a minus sign we have made q_x , positive in the direction of increasing x . It is important to note that this equation is independent of the temperature distribution. Thus, for example, in figure 1.7 (a) $\frac{\partial T}{\partial x} < 0$ and $q_x > 0$, whereas in

figure 1.7 (b) $\frac{\partial T}{\partial x} > 0$ and $q_x < 0$. Both results agree with the second law of thermodynamics in that the heat diffuses from higher to lower temperatures.

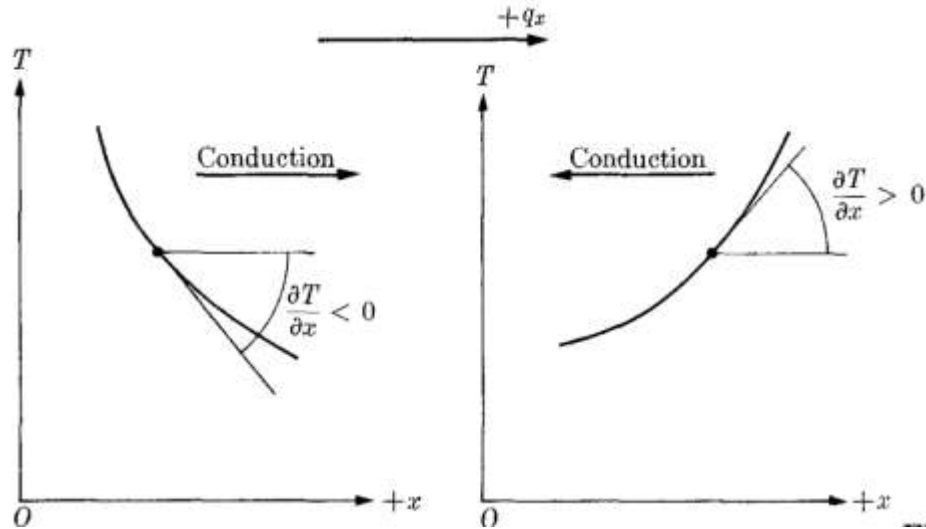


Fig. 1-7: Independent of the temperature distribution.

Equation (1.7) may be readily extended to any isothermal surface if we state that the heat flux across an isothermal surface is

$$q_n = -k \frac{\partial T}{\partial n} \tag{1-8}$$

Generalizing Fourier's law for isotropic continua, we may assume each component of the heat flux vector to be linearly dependent on all components of the temperature gradient at the point. Thus, for example, the Cartesian form of *Fourier's law for heterogeneous anisotropic continua* becomes

$$\left. \begin{aligned} q_x &= -(k_{11} \frac{\partial T}{\partial x} + k_{12} \frac{\partial T}{\partial y} + k_{13} \frac{\partial T}{\partial z}) \\ q_y &= -(k_{21} \frac{\partial T}{\partial x} + k_{22} \frac{\partial T}{\partial y} + k_{23} \frac{\partial T}{\partial z}) \\ q_z &= -(k_{31} \frac{\partial T}{\partial x} + k_{32} \frac{\partial T}{\partial y} + k_{33} \frac{\partial T}{\partial z}) \end{aligned} \right\} \tag{1-9}$$

The value of k for a continuum depends in general on the chemical composition, the physical state, and the structure, temperature, and pressure. In solids the pressure dependency, being very small, is always neglected. For narrow temperature intervals the temperature dependency may also be negligible. Otherwise a linear relation is assumed in the form

$$k = k_o(1 + \beta T) \quad (1-10)$$

where β is small and negative for most solids.

1.3 Equation of Conduction

A major objective in a conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries. That is, we wish to know the temperature distribution, which represents how temperature varies with position in the medium. Once this distribution is known, the conduction heat flux at any point in the medium or on its surface may be computed from Fourier's law. Other important quantities of interest may also be determined.

Consider a homogeneous medium within which there is no bulk motion (*advection*) and the temperature distribution $T(x, y, z)$ is expressed in **Cartesian coordinates**. By applying conservation of energy, we first define an infinitesimally small (*differential*) control volume, $dx.dy.dz$, as shown in figure 1.8. Choosing to formulate the first law at an instant of time, the second step is to consider the energy processes that are relevant to this control volume. In the absence of motion (or with uniform motion), there are no changes in mechanical energy and no work being done on the system. Only thermal forms of energy need be considered. Specifically, if there are temperature gradients, conduction heat transfer will occur across each of the control surfaces. The conduction heat rates perpendicular to each

of the control surfaces at the x -, y -, and z -coordinate locations are indicated by the terms q_x , q_y and q_z , respectively.

- ✓ The conduction heat rates at the opposite surfaces can then be expressed as a Taylor series expansion where, neglecting higher-order terms,

$$\left. \begin{aligned} q_{x+dx} &= q_x + \frac{\partial q_x}{\partial x} dx \\ q_{y+dy} &= q_y + \frac{\partial q_y}{\partial y} dy \\ q_{z+dz} &= q_z + \frac{\partial q_z}{\partial z} dz \end{aligned} \right\} \quad (1-11)$$

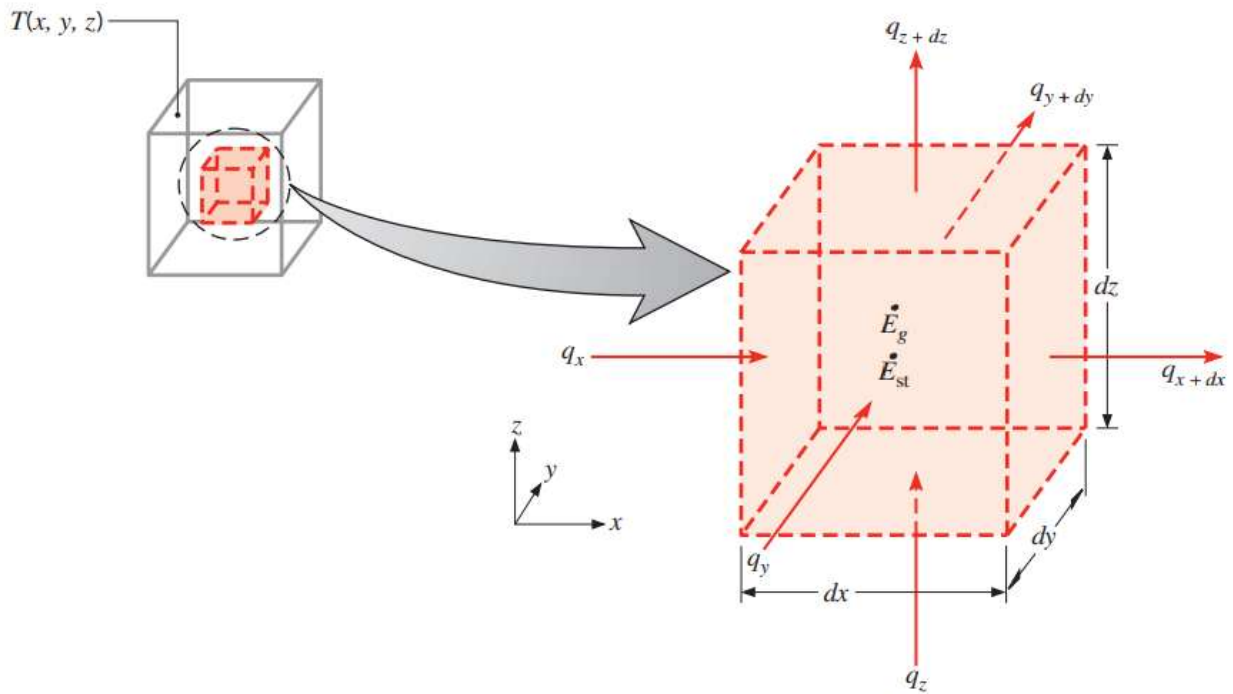


Fig. 1-8: Differential control volume, $dx dy dz$, for conduction analysis in Cartesian coordinates.

- ✓ Within the medium there may also be an energy source term associated with the rate of thermal energy generation. This term is represented as:

$$E_g = \dot{g} dx dy dz \quad (1-12)$$

Where \dot{g} is the rate at which energy is generated per unit volume of the medium (W/m^3).

- ✓ Changes may occur in the amount of the internal thermal energy stored by the material in the control volume and the energy storage term may be expressed as:

$$E_{st} = \rho C_p \frac{\partial T}{\partial t} dx dy dz \quad (1-13)$$

where $\rho C_p \frac{\partial T}{\partial t}$ is the time rate of change of the sensible (thermal) energy of the medium per unit volume.

On a rate basis, the general form of the conservation of energy requirement is:

$$E_{in} + E_g - E_{out} = E_{st} \quad (1-14)$$

Hence, recognizing that the conduction rates constitute the energy inflow E_{in} and outflow E_{out} , and substituting Equations 1.12 and 1.13 into Equation 1.14, we obtain,

$$q_x + q_y + q_z + \dot{g} dx dy dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho C_p \frac{\partial T}{\partial t} dx dy dz \quad (1-15)$$

The conduction heat rates in an isotropic material may be evaluated from Fourier's law,

$$\left. \begin{aligned} q_x &= -k dy dz \frac{\partial T}{\partial x} \\ q_y &= -k dx dz \frac{\partial T}{\partial y} \\ q_z &= -k dx dy \frac{\partial T}{\partial z} \end{aligned} \right\} \quad (1-16)$$

Substituting Equation 1.16 into Equation 1.15 and dividing out the dimensions of the control volume ($dx dy dz$), we obtain

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{g} = \rho C_p \frac{\partial T}{\partial t} \quad (1-17)$$

Equation 1.17 is the general form, in Cartesian coordinates, of the *heat diffusion* equation. This equation, often referred to as the heat equation, provides the basic tool for heat conduction analysis. Equation 1.17, therefore states that at any point

in the medium the *net rate of energy transfer by conduction* into a unit volume plus the *volumetric rate of thermal energy generation* must equal the *rate of change of thermal energy stored* within the volume.

If the **thermal conductivity is constant**, the heat equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1-18)$$

where $\alpha (= \frac{\text{Heat conducted}}{\text{Heat stored}} = k/\rho C_p)$ is the thermal diffusivity.

- ✓ The heat equation under constant the thermal conductivity and steady-state conditions is called **Poisson Equation** as,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = 0 \quad (1-18-a)$$

- ✓ The heat equation under constant the thermal conductivity and no heat generation is called **Diffusion Equation** as,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1-18-b)$$

- ✓ The heat equation under constant the thermal conductivity, no heat generation and steady-state conditions is called **Laplace Equation** as,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (1-18-c)$$

Under **steady-state conditions**, there can be no change in the amount of energy storage; hence Equation 1.17 reduces to

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = 0 \quad (1-19)$$

Moreover, if the heat transfer is *one-dimensional* (e.g., in the *x*-direction) and there is *no energy generation*, Equation 1.19 reduces to

$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0 \quad (1-20)$$

Cylindrical Coordinates

The heat equation may also be expressed in cylindrical coordinates. The differential control volume for this coordinate system is shown in Figure 1.9. In cylindrical coordinates, Fourier's law is

$$\left. \begin{aligned} q_r'' &= -k \frac{\partial T}{\partial r} \\ q_\phi'' &= -\frac{k}{r} \frac{\partial T}{\partial \phi} \\ q_z'' &= -k \frac{\partial T}{\partial z} \end{aligned} \right\} \quad (1-21)$$

Where, $x = r \cos\phi$, $y = r \sin\phi$, $z = z$

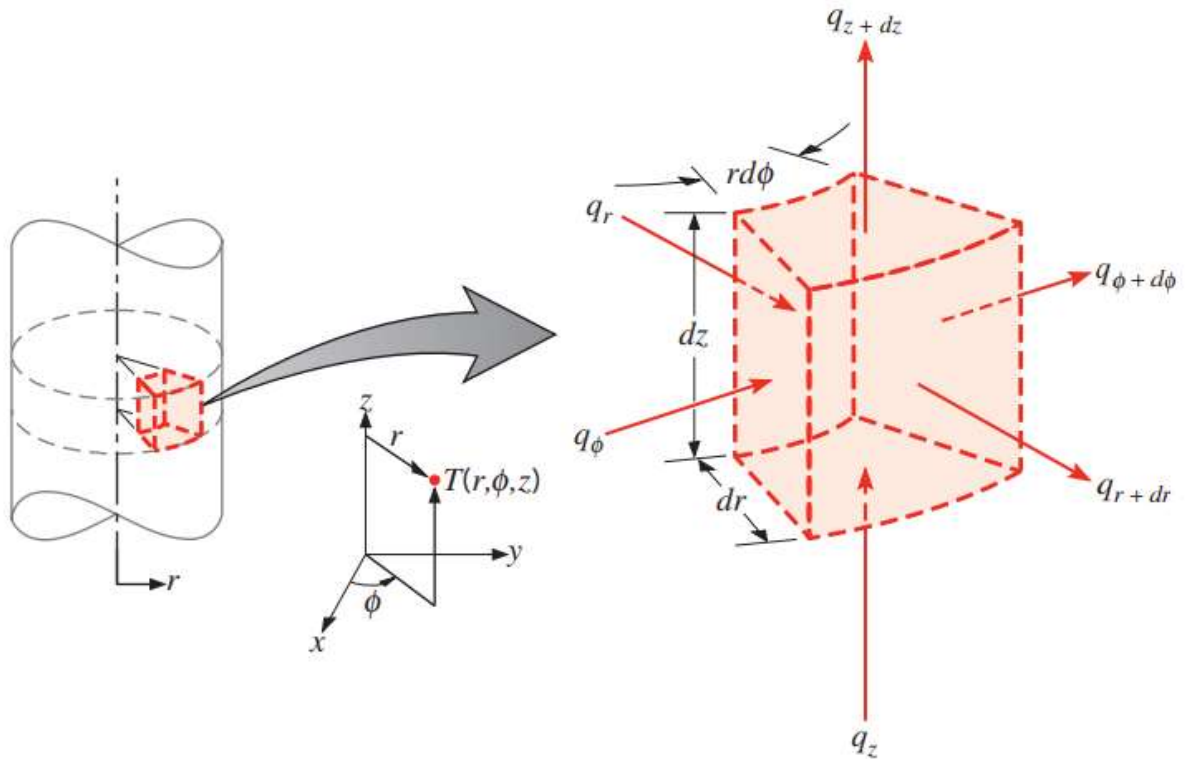


Fig. 1-9: Differential control volume, $dr \ r d \phi \ dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z) .

Applying an energy balance to the differential control volume of Figure 1.9, the following general form of the heat equation is obtained:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{g} = \rho C_p \frac{\partial T}{\partial t} \quad (1-22)$$

If the **thermal conductivity is constant**, the heat equation is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 T}{\partial \phi^2} \right) + \left(\frac{\partial^2 T}{\partial z^2} \right) + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1-23)$$

Spherical Coordinates

The heat equation may also be expressed in spherical coordinates. The differential control volume for this coordinate system is shown in Figure 1.10. In spherical coordinates, Fourier's law is

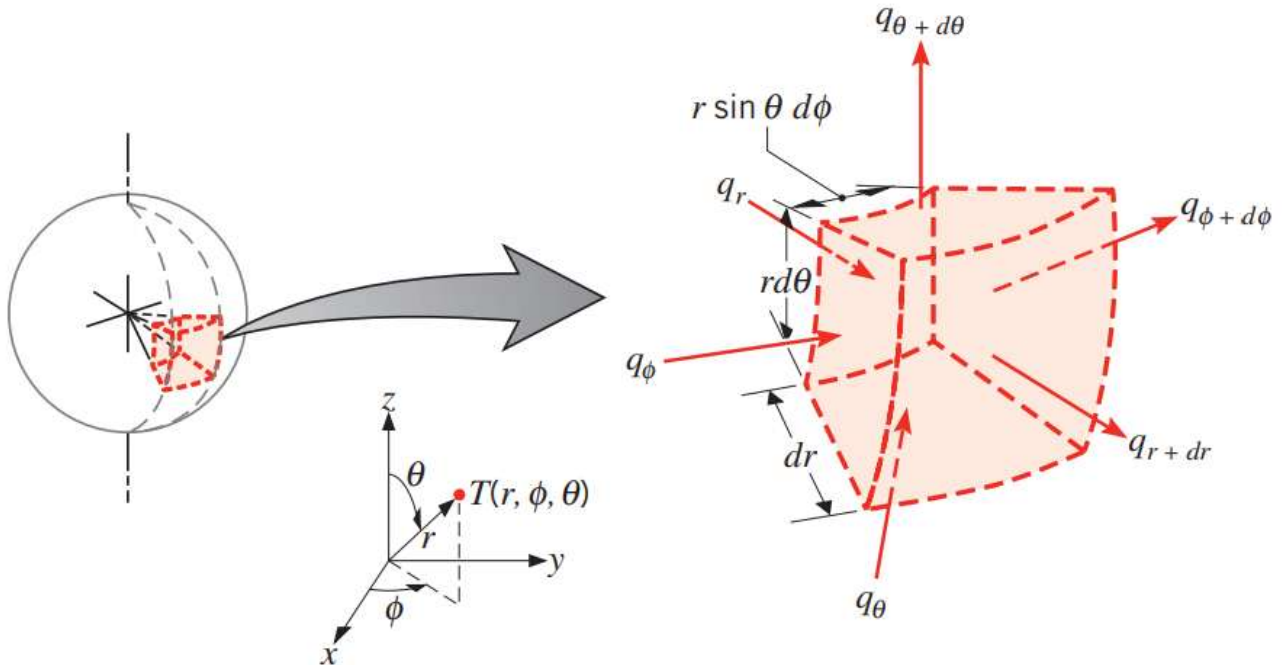


Fig. 1-10: Differential control volume, $dr \cdot r \sin\theta \, d\phi \cdot r d\theta$, for conduction analysis in spherical coordinates (r, θ, ϕ) .

$$\left. \begin{aligned} q_r'' &= -k \frac{\partial T}{\partial r} \\ q_\phi'' &= -\frac{k}{r \sin\theta} \frac{\partial T}{\partial \phi} \\ q_\theta'' &= -\frac{k}{r} \frac{\partial T}{\partial \theta} \end{aligned} \right\} \quad (1-24)$$

Where, $x = r \cos\phi \sin\theta$, $y = r \sin\phi \sin\theta$, $z = r \cos\theta$

Applying an energy balance to the differential control volume of Figure 1.10, the following general form of the heat equation is obtained:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(kr^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{g} = \rho C_p \frac{\partial T}{\partial t} \quad (1-25)$$

If the **thermal conductivity is constant**, the heat equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{\dot{g}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1-26)$$

Exercise 1:

Derive the general 3D heat conduction equation through isotropic media in cylindrical and spherical coordinates using: Coordinate transformation and Energy balance for a finite volume element.

1.4 Boundary and Initial Conditions

1.4.1 Boundary (surface) conditions:

The most frequently encountered boundary conditions in conduction are as follows,

A. Prescribed temperature

The surface temperature of the boundaries is specified to be a constant or a function of space and/or time.

B. Prescribed heat flux

The heat flux across the boundaries is specified to be a constant or a function of space and/or time. The mathematical description of this condition may be given in the light of Kirchhoff's current law; that is, the algebraic sum of heat fluxes at a boundary must be equal to zero. Hereafter the sign is to be assumed positive for the heat flux to the boundary and negative for that from the boundary. Thus, remembering that the statement of Fourier's law, $q_n = -k \frac{\partial T}{\partial n}$, is independent of

the actual temperature distribution, and selecting the direction of q_n , conveniently such that it becomes positive, we have from Figure 1.11.

$$\pm k \frac{\partial T}{\partial n} = \pm q'' \tag{1-27}$$

where $\partial/\partial n$ denotes differentiation along the normal of the boundary. The plus and minus signs of the left-hand side of Equation (1.27) correspond to the differentiations along the inward and outward normals, respectively, and the plus and minus signs of the right-hand side correspond to the heat flux from and to the boundary, respectively.

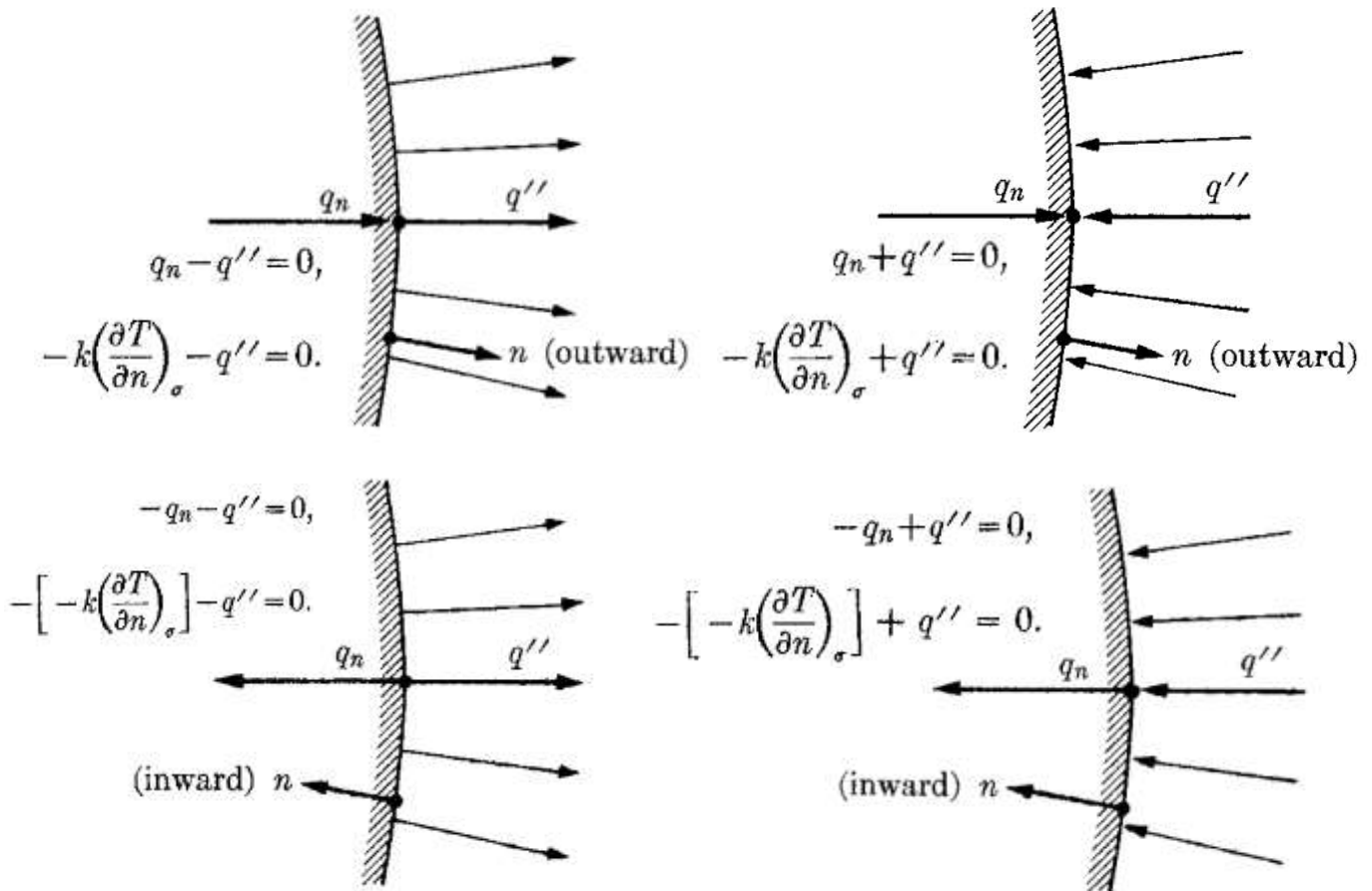


Fig. 1-11: Prescribed surface heat flux boundary conditions.

C. No heat flux (insulation)

This, prescribed is a special form of the previous case, obtained by inserting $q'' = 0$ into Equation (1-27).

$$\frac{\partial T}{\partial n} = 0 \tag{1-28}$$

D. Heat transfer to the ambient by convection

When the heat transfer across the boundaries of a continuum cannot be prescribed, it may be assumed to be proportional to the temperature difference between the boundaries and the ambient. Thus we have

$$q_{conv} = h (T - T_{\infty}) \tag{1-29}$$

where T is the temperature of the solid boundaries, T_{∞} , is the temperature of the ambient at a distance far from the boundaries, and h , the proportionality constant, is the so-called heat transfer coefficient. Equation (1.29) is Newton's cooling law.

The required boundary condition may be stated in the form

$$\pm k \frac{\partial T}{\partial n} = h (T - T_{\infty}) \tag{1-30}$$

Where $\partial/\partial n$ denotes the differentiation along the normal. The plus and minus signs of the left member of Equation (1.30) correspond to the differentiations along the inward and outward normals, respectively (Figure 1.12). It should be kept in mind that q , shown in Figure 1.12 is a positive quantity, obtained by arbitrarily selecting it in the direction of the normal. Actually, Equation (1.30) is independent of the temperature distribution and the direction of the heat transfer.

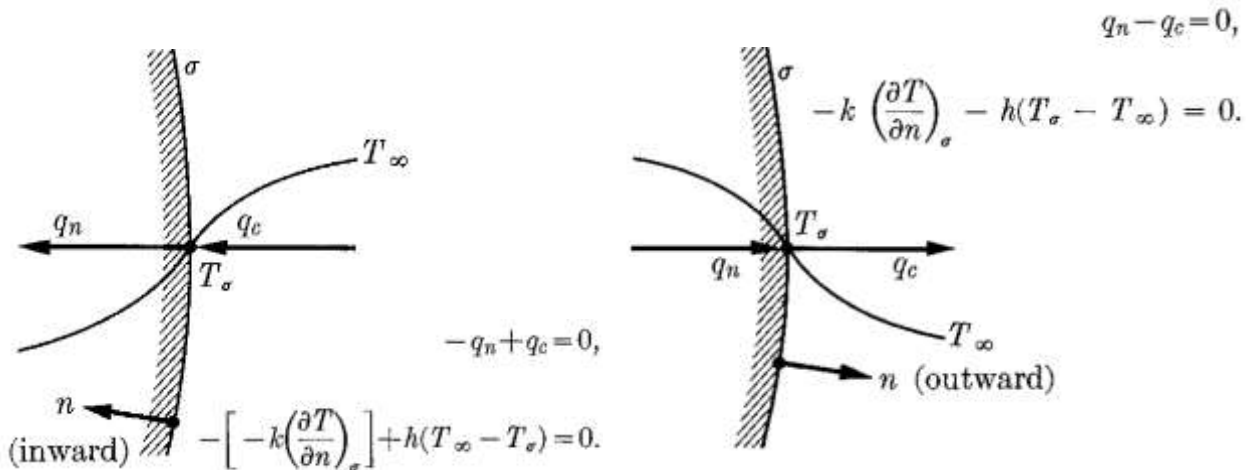


Fig. 1-12: Heat transfer to the ambient by convection surface heat flux boundary conditions.

E. Heat transfer to the ambient by radiation

The boundary condition prescribing heat transfer by radiation from the boundaries of continuum 1. When T_1 is uniform but unspecified, to express the heat flux across the surfaces of 1 by conduction and radiation the required boundary condition may be written in the form

$$\pm k \frac{\partial T}{\partial n} = h_{1-2} \sigma (T_1^4 - T_2^4) \tag{1-31}$$

F. Prescribed heat flux acting at a distance

Consider a continuum that transfers heat to the ambient by convection while receiving the net radiant, heat flux q'' from a distant source (Figure 1.13). The heat transfer coefficient is h , and the ambient temperature T_∞ . This boundary condition may be readily obtained as

$$\pm k \frac{\partial T}{\partial n} + q'' = h_{1-2} \sigma (T_1^4 - T_2^4) \tag{1-32}$$

where the signs of the conduction term depend on the direction of normal in the usual manner.

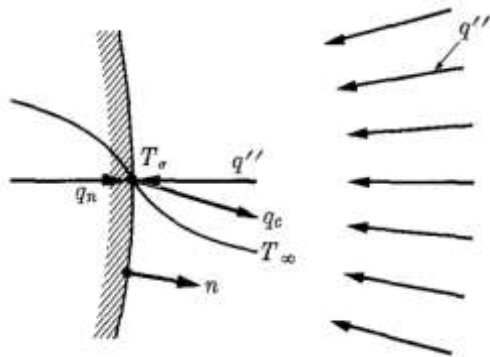


Fig. 1-13: Prescribed heat flux acting at a distance

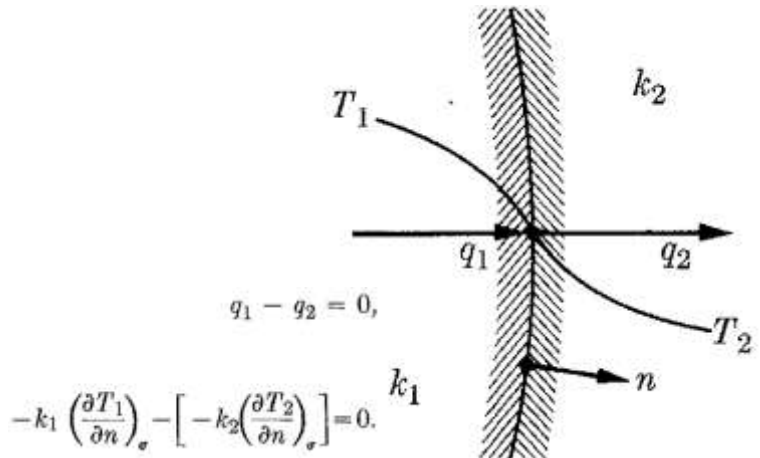


Fig. 1-14: Heat Interface of two continua of different conductivities k_1 and k_2

G. Interface of two continua of different conductivities k_1 and k_2

When two continua have a common boundary (Figure 1.14), the heat flux across this boundary evaluated from both continua, regardless of the direction of normal, gives

$$k_1 \frac{\partial T_1}{\partial n} = k_2 \frac{\partial T_2}{\partial n} \tag{1-33}$$

H. Interface of two continua in relative motion

Consider two solid continua in contact, one moving relative to the other (Fig. 1.15). The local pressure on the common boundary is p , the coefficient of dry friction μ , and the relative velocity V . Noting that the heat transfer to both continua by conduction is equal to the work done by friction, we have

$$\pm k_1 \left(\frac{\partial T}{\partial n} \right) + \mu p V = \pm k_2 \frac{\partial T_2}{\partial n} \tag{1-34}$$

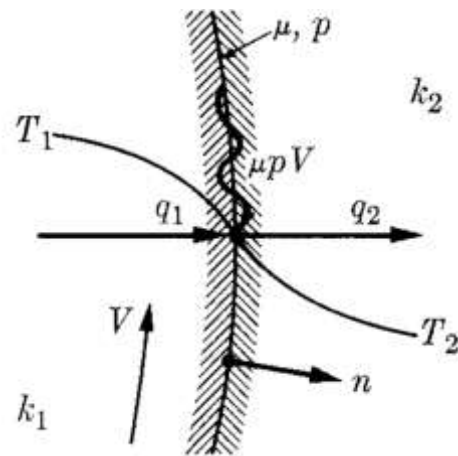


Fig. 1-15: Interface of two continua in relative motion

where the minus signs of the conduction terms correspond to the normal shown in Figure (1.15).

1.4.2 Initial (volume) condition:

For an unsteady problem the temperature of a continuum under consideration must be known at some instant of time. In many cases this instant is most conveniently taken to be the beginning of the problem. Mathematically speaking, if the initial condition is given by $T_o(r)$, the solution of this problem, $T(r, t)$, must be such that at all points of the continuum

$$\lim_{t \rightarrow 0} T(r, t) = T_o \tag{1-35}$$

1.5 Methods of investigation and formulation

Four methods are usually used in conduction problems, these are;

1. Analytical Methods
2. Methods of Analogy
3. Computational Methods
4. Graphical Methods

1.5.1 Analytical Methods

In these methods, a number of assumptions are made to simplify the governing equations and get a solution from them. Analytical solution tends to be lengthy and difficult.

1.5.2 Methods of Analogy

A number of lumped distributed models for conduction problems are available based on mechanical, hydrodynamic, and electrical systems. Networks of electrical resistors, capacitors, and sometimes inductors are the most important simulators of lumped systems; on rare occasions, mechanical simulators systems comprised of masses, springs, and dashpots are also used for this purpose. Electrolytic tanks, conductive papers, stretched membranes; soap film, fluid mappers, and polarized light are some of the distributed models occasionally used.

The direct mathematical similarity between heat and electrical conduction is by far the best known and most widely used analogy for the study of complex problems in both steady and transient heat conduction. The characteristic PDE governing the transient distribution of electric potential (electromotive force) E in an electrically-conducting 2-D region of uniform electrical resistance per unit length ($R_L = \frac{R}{L}$) and uniform electrical capacity per unit length ($C_L = \frac{C}{L}$);

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = R_L C_L \frac{\partial T}{\partial t} \quad (1-36)$$

with the familiar characteristic PDE governing the transient distribution of thermal potential (temperature) T in a thermally conducting 2-D region of uniform diffusivity (α).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{1-37}$$

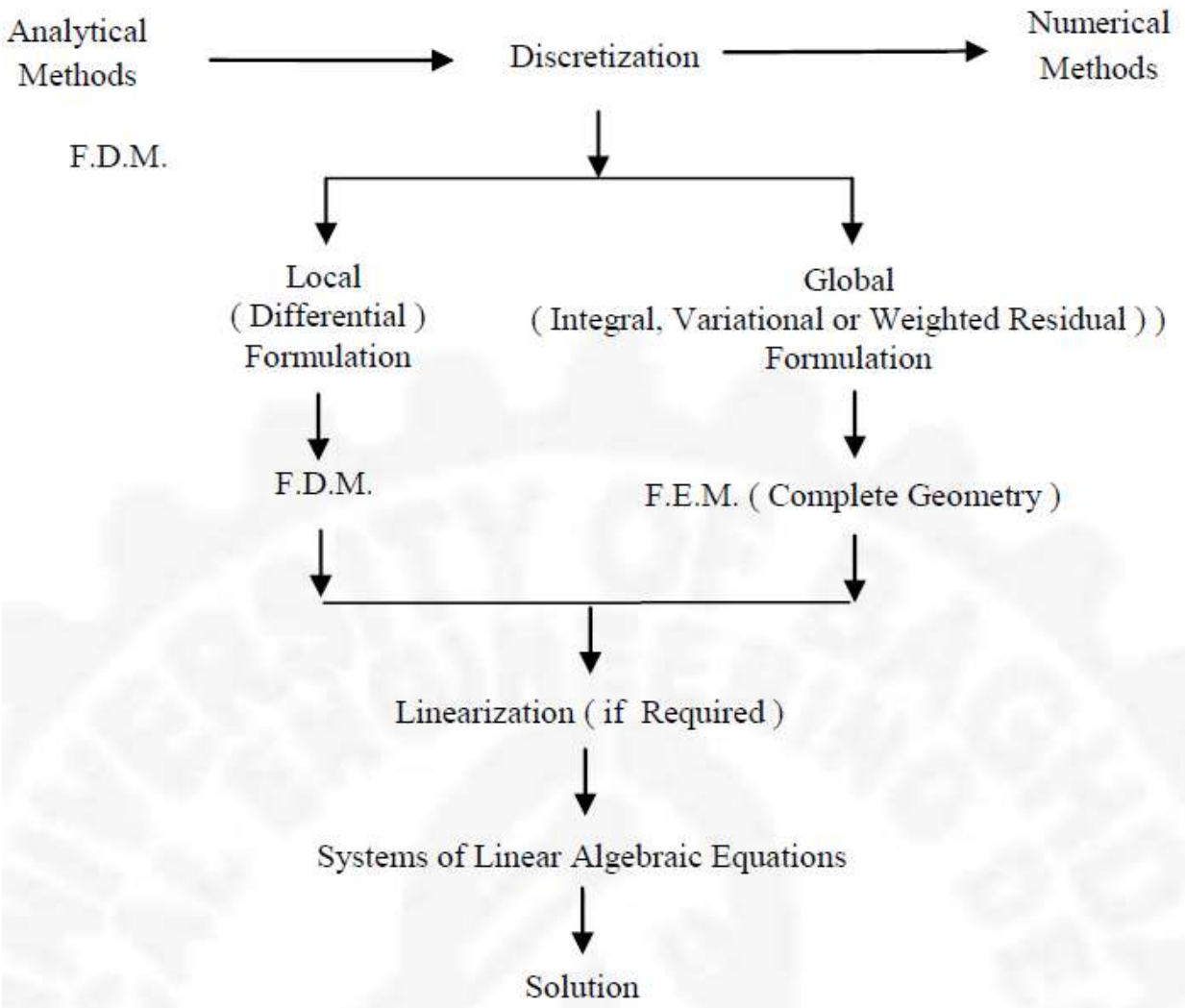
According to previous notations, t represents time. The transient state analogy between electric and temperature potential is therefore complete if on the same time scale the electrical diffusivity ($1/R_L C_L$) and thermal diffusivity (α) are equal. In this state, there is a direct analogy between two laws, the conservation of charge in the electrical system corresponds to the conservation of heat in the thermal system, and the current flow in the electrical system (*Ohm's law*) corresponds to heat flow in the thermal system (*Fourier's law*). The complete electrical thermal analogy is summarized in Table (1.2).

Table 1.2: Analogues Electrical-Thermal Quantities

Electrical	Thermal
Charge = Q_e (coulomb)	Heat = Q (J , Btu)
Voltage = E (volt)	Temperature = T (C° , F°)
Resistance = R (Ohm)	Resistance $R \left(\frac{o_c}{W} \right) = \frac{L}{KA} = \frac{1}{hA}$
Current = I (Ampere) = $\frac{\Delta E}{R}$	Flow = q (J/h_r) = $\frac{\Delta T}{R}$
Capacitance = C (Farad) = $\frac{I}{\frac{dE}{dt}}$	Unit Capacity = $C_P \rho V$ ($J/^\circ c$) = $\frac{q}{dT/dt}$

1.5.3 Computational Methods

Basically, numerical methods are discretization of analytical methods. By this discretization, the local (differential) formulations leads to a finite difference formulation, while the global (integral, variational, or any other methods of weighted residual) formulation leads to finite element formulation. Both numerical methods lead, after linearization if required, to the solution of systems of linear algebraic equations.



1.5.4 Graphical Methods

The graphic method presented in this section can rapidly yield a reasonably good estimate of the temperature distribution and heat flow in geometrically complex two-dimensional systems, but its application is limited to problems with isothermal and insulated boundaries. The object of a graphic solution is to construct a network consisting of isotherms (lines of constant temperature) and constant-flux lines (lines of constant heat flow). The flux lines are analogous to streamlines in a potential fluid flow, that is, they are tangent to the direction of heat flow at any point. Consequently, no heat can flow across the constant-flux lines. The isotherms are analogous to constant-potential lines, and heat flows perpendicular to them. Thus, lines of constant temperature and lines of constant heat flux intersect at right angles. To obtain the temperature distribution one first prepares a scale model and then draws isotherms and flux lines freehand, by trial and error, until they form a network of curvilinear squares. Then a constant amount of heat flows between any two flux lines. The procedure is illustrated in Figure 1.16 for a corner section of unit depth ($\Delta z = 1$) with faces ABC at temperature T_1 , faces FED at temperature T_2 , and faces CD and AF insulated. Figure 1.16 (a) shows the scale model, and Figure 1.16 (b) shows the curvilinear network of isotherms and flux lines. It should be noted that the flux lines emanating from isothermal boundaries are perpendicular to the boundary, except when they come from a corner. Flux lines leading to or from a corner of an isothermal boundary bisect the angle between the surfaces forming the corner.

A graphic solution, like an analytic solution of a heat conduction problem described by the Laplace equation and the associated boundary condition, is unique. Therefore, any curvilinear network, irrespective of the size of the squares, that satisfies the boundary conditions represents the correct solution. For any curvilinear square the rate of heat flow is given by Fourier's law:

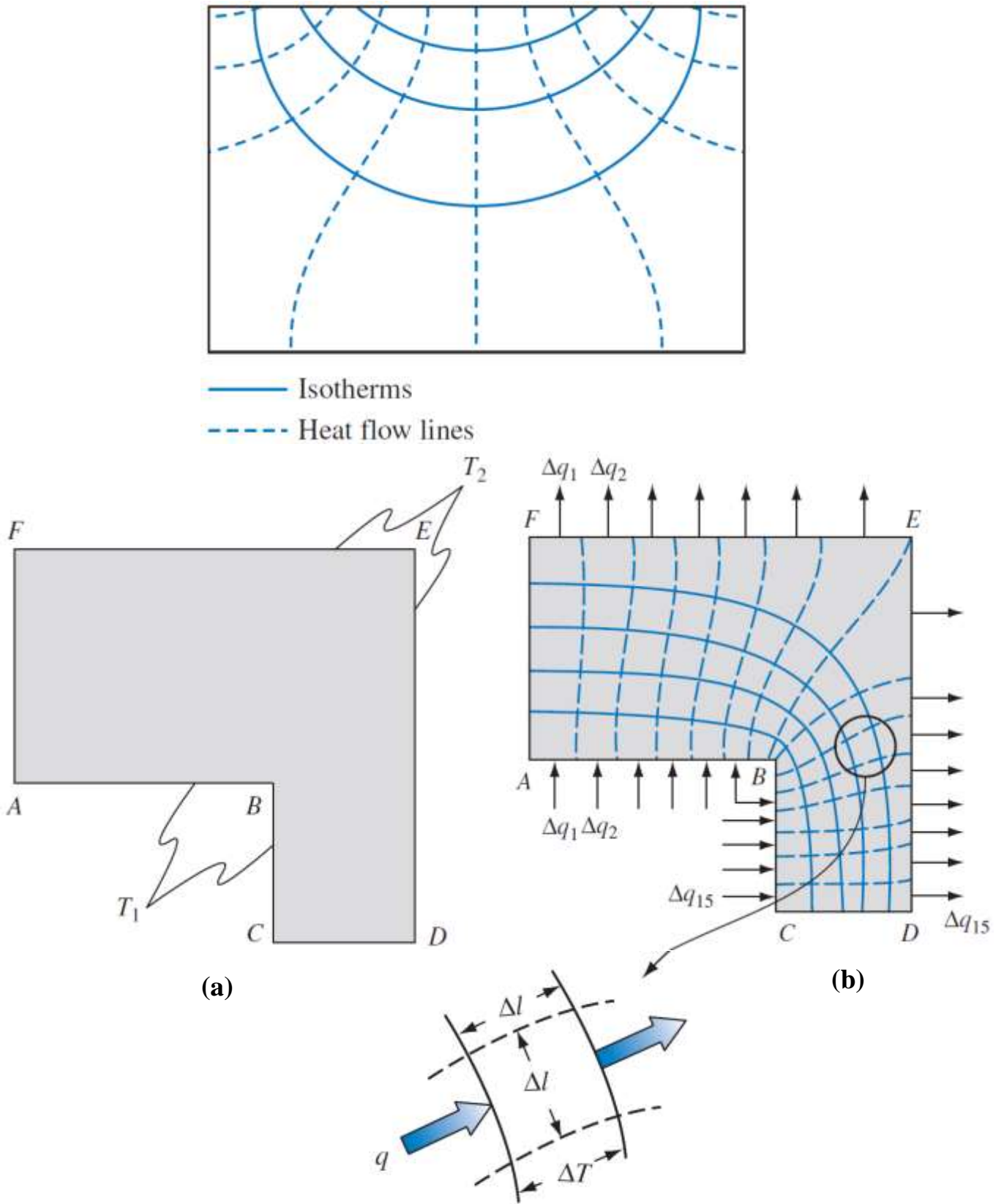


Fig. 1-16: Interface of two continua in relative motion

$$\Delta q = -k(\Delta l \times 1) \frac{\Delta T}{\Delta l} = -k\Delta T \quad (1-38)$$

This heat flow will remain the same across any square within any one heat flow lane from the boundary at T_1 to the boundary at T_2 . The ΔT across any one element in the heat flow lane is therefore

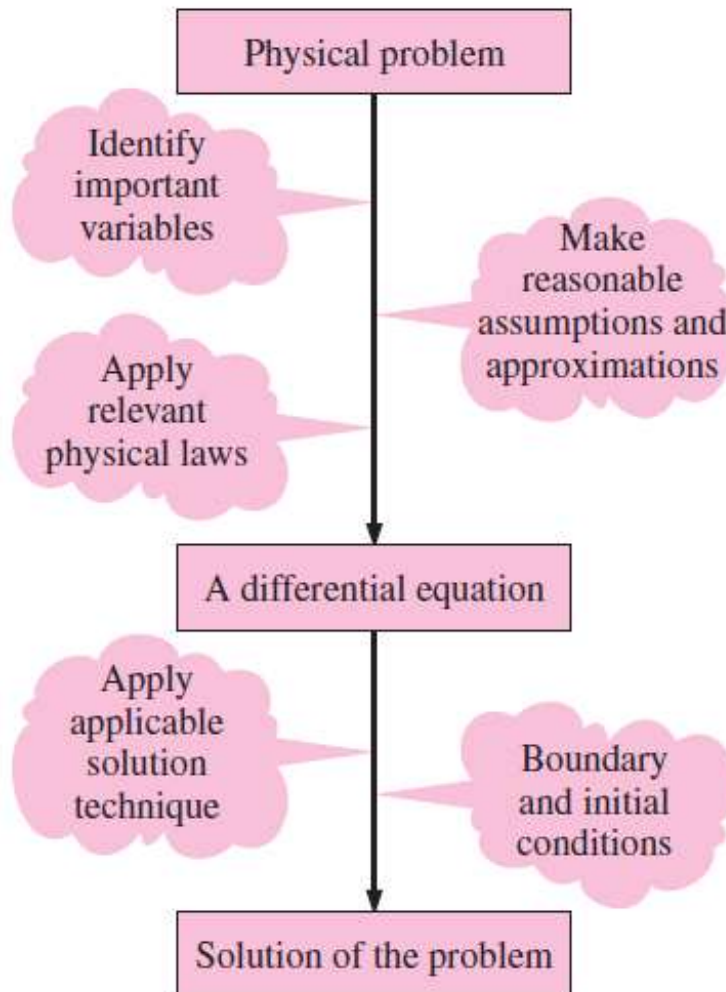
$$\Delta T = \frac{T_2 - T_1}{N} \quad (1-39)$$

where N is the number of temperature increments between the two boundaries at T_1 and T_2 . The total rate of heat flow from the boundary at T_2 to the boundary at T_1 equals the sum of the heat flow through all the lanes. According to the above relations, the heat flow rate is the same through all lanes since it is independent of the size of the squares in a network of curvilinear squares. The total rate of heat transfer can therefore be written

$$q = \sum_{n=1}^{n=M} \Delta q_n = \frac{M}{N} k(T_2 \times T_1) = \frac{M}{N} k\Delta T_{overall} \quad (1-40)$$

where Δq_n is the rate of heat flow through the n^{th} lane, and M is the number of heat flow lanes.

1.6 Mathematical modeling of physical problems



1.7 Homework:

(1) Derive the general three-dimensional conduction heat transfer equation for isotropic heterogeneous medium in cylindrical and spherical coordinates, using energy balance for a finite volume element; obtain the solution for isotropic homogeneous medium ($\alpha = \frac{k}{\rho C}$).

(2) Write down the equation of conduction for the following media in Cartesian coordinates;

a- Heterogeneous anisotropic solids

b- Homogeneous anisotropic solids

c- Heterogeneous isotropic solids

d- Homogeneous isotropic solids

(3) Write down the vectorial and Cartesian forms of the Fourier's law of conduction for heterogeneous anisotropic continua.

(4) What are the most frequently encountered boundary conditions in conduction heat transfer problems? Express these boundary conditions mathematically and mention one application for each boundary condition.

(5) What are the basic modes of heat transfer? And what are the important differences between diffusion and radiation heat transfer?