University of Anbar
College of Engineering
Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Three
Heat Transfer from Extended Surfaces

Course Tutor

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Chapter Three
Heat Transfer from Extended Surfaces

3.1 Introduction

The term "*extended surface*" is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid.

Consider first a wall at temperature T_w transferring heat by convection to an ambient at temperature T_∞. Therefore, the rate of heat transfer from this wall may be evaluated in terms of a heat transfer coefficient in the form,

$$q_{conv.} = hA(T_w - T_\infty) \qquad (3\text{-}1)$$

Clearly, $q_{conv.}$ of Eq. (3.1) may be increased by increasing:

(***i***) The temperature difference between the wall and the ambient.

(***ii***) The heat transfer coefficient.

(***iii***) The heat transfer area.

The first case needs no explanation; the second case is the subject matter of texts on convective heat transfer; the third case is the concern of this section.

Examples of extended surfaces (fins) applications are easy to find around us. Consider the arrangement for cooling engine heads on motorcycles and lawn mowers or for cooling electric power transformers (see Figure 3.1). Consider also the tubes with attached fins used to promote heat exchange between air and the working fluid of an air conditioner. Two common finned-tube arrangements are shown in Figure 3.1.

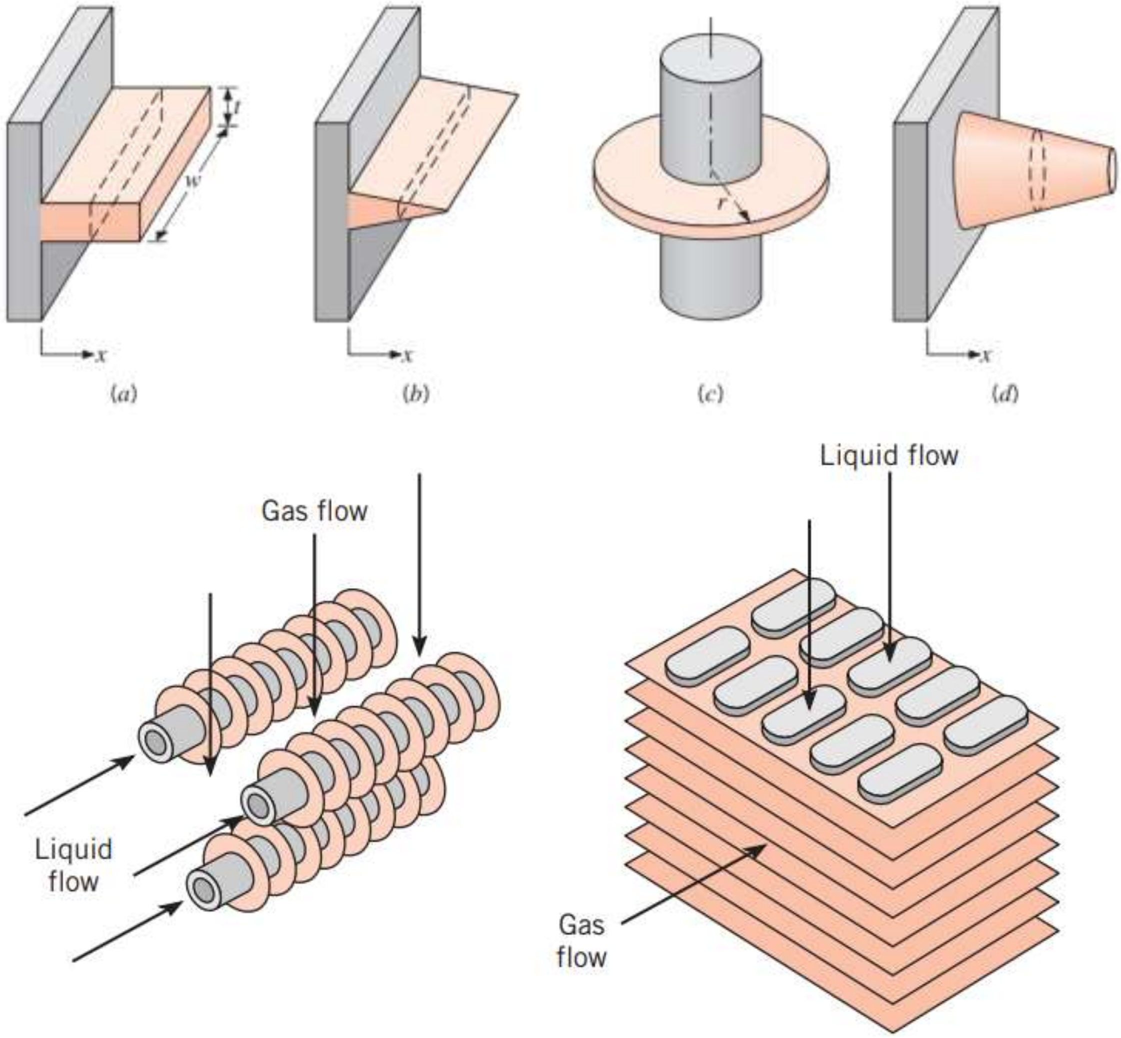


Figure 3.1: Fin configurations. (a) Straight fin of uniform cross section. (b) Straight fin of non-uniform cross section. (c) Annular fin. (d) Pin fin, and Schematic of typical finned-tube heat exchangers.

3.2 Analysis of extended surfaces (fins)

Consider the extended surface of Figure 3.2. The analysis is simplified if certain assumptions are made. We choose to assume one-dimensional conditions in the longitudinal (x-) direction, even though conduction within the fin is actually two-dimensional. The rate at which energy is convected to the fluid from any point on the fin surface must be balanced by the net rate at which energy reaches that point due to conduction in the transverse (y-, z-) direction. However, in practice the fin is thin, and temperature changes in the transverse direction within the fin are small compared with the temperature difference between the fin and the environment. Hence, we may assume that the temperature is uniform across the fin thickness, that is, it is only a function of x.

We will consider steady-state conditions and also assume that the thermal conductivity is constant, that radiation from the surface is negligible, that heat generation effects are absent, and that the convection heat transfer coefficient h is uniform over the surface. Steady state (T , $q \neq f\,(t)$). Homogeneous material. Uniform free stream temperature (T_∞). Uniform base temperature. Negligible contact resistance. No heat generation. By applying the conservation of energy requirement,

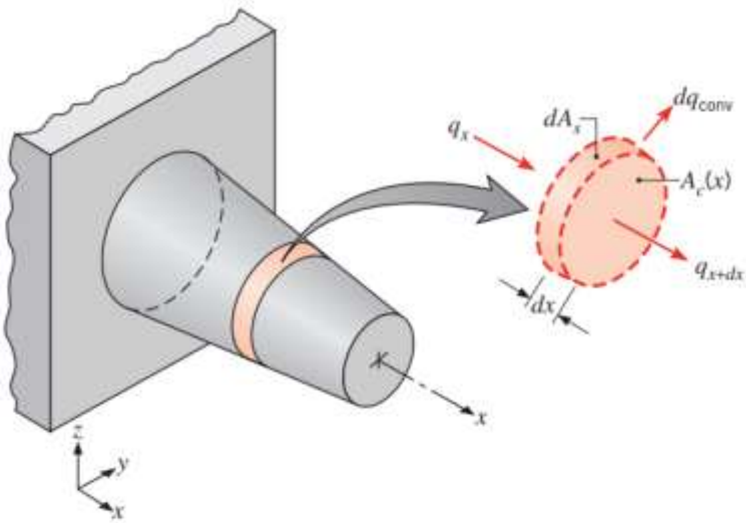


Figure 3.2: Energy balance for an extended surface.

$$q_x = q_{x+dx} + dq_{conv.} \qquad (3\text{-}2)$$

$$-kA_c\frac{dT}{dx} = -kA_c\frac{dT}{dx} - k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx + h\,dA_s(T - T_\infty) \qquad (3\text{-}3)$$

where A_c is the cross-sectional area, which may vary with x. dA_s is the surface area of the differential element.

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{h\,dA_s}{k\,dx}(T - T_\infty) = 0 \qquad (3\text{-}4)$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h\,dA_s}{k\,dx}\right)(T - T_\infty) = 0 \qquad (3\text{-}5)$$

This result provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used to calculate the conduction rate at any x.

3.2.1 Extended surfaces with constant cross sections

To solve Equation 3.5 it is necessary to be more specific about the geometry. We begin with constant (uniform) cross-sectional area such as the simplest case of straight rectangular and pin fins (see Figure 3.1 and Figure 3.3). Each fin is attached to a base surface of temperature $T_{(x=0)} = T_b$ and extends into a fluid of temperature T_∞.

For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to x and P is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, Equation 3.5 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{k\,A_c}(T - T_\infty) = 0 \qquad (3\text{-}6)$$

To simplify the form of this equation, we transform the dependent variable by defining an excess temperature θ as,

$$\theta_{(x)} = T_{(x)} - T_\infty \qquad (3\text{-}7)$$

where, since T_∞ is a constant, $d\theta/dx = dT/dx$, and $m^2 = (hP/kA_c)$, Substituting Equation 3.7 into Equation 3.6, we then obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (3\text{-}8)$$

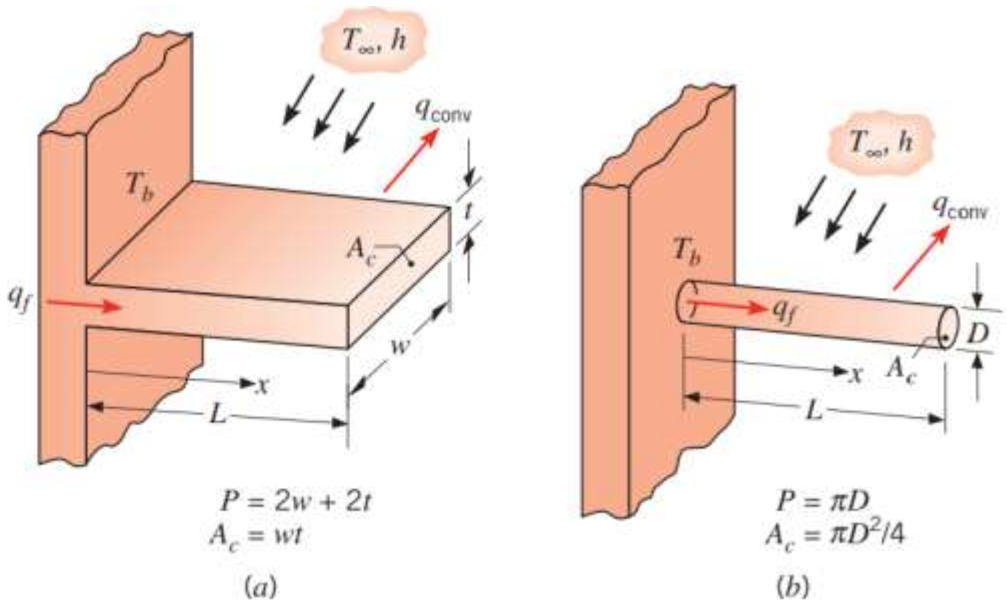


Figure 3.3: Straight fins of uniform cross section. (a) Rectangular fin. (b) Pin fin.

Equation 3.8 is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta_{(x)} = C_1 e^{mx} + C_1 e^{-mx} \tag{3-9}$$

Or, $\theta_{(x)} = C_3 \cosh mx + C_4 \sinh mx$ (3-10)

To evaluate the constants C_1 and C_2 of Equation 3.9 and C_3 and C_4 of Equation 3.10, it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the ***base*** of the fin (x= 0),

$$\theta_{(x=0)} = \theta_b = T_b - T_\infty \tag{3-11}$$

The second condition, specified at the fin ***tip*** (x = L), may correspond to one of four different physical situations.

<u>**Case A**</u>: considers *convection heat transfer from the fin tip*. Applying an energy balance to a control surface about this tip (Figure 3.4), we obtain

$$hA_c\left(T_{(x=L)} - T_\infty\right) = -kA_c \frac{dT}{dx}\bigg|_{x=L} \qquad \text{or} \qquad h\theta_{(L)} = -k\frac{d\theta}{dx}\bigg|_{x=L}$$

Solving for C_1 and C_2, it may be shown, after some manipulation, that

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$$

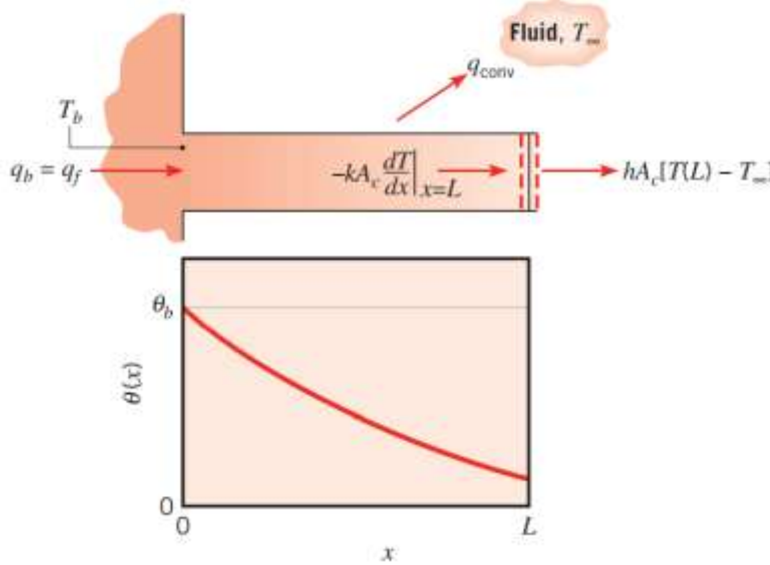


Figure 3.4: Conduction and convection in a fin of uniform cross section.

The form of this temperature distribution is shown schematically in Figure 3.4. Note that the magnitude of the temperature gradient decreases with increasing x. This trend is a consequence of the reduction in the conduction heat transfer $q_x(x)$ with increasing x due to continuous convection losses from the fin surface. We are particularly interested in the amount of heat transferred from the entire fin. From Figure 3.4, it is evident that the fin heat transfer rate q_f may be evaluated in two alternative ways, both of which involve use of the temperature distribution. The simpler procedure, and the one that we will use, involves applying *Fourier's law* at the fin base. That is,

$$q_f = q_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0} \qquad (3\text{-}13)$$

Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+ (h/mk)\ \sinh mL} \qquad (3\text{-}14)$$

Case B: corresponds to the assumption that the convective heat loss from the fin tip is negligible, in which case *the tip may be treated as adiabatic* and

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

Using this boundary condition to solve for C_1 and C_2 and substituting the results into Equation 3.9, we obtain

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL} \qquad (3\text{-}15)$$

Using this temperature distribution with Equation 3.13, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\theta_b \tanh mL \qquad (3\text{-}16)$$

Case C: where *the temperature is prescribed at the fin tip*. That is, the second boundary condition is $\theta_{(L)} = \theta_L$, and the resulting expressions are of the form

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL} \qquad (3\text{-}17)$$

Using this temperature distribution with Equation 3.13, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\theta_b \, \frac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL} \qquad (3\text{-}18)$$

Case D: is an interesting extension of these results. In particular, as *Infinite fin* ($L \to \infty$, $\theta_{(L)} \to 0$) and it is easily verified that

$$\frac{\theta}{\theta_b} = e^{-mx} \qquad (3\text{-}19)$$

$$q_f = \sqrt{hPkA_c}\theta_b \qquad (3\text{-}20)$$

Homework 1:

Prove the following Equations,

Equation 3.12, Equation 3.14, Equation 3.15, Equation 3.16, Equation 3.17, Equation 3.18, Equation 3.19, Equation 3.20.

Examples:

Q1: A metal rod of length $2L$, diameter D, and thermal conductivity k is inserted into a perfectly insulating wall, exposing one-half of its length to an airstream that is of temperature T_∞ and provides a convection coefficient h at the surface of the rod as shown in Figure 3.5. An electromagnetic field induces volumetric energy generation at a uniform rate $\dot{g}$ within the embedded portion of the rod.

(**a**) Derive an expression for the steady-state temperature T_b at the base of the exposed half of the rod. The exposed region may be approximated as a very long fin.

(**b**) Derive an expression for the steady-state temperature T_o at the end of the embedded half of the rod.

(**c**) Using numerical values provided in the schematic, plot the temperature distribution in the rod and describe key features of the distribution. Does the rod behave as a very long fin?

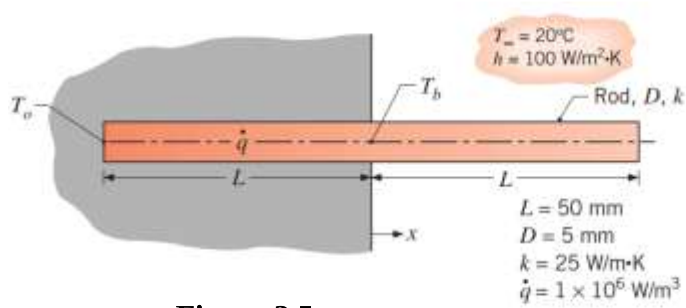


Figure 3.5

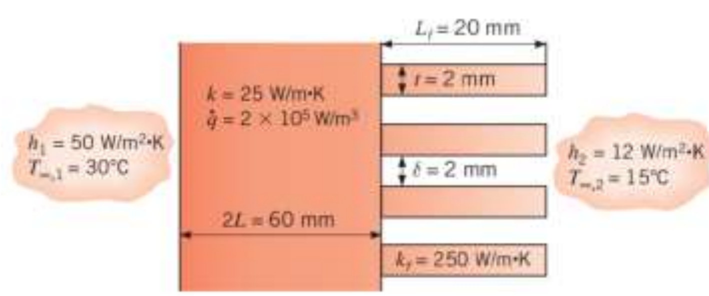


Figure 3.6

Q2: Heat is uniformly generated at the rate of 2×10^5 W/m^3 in a wall of thermal conductivity 25 W/m K and thickness 60 mm. The wall is exposed to convection on both sides, with different heat transfer coefficients and temperatures as shown in Figure 3.6. There are straight rectangular fins on the right-hand side of the wall, with dimensions as shown and thermal conductivity of 250 W/mK. What is the maximum temperature that will occur in the wall?

Q3: A constant-area fin between surfaces at temperatures T_1 and T_2 is shown in Figure 3.7. If the external temperature, $T_\infty(x)$, is a function of the coordinate x, find the general steady-state solution of the fin temperature $T(x)$ for (**a**) $T_1 = T_2$ and (**b**) $T_1 \neq T_2$.

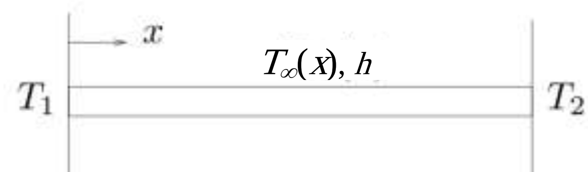


Figure 3.7

3.2.2 Bessel Functions

In section 3.2.3, a class of one-dimensional problems associated with extended surfaces (fins, pins, or spines) will be discussed. When the cross section of an extended surface is variable, the formulation of the problem results in a *second-order* linear differential equation with variable coefficients. This differential equation is a form of *Bessel's equation*, except in a special case which leads to the so-called *equidimensional equation*. The solution methods suitable to *second-order* linear differential equations with constant coefficients are not suitable to those with variable coefficients. We may, however, recall that equations with variable coefficients possess solutions expressible, over an appropriate interval, in terms of power series. This section is therefore devoted to a brief review of the power series solution of *Bessel's equation*.

The general *Bessel's equation* is,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

The solution of *Bessel's equation*:

$$y = X^A e^{BX}[C_1 J_n(DX^C) + C_2 Y_n(DX^C)] \qquad \text{(when } D \text{ is real)}$$

$$y = X^A e^{BX}[C_1 I_n(DX^C) + C_2 K_n(DX^C)] \qquad \text{(when } D \text{ is imaginary)}$$

Where:

J_n : Ordinary *Bessel function* of first kind of order *n*.

Y_n : Ordinary *Bessel function* of second kind of order *n*.

I_n : Modified *Bessel function* of first kind of order *n*.

K_n : Modified *Bessel function* of second kind of order *n*.

Ordinary Bessel Function:

$J_o(x), J_1(x)$: Ordinary *Bessel function* of first kind of order, zero and first order respectively.

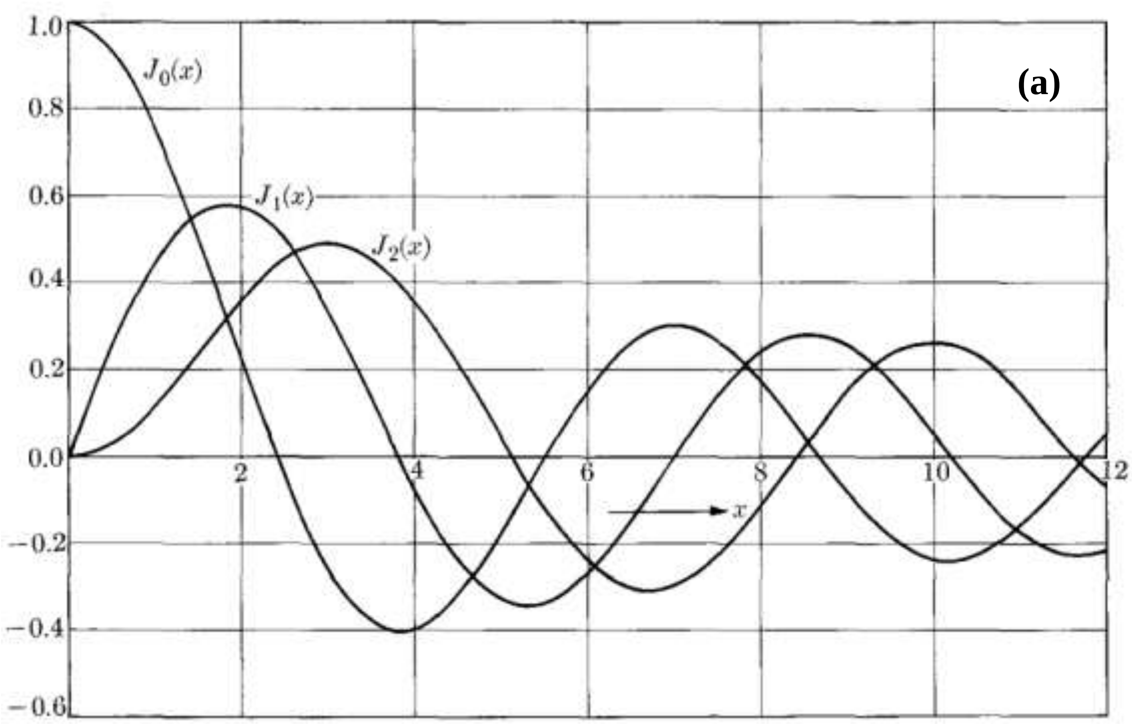
$Y_o(x), Y_1(x)$: Ordinary *Bessel function* of second kind of order, zero and first order respectively.

Modified Bessel function:

$I_o(x), I_1(x)$: *Modified Bessel function* of first kind of order, zero and first order respectively.

$K_o(x), K_1(x)$: *Modified Bessel function* of second kind of order, zero and first order respectively.

Graphical representation of the general behavior of *Bessel functions*. Graphs of the general behavior of *Bessel functions* are shown in Figure 3.8. Having thus completed our review of *Bessel functions* we may now proceed to demonstrate the use of these functions in the solution of problems related to extended surfaces.



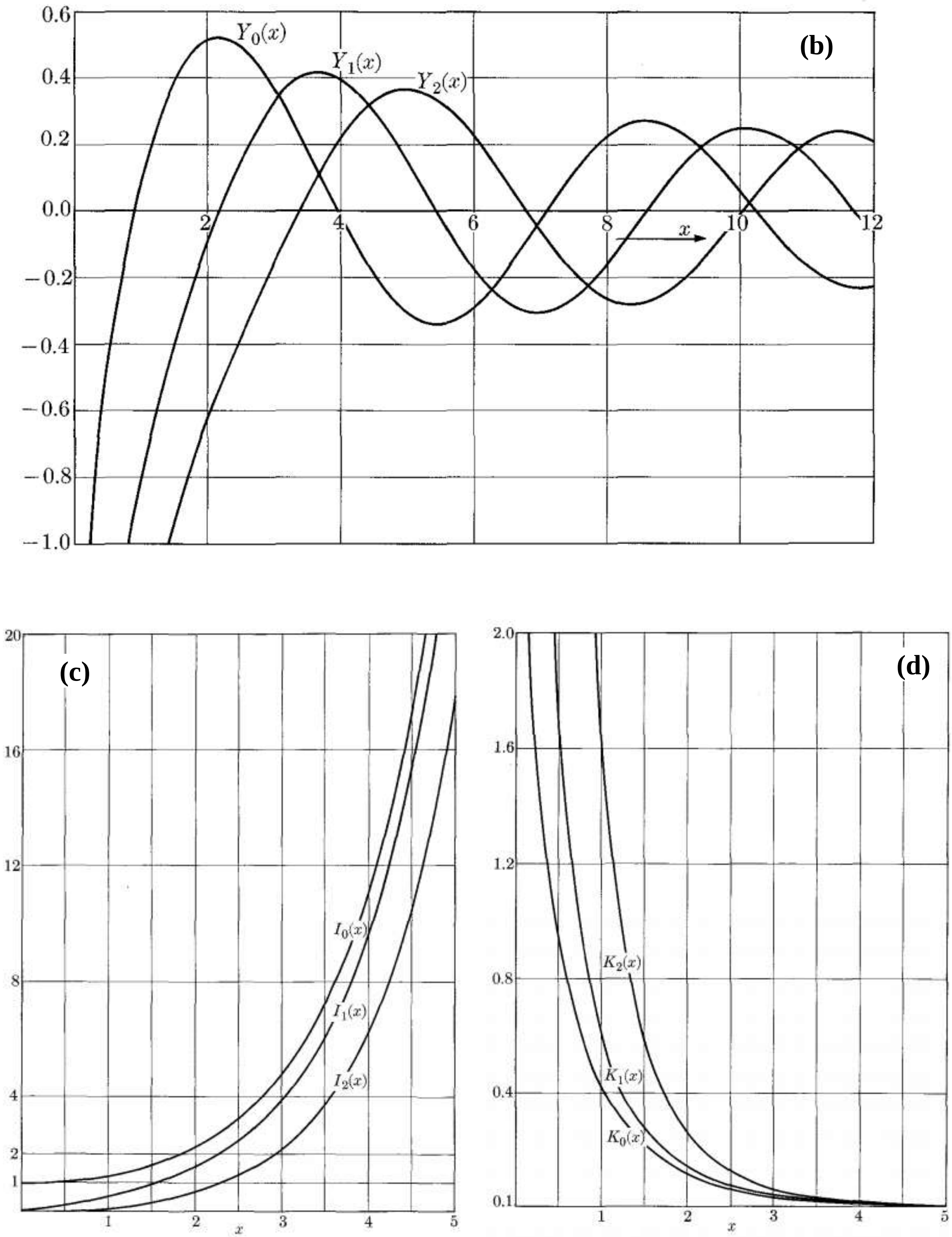

(b)
$Y_0(x)$
$Y_1(x)$
$Y_2(x)$
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
2
4
6
8
10
12
x
(c)
$I_0(x)$
$I_1(x)$
$I_2(x)$
20
16
12
8
4
2
1
1
2
3
4
5
x
(d)
$K_2(x)$
$K_1(x)$
$K_0(x)$
2.0
1.6
1.2
0.8
0.4
0.1
1
2
3
4
5
x

Figure 3.8

Example-1:

$$X^2 \frac{d^2y}{dx^2} + X\frac{dy}{dx} + \lambda^2 X^2 y = 0$$

Solution:

The comparison between the above equation and *Bessel functions* yields to,

$$X^2 \frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

Therefore,

$$1-2A = 1 \xrightarrow{yields} A = 0$$

$$2B = 0 \xrightarrow{yields} B = 0$$

$$2C = 2 \xrightarrow{yields} C = 1$$

$$C^2D^2 = \lambda^2 \xrightarrow{yields} D = \lambda$$

$$C^2n^2 = 0 \xrightarrow{yields} n = 0$$

$$y_{(x)} = C_1 J_0(\lambda X) + C_2 Y_0(\lambda X)$$

Example-2:

$$y'' + \frac{1}{X}y' - \mu^2 y = 0 \qquad (\times X^2)$$

$$X^2y'' + Xy' - \mu^2X^2y = 0$$

Solution:

The comparison between the above equation and *Bessel functions* yields to,

$$X^2 \frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

Therefore,

$$1-2A = 1 \xrightarrow{yields} A = 0$$

$$2B = 0 \xrightarrow{yields} B = 0$$

$$2C = 2 \xrightarrow{yields} C = 1$$

$$C^2D^2 = -\mu^2 \xrightarrow{yields} D = i\mu$$

$$C^2n^2 = 0 \xrightarrow{yields} n = 0$$

$$y_{(x)} = C_1 I_0(\mu X) + C_2 K_0(\mu X)$$

3.2.3 Extended surfaces with variable cross sections

The general formulation of problems of extended surfaces with variable cross sections has already been given by Eq. (3.4) [$\frac{d}{dx}\left(A_c\frac{d\theta}{dx}\right)+\frac{hdA_s}{k\,dx}\theta=0$]. Since A_c and A_s (Pdx) are no longer constant, this equation now becomes a differential equation with variable coefficients whose general solution can be determined only when A_c and A_s are specified. In most cases Eq. (3.4) is reduced to a form of *Bessel's equation*; a special case is that leading to the *equidimensional equation*. Cases which do not lead to either of these equations may be treated individually by employing the *power series solutions* of differential equations.

Example 1: The geometry of a straight fin of triangular profile is described in Figure 3.8. The base temperature T_o of the fin is specified. The temperature distribution and the rate of heat transfer from this triangular fin can be determined as,

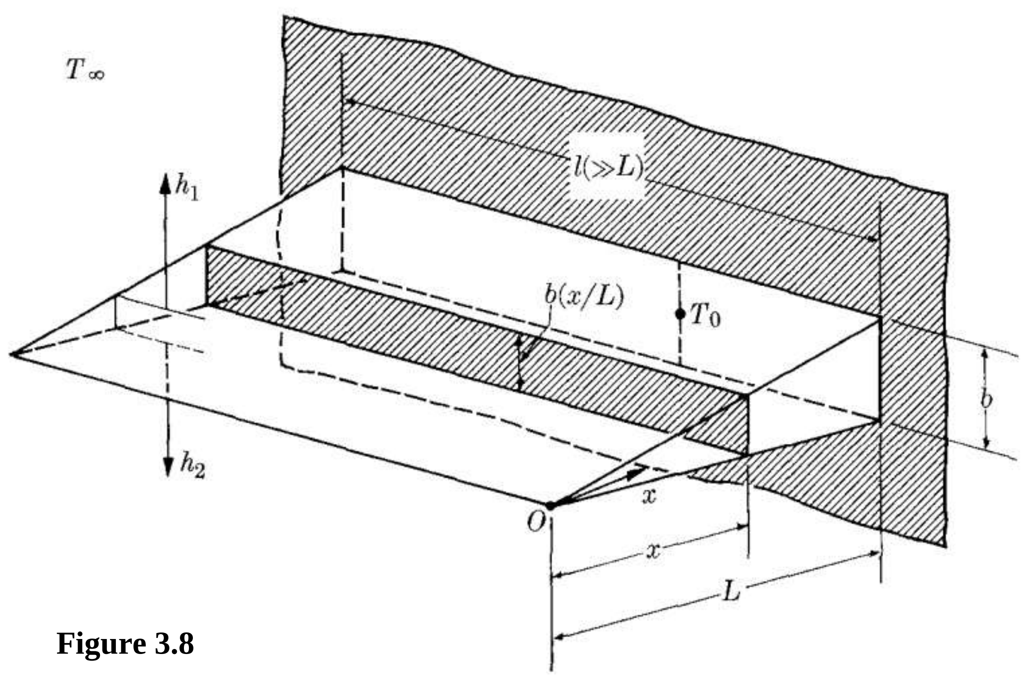


Figure 3.8

Solution:

$b/L << 1$

$L/l << 1$ (The temperature distribution of the present problem is One-dimensional or that the ends in the l-direction are insulated)

Noting from Figure 3.8 that $\theta = (T - T_\infty)$, $A_c = (b\frac{x}{L}) \times l$ and $hP = (\frac{h_1+h_2}{2}) \times 2l$, inserting these values into Eq. (3.4), $\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{hdA_s}{k\,dx}(T - T_\infty) = 0$, and rearranging the result, we get

$$\frac{d}{dx}\left(\left(b\frac{x}{L}\right) \times l\frac{d\theta}{dx}\right) - \frac{\left(\frac{h_1+h_2}{2}\right)\times 2ldx}{k\,dx}\theta = 0 \qquad \rightarrow \qquad \frac{d}{dx}\left[\left(\frac{bl}{L}x\right)\frac{d\theta}{dx}\right] - \frac{(h_1+h_2)l}{k}\theta = 0$$

$$\frac{d}{dx}\left[x\frac{d\theta}{dx}\right] - \frac{(h_1+h_2)L}{bk}\theta = 0$$

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0 \qquad \rightarrow \qquad x^2\,\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2x\theta = 0 \qquad (\times X) \qquad (3\text{-}21)$$

where, $m^2 = (h_1+h_2)L/kb$. Comparison of Eq. (3.21) with the general form of *Bessel function* gives,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

$1 - 2A = 1 \quad \xrightarrow{yields} A = 0$

$2B = 0 \quad \xrightarrow{yields} B = 0$

$2C = 1 \quad \xrightarrow{yields} C = \frac{1}{2}$

$C^2D^2 = -m^2 \qquad (\frac{1}{2})^2D^2 = -m^2 \qquad D = \sqrt{-m^2} \times \sqrt{4} \quad \xrightarrow{yields} D = im \times 2$

$C^2n^2 = 0 \quad \xrightarrow{yields} n = 0$

$y = X^Ae^{BX}[C_1I_n(DX^C) + C_2K_n(DX^C)]$ (Because D is *imaginary*)

$$\theta_{(x)} = C_1I_0\left(2mX^{\frac{1}{2}}\right) + C_2K_0\left(2mX^{\frac{1}{2}}\right) \qquad (3\text{-}22)$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad \theta_{(0)} = finite\ value\ of\ temperature\ at\ the\ fin\ tip$$

Since we have from Figure 3.8 (d), $lim_{x \to 0} K_o(x) \to \infty$

Therefore, the finiteness of tip temperature implies C_2= 0.

Next, the use of the base temperature, $x = L \quad \to \quad \theta_{(L)} = \theta_o = (T_o - T_\infty)$

So, $C_1 = \theta_o / I_o(2mL^{1/2})$. Inserting the values of C_l and C_2 into Eq. (3.22), we find that the temperature distribution in the fin is

$$\frac{\theta_{(x)}}{\theta_o} = \frac{I_o(2mx^{1/2})}{I_o(2mL^{1/2})} \qquad (3\text{-}23)$$

Again, the heat transfer from the fin may conveniently be obtained by considering the conduction through its base. Thus

$$q = -[-kA(d\theta/dx)_{x=L}]$$

which may be evaluated from Eq. (3.23). It follows, in terms of ($\xi = 2mx^{1/2}$), that

$\frac{d}{dx}[I_o(\xi)] = \frac{d}{d\xi}[I_o(\xi)]\frac{d\xi}{dx} = \frac{m}{x^{1/2}} I_1(2mx^{1/2})$, and the heat transfer from the fin is

$$q = kA\theta_o \, \frac{(m/L^{1/2})\, I_1(2mL^{1/2})}{I_o(2mL^{1/2})} \quad \to \quad \frac{q}{\frac{kA\theta_o}{L}} = \frac{(mL^{1/2})\, I_1(2mL^{1/2})}{I_o(2mL^{1/2})}$$

Example 2: Consider a straight fin of parabolic profile as shown in Figure 3.9. The thermal conductivity, base thickness, and length of the fin are k, 2b, and L, respectively. The heat transfer coefficient is h and the ambient temperature T_∞. Find the steady temperature and the total heat transfer from the fin, assuming that parabola is given by $y = Cx^{1/2}$, where C is a constant.

Solution:

$y = Cx^{1/2}$, at $x = L$ then $y = b$

$C = \frac{b}{\sqrt{L}} \quad \to \quad y = \left(\frac{b}{\sqrt{L}}\right) x^{1/2}$

$A_c = 2y \times 1 = 2y$

$A_s = Pdx = 2\,\underbrace{(2y+1)}_{\substack{very\ small \\ negligible}}\,dx = 2dx$

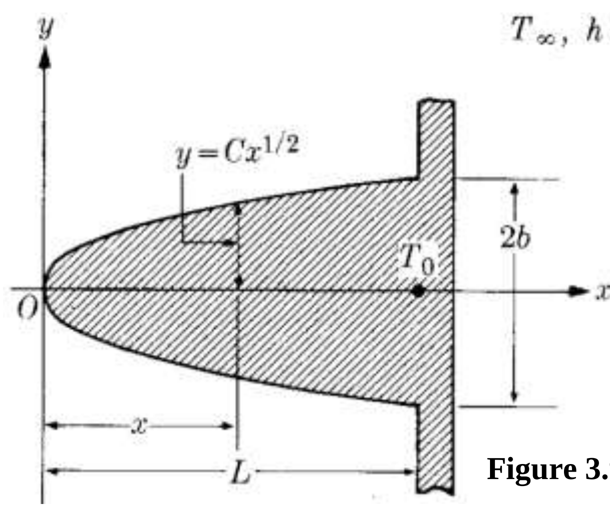


Figure 3.9

From energy balance,

$$q_x = q_{x+dx} + q_{conv.}$$

$$\frac{\partial}{\partial x}\left(-kA_c\frac{\partial T}{\partial x}\right)dx + hPdx(T - T_\infty) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{2b}{\sqrt{L}}x^{1/2}k\frac{\partial T}{\partial x}\right) - 2h(T - T_\infty) = 0 \qquad \text{Let } \theta = (T - T_\infty)$$

$$\frac{\partial}{\partial x}\left(x^{1/2}\frac{\partial \theta}{\partial x}\right) - \frac{h\sqrt{L}}{kb}\theta = 0 \qquad \text{Let } m^2 = \frac{h\sqrt{L}}{kb}$$

Or Eq. (3.4) can be applied directly as,

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{hdA_s}{k\,dx}(T - T_\infty) = 0$$

$$\frac{d}{dx}\left(2y\frac{d\theta}{dx}\right) - \frac{h2dx}{k\,dx}\theta = 0 \qquad \frac{d}{dx}\left(2\left(\frac{b}{\sqrt{L}}\right)x^{1/2}\frac{d\theta}{dx}\right) - \frac{2h}{k}\theta = 0$$

$$\frac{d}{dx}\left(x^{1/2}\frac{d\theta}{dx}\right) - \frac{h\sqrt{L}}{kb}\theta = 0$$

Therefore,

$$x^{1/2}\frac{\partial^2\theta}{\partial x^2} + \frac{1}{2\sqrt{x}}\frac{\partial\theta}{\partial x} - m^2\theta = 0 \qquad \times (x\sqrt{x})$$

$$x^2\frac{\partial^2\theta}{\partial x^2} + \frac{1}{2}x\frac{\partial\theta}{\partial x} - m^2x^{3/2}\theta = 0$$

By comparing the above equation with *Bessel functions* as,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

Therefore,

$$1 - 2A = \frac{1}{2} \quad \xrightarrow{yields} A = \frac{1}{4}$$

$$2B = 0 \quad \xrightarrow{yields} B = 0$$

$$2C = \frac{3}{2} \quad \xrightarrow{yields} C = \frac{3}{4}$$

$$C^2D^2 = -m^2 \qquad \left(\frac{3}{4}\right)^2 D^2 = -m^2 \qquad D = \sqrt{-m^2} \times \sqrt{\left(\frac{4}{3}\right)^2} \quad \xrightarrow{yields} D = im \times \frac{4}{3}$$

$$A^2 - C^2n^2 = 0 \qquad (\frac{1}{4})^2 - (\frac{3}{4})^2n^2 = 0 \qquad \frac{1}{16} \times \frac{16}{9} = n^2 \quad \xrightarrow{yields} n = \frac{1}{3}$$

$$\theta_{(x)} = X^{\frac{1}{4}}\left[C_1 I_{\frac{1}{3}}\left(\frac{4}{3}mX^{\frac{3}{4}}\right) + C_2 K_{\frac{1}{3}}\left(\frac{4}{3}mX^{\frac{3}{4}}\right)\right]$$

Let, $u=\frac{4}{3}mX^{\frac{3}{4}}$ $X=\left(\frac{3}{4m}\right)^{\frac{4}{3}}u^{\frac{4}{3}}$ Now, substituting into $\theta_{(x)}$equation as,

$$\theta_{(u)}=\left(\left(\frac{3}{4m}\right)^{\frac{4}{3}}u^{\frac{4}{3}}\right)^{\frac{1}{4}}\ [C_1 I_{\frac{1}{3}}(u)+C_2 K_{\frac{1}{3}}(u)]$$

$$\theta_{(u)}=\left(\left(\frac{3}{4m}\right)^{\frac{1}{3}}u^{\frac{1}{3}}\right)\ [C_1 I_{\frac{1}{3}}(u)+C_2 K_{\frac{1}{3}}(u)]$$

Boundary conditions:

$x=0 \quad \rightarrow \ u=0 \quad \rightarrow \quad \theta_{(0)}=finite\ value\ of\ temperature$

Since we have from Figure 3.8 (d), $lim_{u\rightarrow 0}\, K_{\frac{1}{3}}(u)\rightarrow\ \infty$

The finiteness of tip temperature implies C_2= 0.

Next, the use of the base temperature, $x=L\ \rightarrow\ u=\frac{4}{3}mL^{\frac{3}{4}}\ \rightarrow\ \theta_{(L)}=\theta_o$

$$\theta_o=\left(\left(\frac{3}{4m}\right)^{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)^{\frac{1}{3}}\right)\ C_1 I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) \qquad \theta_o=L^{\frac{1}{4}}\,C_1 I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)$$

$$\frac{\theta_o}{L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)}=\ C_1 \qquad (3\text{-}24)$$

Inserting the values of C_l and C_2 into $\theta_{(u)}$ equation, we find that the temperature distribution in the fin is

$$\theta_{(u)}=\left(\left(\frac{3}{4m}\right)^{\frac{1}{3}}u^{\frac{1}{3}}\right)\ [\frac{\theta_o}{L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} I_{\frac{1}{3}}(u)] \qquad \frac{\theta_{(u)}}{\theta_o}=\left(\left(\frac{3}{4m}\right)^{\frac{1}{3}}u^{\frac{1}{3}}\right)\ [\frac{L^4 I_{\frac{1}{3}}(u)}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)}] \quad (3\text{-}25)$$

$$\frac{\theta_{(x)}}{\theta_o}=\left(\frac{X}{L}\right)^{\frac{1}{4}}\ \ [\frac{I_{\frac{1}{3}}\left(\frac{4}{3}mX^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)}] \qquad (3\text{-}26)$$

Again, the heat transfer from the fin may conveniently be obtained by considering the conduction through its base. Thus

$$q=\ -[-kA(d\theta/dx)_{x=L}]$$

which may be evaluated from Eq. (3.26). It follows, in terms of ($\xi=\frac{4}{3}mx^{\frac{3}{4}}$), that

$\frac{d}{dx}\left[I_{\frac{1}{3}}(\xi)\right] = \frac{d}{d\xi}\left[I_{\frac{1}{3}}(\xi)\right]\frac{d\xi}{dx} = \frac{4m}{3x^{1/4}} I_{\frac{4}{3}}(\frac{4}{3}mx^{\frac{3}{4}})$, and the heat transfer from the fin is

$$\left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [L^{\frac{1}{4}} \times \frac{4m}{3L^{1/4}} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) \times \frac{L^{\frac{-3}{4}}}{4}]$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [\frac{4m}{3} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + \frac{1}{4L^{\frac{3}{4}}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)]$$

$$q = kA \, \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [\frac{4m}{3} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + \frac{1}{4L^{\frac{3}{4}}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)] \qquad (3\text{-}27)$$

3.2.4 Extended Surface Efficiency

The maximum rate at which a fin could dissipate energy is the rate that would exist if the entire fin surface were at the base temperature. However, since any fin is characterized by a finite conduction resistance, a temperature gradient must exist along the fin and the preceding condition is an idealization. A logical definition of *extended surface efficiency* is therefore, the ratio of the actual to a hypothetical (Ideal) heat transfer:

$$\eta_f = \frac{\text{Actual heat transfer from extended surface}}{\text{Ideal Heat transfer from extended surface}} = \frac{q_{fin}}{q_{Max.}} \qquad (3\text{-}28)$$

Ideal heat transfer from extended surface, it is the heat transfer at the base temperature, $q_{Ideal} = hA_s(T_b - T_\infty)$ where A_s total surface area of the fin

The denominator of Eq. (3.28) denotes the heat transfer from an area of the wall equivalent to the base area of the extended surface; the heat transfer to be evaluated by numerator and denominator together is based on the same temperature difference, base minus ambient. Since the temperature of a wall and the heat transfer coefficient between the wall and the ambient are somewhat changed when an extended surface is attached to the wall, the efficiency defined by Eq. (3.28) is quite approximate. The error involved in this approximation depends on the length of the extended surfaces and the space between them. Therefore, rather than to demonstrate the increased heat transfer from a wall by the use of extended surfaces, this efficiency may better be employed to compare different extended surfaces. The particular values of these efficiencies for specific cases are,

Case D: $$\eta_f\Big|_{\text{Infinite fin (L}\to\infty)} = \frac{\theta_o\sqrt{hPkA_c}}{\theta_o hA_c} = \sqrt{\frac{kP}{hA_c}} \quad (3\text{-}29)$$

Case B: $$\eta_f\Big|_{\text{Adiabatic fin tial}} = \frac{\theta_o\sqrt{hPkA_c}\tanh mL}{\theta_o hA_c} = \sqrt{\frac{kP}{hA_c}}\tanh mL \quad (3\text{-}30)$$

The efficiencies of extended surfaces have been extensively investigated in the literature. In practice, however, the technology involved may be a more important consideration than finding a 5-10% more efficient profile which is expensive to manufacture.

In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the overall surface efficiency η_o characterizes an array of fins and the base surface to which they are attached. Representative arrays are shown in Figure 3.10, where S designates the fin pitch. In each case the overall efficiency is defined as,

$$\eta_f = \frac{q_t}{q_{max}} = \frac{q_t}{hA_f\theta_b} \quad (3\text{-}31)$$

where q_t is the total heat rate from the surface area A_t associated with both the fins and the exposed portion of the base (often termed the prime surface). If there are N fins in the array, each of surface area A_f, and the area of the prime surface is designated as A_b, the total surface area is

$$A_t = NA_f + A_b \quad (3\text{-}32)$$

The maximum possible heat rate would result if the entire fin surface, as well as the exposed base, were maintained at T_b. The total rate of heat transfer by convection from the fins and the prime (unfinned) surface may be expressed as

$$q_t = N\,\eta_f hA_f\theta_b + hA_b\theta_b \quad (3\text{-}33)$$

where the convection coefficient h is assumed to be equivalent for the finned and prime surfaces and η_f is the efficiency of a single fin. Hence

$$q_t = h[N\,\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)]\theta_b \qquad (3\text{-}34)$$

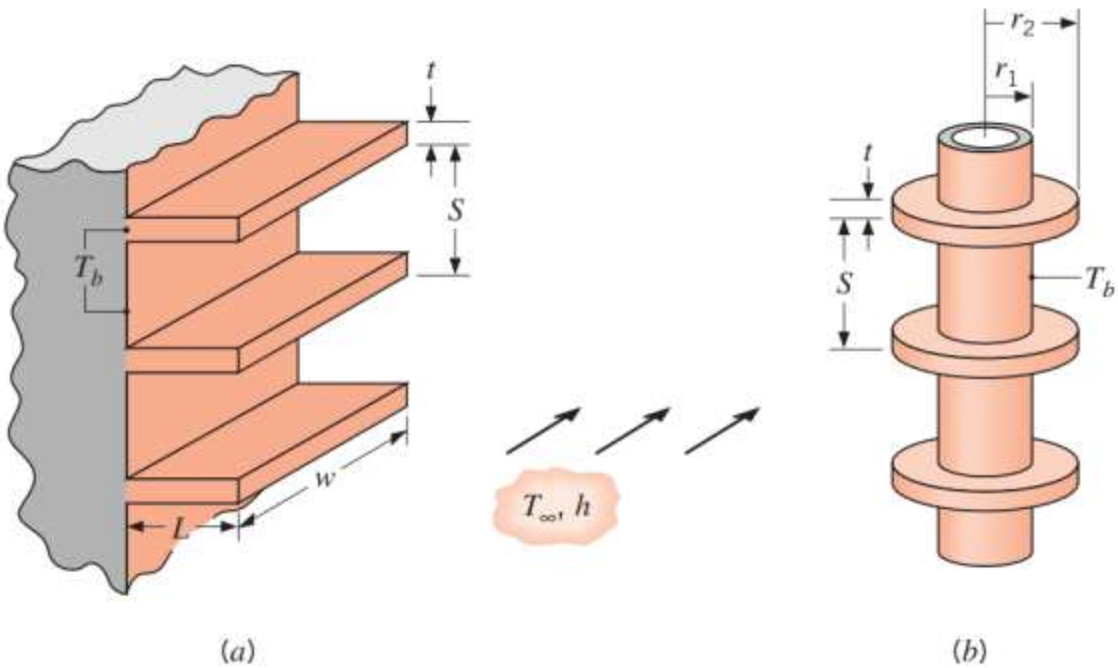


Figure 3.10: Representative fin arrays. (a) Rectangular fins. (b) Annular fins.

Substituting Equation (3.34) into (3.31), it follows that

$$\eta_o = 1 - \frac{NA_f}{A_t}(1-\eta_f) \qquad (3\text{-}35)$$

From knowledge of η_o, Equation 3.31 may be used to calculate the total heat rate for a fin array.

3.2.5 Extended Surface Effectiveness

Recall that fins are used to increase the heat transfer from a surface by increasing the effective surface area. However, the fin itself represents a conduction resistance to heat transfer from the original surface. For this reason, there is no assurance that the heat transfer rate will be increased through the use of fins. An assessment of this matter may be made by evaluating the fin effectiveness ε_f. It is defined as *the* *ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin*. Therefore

$$\varepsilon_f = \frac{\text{Actual heat transfer from extended surface}}{Rate\ of\ heat\ transfer\ from\ the\ base\ area\ (without\ fin)} = \frac{q_f}{hA_{c,b}\theta_b} \qquad (3\text{-}36)$$

where $A_{c,b}$ is the fin cross-sectional area at the base. In any rational design the value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless $\varepsilon_f \geq 2$.

Subject to any one of the four tip conditions that have been considered, the effectiveness for a fin of uniform cross section may be obtained by dividing the appropriate expression for q_f by $hA_{c,b}\theta_b$. Although the installation of fins will alter the surface convection coefficient, this effect is commonly neglected. Hence, assuming the convection coefficient of the finned surface to be equivalent to that of the unfinned base, it follows that, for the infinite fin approximation (**Case D**), the result is,

$$\varepsilon_f = \sqrt{\frac{kP}{hA_c}} \qquad (3\text{-}37)$$

Several important trends may be inferred from this result. Obviously, fin effectiveness is enhanced by the choice of a material of high thermal conductivity. Aluminum alloys and copper come to mind. However, although copper is superior from the standpoint of thermal conductivity, aluminum alloys are the more common choice because of additional benefits related to lower cost and weight. Fin effectiveness is also enhanced by increasing the ratio of the perimeter to the cross-sectional area. For this reason, the use of thin, but closely spaced fins is preferred, with the proviso that the fin gap not be reduced to a value for which flow between the fins is severely impeded, thereby reducing the convection coefficient. Equation 3.37 also suggests that the use of fins can be better justified under conditions for which the convection coefficient h is small. Hence it is evident that the need for fins is stronger when the fluid is a gas rather than a liquid and when the surface heat transfer is by free convection. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side, which is the side of lower convection coefficient. A common example is the tubing in an automobile radiator. Fins are applied to the outer tube surface, over which there is flow of ambient air (*small h*), and not to the inner surface, through which there is flow of water (*large h*). Note that, if $\varepsilon_f \geq 2$ is used as a criterion to justify the implementation of fins, Equation 3.37 yields the requirement that $(kP/hA_c) \geq 4$.

We can draw several important conclusions from equation (3.37) for consideration in the design and selection of the fins:

1. The thermal conductivity of the fin material (k) should be as high as possible. The most widely used fins are made of aluminum.
2. The ratio of the perimeter to the cross-section area of the fin should be as high as possible. This condition is satisfied by thin plate fins or slender pin fins.
3. The use of fins is most effective in applications involving a low convective heat transfer coefficient (h). In liquid-to-gas heat exchanger such as the car radiator, fins are placed on the gas side.

Homework 2: Consider a straight fin of parabolic profile as shown in Figure 3.11. The thermal conductivity, base thickness, and length of the fin are k, $2b$, and L, respectively. The heat transfer coefficient is h and the ambient temperature T_∞. Find the steady temperature of and the total heat transfer from the fin, assuming that parabola is given by $y = Cx^2$, where C is a constant.

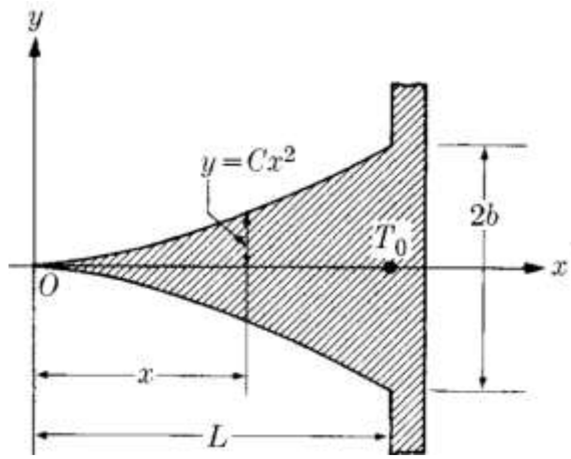


Figure 3.11

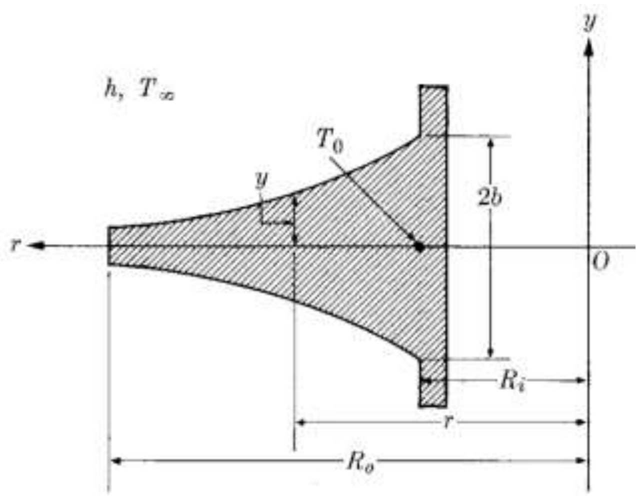


Figure 3.12

Homework 3: Consider a straight fin of parabolic profile as shown in Figure 3.12. The thermal conductivity, base thickness, and inner and outer radii of the fin are k, $2b$, R_i and R_o, respectively. The heat transfer coefficient is h and the ambient temperature T_∞. Find the steady temperature of and the total heat transfer from the fin, assuming that hyperbola is given by (a) $yr^{1/2} = C$, (b) $yr^2 = C$ where C is a constant.